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ON THE YAGER TYPE ENTROPY OF DYNAMICAL SYSTEMS

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Abstract. The notion of the Kolmogorov-Sinaj entropy of a dynamical system $(X, \mathcal{S}, P, T)$ is based on measurable set partitions $\mathcal{A} = \{A_1, \dots, A_n\}$ and their entropy $H(\mathcal{A}) = -\Sigma P(A_i) \log P(A_i)$. The notion was extended by many authors using partitions of unity, i.e., such collections $(f_1, \dots, f_n)$ of non-negative measurable functions that $\sum_{i=1}^n f_i = 1_\Omega$. In this contribution we extend this approach considering such collections $(f_1, \dots, f_n)$ of non-negative measurable functions that $\sum_{i=1}^n f_i^k = 1_\Omega$.

1. assumptions

There is given a dynamical system $(X, \mathcal{S}, P, T)$, where $(X, \mathcal{S}, P)$ is a probability space and $T \colon X \to X$ is a measure preserving transformation (i.e., $A \in \mathcal{S}$ implies $T^{-1}(A) \in \mathcal{S}$ and $P(A) = P\big(T^{-1}(A)\big)$). By $\mathcal{F}$ we denote the family of all $\mathcal{S}$-measurable functions from X to $< 0, 1 >$. Consider $k \geq 1$. We define a mapping $m \colon \mathcal{F} \to < 0, 1 >$ by the formula $m(f) = \int_X f^k dP$.

By a Yager partition (in the paper) we mean a finite collection $f_1, \dots, f_n)$ of functions such that

$$\sum_{i=1}^n f_i^k = 1_X .$$

Denote by $\mathcal{P}$ the family of all Yager partitions. If $\mathcal{A}, \mathcal{B} \in \mathcal{P}$, $\mathcal{A} = (f_1, \dots, f_n)$, $\mathcal{B} = (g_1, \dots, g_n)$, then we define

$$\mathcal{A} \vee \mathcal{B} = \{f_i \cdot g_j;\ i = 1, \dots, n,\ j = 1, \dots n\} .$$

Since

$$\sum_{i=1}^n \sum_{j=1}^m (f_i \cdot g_j)^k = \Big(\sum_{i=1}^n f_i^k\Big) \Big(\sum_{j=1}^m g_j^k\Big) = 1_X ,$$

$\mathcal{A} \vee \mathcal{B}$ is a Yager partition again. Finally define $\varphi \colon < 0, \infty) \to < 0, \infty)$ by the prescription

$$\varphi(x) = \begin{cases} -x \log x , & \text{if } x > 0 , \\ 0 & \text{if } x = 0 , \end{cases}$$

and for $\mathcal{A}, \mathcal{B} \in \mathcal{P}$ the entropy

$$H(\mathcal{A}) = \sum_{i=1}^n \varphi\big(m(f_i)\big)$$

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and the conditional entropy

$$H(\mathcal{A}|\mathcal{B}) = \sum_i \sum_j m(g_j)\,\varphi\Big(\frac{m(f_i \cdot g_j)}{m(g_j)}\Big)\,,$$

where the summands with $m(g_j) = 0$ are ommited.

2. Basic lemmas

Lemma 1. *$H(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) \leq H(\mathcal{B}|\mathcal{B})$ for any $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{P}$.*

Proof. Let $\mathcal{A} = (f_1, \dots, f_n)$, $\mathcal{B} = (g_1, \dots, g_p)$, $\mathcal{C} = (h_1, \dots, h_q)$. Then

$$\begin{aligned} H(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) &= \sum_i \sum_j \sum_\ell m(g_j \cdot h_\ell)\varphi\Big(\frac{m(f_i \cdot g_j \cdot h_\ell)}{m(g_j \cdot h_\ell)}\Big) \\ &= \sum_i \sum_j m(g_j) \sum_\ell m\frac{(g_j \cdot h_\ell)}{m(g_j)}\varphi\Big(\frac{m(f_i \cdot g_j \cdot h_\ell)}{m(g_j \cdot h_\ell)}\Big)\,. \end{aligned}$$

For fixed i, j put

$$\alpha_\ell = \frac{m(g_j \cdot h_\ell)}{m(g_j)}\,,\ x_\ell = \frac{m(f_i \cdot g_j \cdot h_\ell)}{m(g_j \cdot h_\ell)}\,.$$

Since

$$\sum_{\ell=1}^q g_j^k \cdot h_\ell^k = g_j^k \sum_{\ell=1}^q h_\ell^k = g_j^k \cdot 1_X = g_j^k\,,$$

we obtain

$$m(g_j) = \int_X g_j^k dP = \sum_{\ell=1}^q \int_X g_j^k \cdot h_\ell^k dP = \sum_{\ell=1}^q m(g_j \cdot h_\ell)\,,$$

hence

$$\sum_{\ell=1} \alpha_\ell = \frac{1}{m(g_j)} \sum_{\ell=1}^q m(g_j \cdot h_\ell) = 1\,.$$

Using Jensen inequality $\sum_{\ell=1}^a \alpha_\ell \varphi(x_\ell) \leq \varphi\Big(\sum_{\ell=1}^q \alpha_\ell x_\ell)$, we obtain

$$H(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) \leq \sum_i \sum_j m(g_j)\varphi\Big(\sum_\ell \frac{m(f_i \cdot g_j \cdot h_\ell)}{m(g_j)}\Big)\,.$$

Again

$$\sum_\ell m(f_i \cdot g_j \cdot h_\ell) = \sum_\ell \int_X f_i^k \cdot g_j^k \cdot h_\ell^k dP = \int_X f_i^k dP = m(f_i \cdot g_j)\,,$$

hence

$$H(\mathcal{A}|\mathcal{B} \vee \mathcal{C}) \leq \sum_i \sum_j m(g_j)\varphi\Big(\frac{m(f_i \cdot g_j)}{m(g_j)}\Big) = H(\mathcal{A}|\mathcal{B})\,.$$

Lemma 2. *$H(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) = H(\mathcal{A}|\mathcal{C}) + H(\mathcal{B}|\mathcal{A} \vee \mathcal{C})$.*

Proof.

$$\begin{aligned}
H(\mathcal{A} \vee \mathcal{B}|\mathcal{C}) = & \sum_i \sum_j \sum_\ell m(h_\ell)\, \varphi\Big) \frac{m(f_i \cdot g_j \cdot h_\ell)}{m(h_\ell)}\Big) \\
= & \sum_i \sum_j \sum_\ell m(h_\ell)\, \varphi\Big(\frac{m(f_i \cdot g_j \cdot h_\ell)}{m(f_i \cdot h_\ell)} \cdot \frac{m(f_i \cdot h_\ell)}{m(h_\ell)}\Big) \\
= & -\sum_i \sum_j \sum_\ell m(f_i \cdot g_j \cdot h_\ell) \log \frac{m(f_i \cdot g_j \cdot h_\ell)}{m(f_i \cdot h_\ell)} \\
& -\sum_i \sum_j \sum_\ell m(f_i \cdot h_\ell) \log \frac{m(f_i \cdot h_\ell)}{m(h_\ell)} \\
= & \sum_i \sum_j \sum_\ell m(f_i \cdot h_\ell)\, \varphi\Big(\frac{m(f_i \cdot g_j \cdot h_\ell)}{m(f_i \cdot h_\ell)}\Big) \\
& -\sum_i \sum_\ell \Big(\log \frac{m(f_i \cdot h_\ell)}{m(h_\ell)}\Big) \Big(\sum_j m(f_i \cdot g_j \cdot h_\ell)\Big) \\
= & \, H(\mathcal{B}|\mathcal{A} \vee \mathcal{C}) + \sum_i \sum_\ell m(h_\ell)\, \varphi\Big(\frac{m(f_i \cdot h_\ell)}{m(h_\ell)}\Big) \\
= & \, H(\mathcal{B}|\mathcal{A} \vee \mathcal{C}) + H(\mathcal{A}|\mathcal{C}) .
\end{aligned}$$

Lemma 3. *Put $\mathcal{E} = \{1_X\} \in \mathcal{P}$. Then $\mathcal{A} \vee \mathcal{E} = \mathcal{A}$ and $H(\mathcal{A}|\mathcal{E}) = H(\mathcal{A})$ for any $\mathcal{A} \in \mathcal{P}$.*

Proof. By the definition

$$\begin{aligned}
\mathcal{A} \vee \mathcal{E} &= \{f_i \cdot 1_X;\ i = 1, \dots, n\} = \{f_i;\ i = 1, \dots, n\} = \mathcal{A}, \\
H(\mathcal{A}|\mathcal{E}) &= \sum_i m(1_X)\, \varphi\Big(\frac{m(f_i \cdot 1_X)}{m(1_X)}\Big) = \sum_i \varphi\big(m(f_i)\big) = H(\mathcal{A}) .
\end{aligned}$$

Lemma 4. *Define $U(\mathcal{A}) = (f_i \circ T, \dots, f_n \circ T)$ for any $\mathcal{A} \in \mathcal{P}$. Then $U(\mathcal{A}) \in \mathcal{P}$ and*

$$\begin{aligned}
U(\mathcal{A} \vee \mathcal{B}) = U(\mathcal{A}) \vee U(\mathcal{B}),\ & U(\mathcal{E}) = \mathcal{E}, \\
H\big(U(\mathcal{A})|U(\mathcal{B})\big) = H(\mathcal{A}|\mathcal{B}) . &
\end{aligned}$$

for any $\mathcal{A}, \mathcal{B} \in \mathcal{P}$.

Proof. Evidently $\sum_{i=1}^n (f_i \circ T)^k(x) = \sum_{i=1}^n \big(f_i(T(x))\big)^k = 1_X\big(T(x)\big) = 1$ for any $x \in X$, hence $U(\mathcal{A}) \in \mathcal{P}$. Further

$$\begin{aligned}
U(\mathcal{A} \vee \mathcal{B}) &= \big\{f_i \cdot g_j) \circ T;\ i, j\big\} = \big\{(f_i \circ T) \cdot (g_j \circ T);\ i, j\big\} \\
&= U(\mathcal{A}) \vee U(\mathcal{B}),
\end{aligned}$$

$$U(\mathcal{E}) = \{1_X \circ T\} = \{1_X\} = \mathcal{E}$$

and

$$H\big(U(\mathcal{A})|U(\mathcal{B})\big) = \sum_i \sum_j m(g_j \circ T)\,\varphi\Big(\frac{m((f_i \circ T)\cdot(g_j \circ T))}{m(g_j \circ T)}\Big)\,.$$

Of course

$$m\big((f_i \circ T)\cdot(g_j \circ T)\big) = \int_X \big((f_i \cdot g_j)\circ T\big)^k dP = \int_X (f_i \cdot g_j)^k dP = m(f_i \cdot g_j)$$

since T is P-measure preserving. Similarly $m(g_j \circ T) = m(g_j)$. Therefore

$$H\big(U(\mathcal{A})|(\mathcal{B})\big) = H(\mathcal{A}|\mathcal{B})\,.$$

3. General construction

In this section we shall assume that there is given a structure $(\mathcal{P}, \vee, H, U, \mathcal{E})$, where $\mathcal{P}$ is a set, $\vee$ is a commutative, associative operation on $\mathcal{P}$, $U\colon \mathcal{P} \to \mathcal{P}$, $H\colon \mathcal{P}\times\mathcal{P} \to <0,\infty)$, $\mathcal{E} \in \mathcal{P}$ is a fixed element such that

$H(\mathcal{A}|\mathcal{B}\vee\mathcal{C}) \le H(\mathcal{A}|\mathcal{B})$ for any $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathcal{P}$,
$H(\mathcal{A}\vee\mathcal{B}|\mathcal{E}) = H(\mathcal{A}|\mathcal{C}) + H(\mathcal{B}|\mathcal{A}\vee\mathcal{C})$ for any $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathcal{P}$,
$H(\mathcal{E}) = \mathcal{E}$, $\mathcal{A}\vee\mathcal{E} = \mathcal{A}$ for any $\mathcal{A} \in \mathcal{P}$,
$U(\mathcal{A}\vee\mathcal{B}) = U(\mathcal{A})\vee U(\mathcal{A})\vee U(\mathcal{B})$ for any $\mathcal{A},\mathcal{B} \in \mathcal{P}$.

We have seen that if $\mathcal{P}$ is the set of all Yager partitions (with fixed power k), then all the assumptions are satisfied.

Define in our general scheme

$$H(\mathcal{A}) = H(\mathcal{A}|\mathcal{E})$$

and $U^i(\mathcal{A})$ by iteration, i.e., $U^0(\mathcal{A}) = \mathcal{A}$, $U^{i+1}(\mathcal{A}) = U\big(U^i(\mathcal{A})\big)$. Instead of $\mathcal{A}_1 \vee \mathcal{A}_2 \vee \cdots \vee \mathcal{A}_n$ we shall use the notation $\bigvee_{i=1}^n \mathcal{A}_i$.

Theorem 1. *Put $a_n = H\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big)$. Then there exists $\lim_{n\to\infty} \frac{1}{n} a_n$.*

Proof. It follows by the inequality $a_{n+m} \le a_n + a_m$. Indeed,

$$\begin{aligned} a_{n+m} &= H\Big(\bigvee_{i=o}^{n+m-1} U^i(\mathcal{A})\Big) = H\bigg(\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big) \vee U^n\Big(\bigvee_{j=0}^{m-1} U^j(\mathcal{A})\Big)\bigg) \\ &\le H\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big) + H\bigg(U^n\Big(\bigvee_{j=0}^{n-1}(\mathcal{A})\Big)\bigg) = a_n + a_m\,. \end{aligned}$$

Definition. For $\mathcal{A} \in \mathcal{P}$ define

$$h(\mathcal{A}, U) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big)\,.$$

Theorem 2. *For any $\mathcal{A}, \mathcal{B} \in \mathcal{P}$ we have*

$$h(\mathcal{A}, U) \leq h(\mathcal{B}, U) + H(\mathcal{A}|\mathcal{B}).$$

Proof. We have

$$\begin{aligned} H\Big(\bigvee_{i=o}^{n-1} U^i(\mathcal{A})\Big) &\leq H\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A}) \vee \bigvee_{j=0}^{n-1} U^j(\mathcal{B})\Big) \\ &= H\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{B})\Big) + H\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A}) \Big| \bigvee_{j=0}^{n-1} U^j(\mathcal{B})\Big). \end{aligned}$$

Of course

$$\begin{aligned} H\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A}) \Big| \bigvee_{j=0}^{n-1} U^j(\mathcal{B})\Big) &\leq \sum_{i=0}^{n-1} H\Big(U^i(\mathcal{A}) \Big| \bigvee_{j=0}^{n-1} U^j(\mathcal{B})\Big) \\ &\leq \sum_{i=0}^{n-1} H\Big(U^i(\mathcal{A}) \Big| U^i\mathcal{B}\Big) = \sum_{i=o}^{n-1} H(\mathcal{A}|\mathcal{B}) \\ &= nH(\mathcal{A}|\mathcal{B}). \end{aligned}$$

Theorem 3. $h(\mathcal{B}, U) = h\Big(\bigvee_{j=0}^{t} U^j(\mathcal{B}), U\Big)$ *for any $\mathcal{B} \in \mathcal{P}$ and any $t \in \mathbb{N}$.*

Proof. We obtain immediately

$$\lim_{n\to\infty} \frac{1}{n}\Big(\bigvee_{i=0}^{n-1} U^i\Big(\bigvee_{j=0}^{t} U^j(\mathcal{B})\Big)\Big) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{s=0}^{n+t-1} U^s(\mathcal{B})\Big)$$

$$\lim_{p\to\infty} \frac{p}{p-t}\frac{1}{p}\Big(\bigvee_{s=0}^{p-1} U^s(\mathcal{B})\Big) = h(u, \mathcal{B}).$$

4. Hudetz correction

Let us return to the original assumptions about $(X, \mathcal{S}, P, T)$ and $\mathcal{F}$. To obtain the Kolmogorov-Sinaj entropy one can define

$$h(T) = \sup\big\{h(\mathcal{A}, U);\ \mathcal{A} \in \mathcal{K}\big\},$$

where $\mathcal{K}$ is the family of all partitions of the form $\big(\chi_{A_1}, \chi_{A_2}, \dots, \chi_{A_n}\big)$. It could be possible to define analogously the (Yager) generalized entropy as the supremum $\sup\big\{h(\mathcal{A}, U)\ \mathcal{A} \in Q\big\}$, where $Q \in \mathcal{F}$. Of course, if Q obtains all constant functions, then the supremum is ∞. Therefore Hudetz suggested the following correction coefficient: for $\mathcal{A} = (f_1, \dots, f_q) \in \mathcal{P}$ put

$$\Phi(\mathcal{A}) = \sum_{i=1}^{q} \int_X \varphi(f_i^k)dP,\ H^b(\mathcal{A}) = H(\mathcal{A}) - \Phi(\mathcal{A}).$$

Proposition 1. $\Phi\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big) = n\Phi(\mathcal{A})$.

Proof. Let $\mathcal{A} = (f_1, \dots, f_q)$. Put $g_j^{(i)} = (f_j \circ T^i)^k$. Then

$$\Phi\Big(\bigvee_{i=1}^{n-1} u^i(\mathcal{A})\Big) = \sum_{p_0=1}^{q} \sum_{p_1=1}^{q} \cdots \sum_{p_{n-1}=1}^{q} \int_X \varphi\big(g_{p_1}^{(0)} g_{p_1}^{(1)} \cdots g_{p_{n-1}}^{(n-1)}\big)\, dP\,.$$

We have

$$\begin{aligned}
&\sum_{p_0} \sum_{p_1} \cdots \sum_{p_{n-1}} \varphi\big(g_{p_0}^{(0)} g_{p_1}^{(1)} \cdots g_{p_{n-1}}^{(n-1)}\big) \\
&\quad = -\sum_{p_0} \sum_{p_1} \cdots \sum_{p_{n-1}} g_{p_0}^{(0)} g_{p_1}^{(1)} \cdots g_{p_{n-1}}^{(n-1)} \big(\log g_{p_0}^{(0)} + \log g_{p_1}^{(1)} + \cdots + \log g_{p_{n-1}}^{(n-1)}\big) \\
&\quad = \sum_{p_0} \varphi\big(g_{p_0}^{(0)}\big) \sum_{p_1} \cdots \sum_{p_{n-1}} g_{p_1}^{(1)} \cdots g_{p_{n-1}}^{(n-1)} \\
&\qquad + \sum_{p_1} \varphi\big(g_{p_1}^{(1)}\big) \sum_{p_0} \sum_{p_2} \cdots \sum_{p_{n-1}} g_{p_0}^{(0)} g_{p_2}^{(2)} \cdots g_{p_{n-1}}^{(n-1)} \\
&\qquad \cdots \\
&\qquad + \sum_{p_{n-1}} \varphi\big(g_{p_{n-1}}^{(n-1)}\big) \sum_{p_0} \cdots \sum_{p_{n-2}} g_{p_0}^{(0)} \cdots g_{p_{n-2}}^{(n-2)}\,.
\end{aligned}$$

Of course

$$\sum_{p_i=1}^{q} g_{p_i}^{(i)} = \sum_{p_i=1}^{q} \big(f_{p_i} \circ T^i\big)^k = 1_X\,,$$

hence the last sum equals

$$\sum_{i=0}^{n-1} \sum_{p_i=1}^{q} \varphi\big(g_{p_i}^{(i)}\big)\,,$$

and therefore

$$\Phi\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big) = \sum_{i=0}^{n-1} \sum_{p_i=1}^{q} \int_X \varphi\big(g_{p_i}^{(i)}\big)\, dP\,.$$

On the other hand, because T is measure preserving, we have

$$\int_X \varphi\big(g_{p_i}^{(i)}\big)\, dP = \sum_X \varphi\big((f_{p_i} \circ T^i)^k\big)\, dP = \int_X \varphi(f_{p_i}^k)\, dP\,,$$

hence

$$\begin{aligned}
\Phi\Big(\bigvee_{i=0}^{n-1} u^i(\mathcal{A})\Big) &= \sum_{i=0}^{n-1} \sum_{p_i=1}^{q} \int_X \varphi(f_j^k)\, dP \\
&= n \sum_{j=1}^{q} \int_X \varphi(f_j)\, dP = n\Phi(\mathcal{A})\,.
\end{aligned}$$

Proposition 2. *For any Yager partition $\mathcal{A} = (f_1, \dots, f_q)$ there exists*

$$\lim_{n\to\infty} \frac{1}{n} H^b\Big(\bigvee_{i=0}^{n-1} u^i(\mathcal{A})\Big) = h(\mathcal{A}, U) - \sum_{j=1}^{q} \int_X \varphi(f_j^k)\,dP\,.$$

Proof. By Proposition 1

$$\frac{1}{n} H^b\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big) = \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} u^i(\mathcal{A})\Big) - \sum_{j=1}^{q} \int_X \varphi(f_j^k)\,dP\,.$$

Definition. For any $\mathcal{A} \in \mathcal{P}$ define

$$h^b(\mathcal{A}, U) = \lim_{n\to\infty} \frac{1}{n} H^b\Big(\bigvee_{i=0}^{n-1} U^i(\mathcal{A})\Big)\,.$$

Proposition 3. *If $\mathcal{A} = (\chi_{A_1}, \dots \chi_{A_q})$, then*

$$h^b(\mathcal{A}, U) = h(\mathcal{A}, U)\,.$$

Proof. Since $\varphi(\chi_{A_j}^k) = \varphi(\chi_{A_j}) = 0$ for any j, we obtain the equality by Proposition 2.

5. Conditional expectation

Let us consider a partition $\mathcal{B} = \{\chi_{B_1}, \dots, \chi_{B_2}\}$ and an integrable function $g\colon X \to \mathbb{R}$. Recall that the conditional expectation of g with respect to the σ-algebra $\sigma(\mathcal{B})$ generated by $\mathcal{B}$ can be constructed by the formula

$$E\big(g|\sigma(\mathcal{B})\big) = \sum_j \bigg(\frac{1}{P(B_j)} \int_{B_j} g\,dP\bigg)\chi_{B_j}\,.$$

Proposition 4. *If $\mathcal{B} = \big(\chi_{B_1}, \dots \chi_{B_q}\big)$, $\mathcal{A} = (f_1, \dots, f_p)$, then*

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{p} \int_X \big(E(f_i^k|\sigma(\mathcal{B}))\big)dP\,.$$

Proof. By the definition

$$H(\mathcal{A}|\mathcal{B}) = \sum_{i=1}^{p} \sum_{j=1}^{q} m(\chi_{B_j})\varphi\Big(\frac{m(f_i \cdot \chi_{B_j})}{m(\chi_{B_j})}\Big)\,.$$

Of course,

$$m(f_k \chi_{B_j}) = \int_X f_i^k \chi_{B_j}^k\,dP = \int_{B_j} f_i^k\,dP\,,$$

hence

$$\begin{aligned} H(\mathcal{A}|\mathcal{B}) &= \sum_{i=1}^{p}\sum_{j=1}^{q} P(B_j)\varphi\Big(\frac{1}{P(B_j)}\int_{B_j} f_i^k\,dP\Big) \\ &= \sum_{i=1}^{p}\int_X \sum_{j=1}^{q}\Big(\varphi\Big(\frac{1}{p(B_j)}\int_{B_j} f_i^k\,dP\Big)\Big)\chi_{B_j}\,dP \\ &= \sum_{i=1}^{p}\int_X \varphi\Big(\sum_{j=1}^{q}\Big(\frac{1}{P(B_j)}\int_{B_j} f_i^k\,dP\Big)\chi_{B_j}\Big)\,dP \\ &= \sum_{i=1}^{p}\int_X \varphi\big(E(f_i^k|\sigma(\mathcal{B}))\big)\,dP\,. \end{aligned}$$

Theorem 4. *If* $\mathcal{A} = (f_1, \dots, f_p)$ *and* $\mathcal{B} = (\chi_{B_1}, \dots, \chi_{B_q})$ *is such that* $\sigma\Big(\bigcup_{i=0}^{\infty} U^i(\mathcal{B})\Big) = \mathcal{S}$, *then*

$$h(\mathcal{A}, U) \le h(\mathcal{B}, U) + \sum_{i=1}^{p}\int_X \varphi(f_i^k)\,dP\,.$$

Proof. Put $\mathcal{B}_n = \bigvee_{i=0}^{n-1} U^i(\mathcal{B})$. By Theorem 2 we obtain

$$h(\mathcal{A}, U) \le h(\mathcal{B}_n, U) + H(\mathcal{A}|\mathcal{B}_n)\,.$$

By Theorem 3

$$h(\mathcal{B}_n, U) = h(\mathcal{B}, U)\,,$$

hence by Proposition 4

$$h(\mathcal{A}, U) \le h(\mathcal{B}, U) + \sum_{i=1}^{p}\int_X \varphi\big(E(f_i^k|\sigma(\mathcal{B}_n))\big)\,dP\,.$$

Since $\sigma\Big(\bigcup_{n=1}^{\infty} \mathcal{B}_n\Big) = \mathcal{S}$, we have by the martingale convergence theorem $(n \to \infty)$.

$$E\big(f_i^q|\sigma(\mathcal{B}_n)\big) \nearrow E(f_i^k|\mathcal{S}) = f_i^k\,.$$

Since φ is a continuous function, we obtain

$$\varphi\big(E(f_i^k|\sigma(\mathcal{B}_n))\big) \to \varphi(f_i^k)$$

$(n \to \infty)$ P-almost everywhere. Finally by the Lebesgue dominance convergence theorem

$$\lim_{n\to\infty}\int_X \varphi\big(E(f_i^k|\sigma(\mathcal{B}_n))\big)\,dP = \int_X \varphi(f_i^k)dP$$

and therefore

$$h(\mathcal{A}, U) \le h(\mathcal{B}, U) + \sum_{i=1}^{p}\int_X \varphi(f_i^k)\,dP\,.$$

6. Kolmogorov theorem on generators

The Kolmogorov theorem states that if $\mathcal{B} = \{\chi_{b_1}, \dots \chi_{B_p?}\}$ is a generator (i.e., $\sigma\Big(\bigcup_{i=0}^{\infty} U^i(\mathcal{B}) = \mathcal{S}\Big)$, then

$$h(\mathcal{B}, U) = \sup\big\{h(\mathcal{A}, U);\ \mathcal{A} \in \mathcal{K}\big\}.$$

We shall prove the theorem for the general entropy with the Hudetz correction.

Theorem 5. *Let* $\mathcal{B} = \{\chi_{B_1}, \dots, \chi_{B_q}\}$ *be a generator, i.e.,* $\sigma\Big(\bigcup_{i=1}^{\infty} U^i(\mathcal{B})\Big) = \mathcal{S}$. *Then for any* $\mathcal{A} = (f_1, \dots, f_p) \in \mathcal{P}$

$$h^b(\mathcal{A}, U) \le h^b(\mathcal{B}, U) = h(\mathcal{B}, U),$$

hence

$$\sup\big\{h^b(\mathcal{A}, U);\ \mathcal{A} \in \mathcal{P}\big\} = h^b(\mathcal{B}, U) = h(\mathcal{B}, U).$$

Proof. By Proposition 2 we have

$$h^b(\mathcal{A}, U) = h(\mathcal{A}, U) - \sum_{j=1}^{p} \int_X \varphi(f_j^k)\, dP,$$

hence by Theorem 4

$$h^b(\mathcal{A}, U) \le h(\mathcal{B}, U).$$

Of course, $h(\mathcal{B}, U) = h^b(\mathcal{B}, U)$ by Proposition 3.

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