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Coexistence vs. functional coexistence of quantum observables

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COEXISTENCE VS. FUNCTIONAL COEXISTENCE OF QUANTUM OBSERVABLES

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Abstract. We consider the question when coexistent observables are functionally coexistent and give partial answers for special classes of observables on effect algebras, in particular on the Hilbert space effects.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded operators, $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{L}(\mathcal{H}) \,|\, O \leq A \leq I\}$ the set of effects, and $\mathcal{P}(\mathcal{H}) = \{P \in \mathcal{P}(\mathcal{H}) \,|\, P = P^2 = P^*\}$ the set of projections on $\mathcal{H}$. Let Ω be a nonempty set and $\mathcal{A}$ a σ-algebra of subsets of Ω. Let $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ be an operator measure, which is positive, that is, $E(X) \geq O$ for all $X \in \mathcal{A}$, and normalized, that is, $E(\Omega) = I$. We call such measures *observables*.

Observables $E_1 : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ are *coexistent* if there is an observable $F : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ such that $\mathrm{ran}\,(E_1) \cup \mathrm{ran}\,(E_2) \subseteq \mathrm{ran}\,(F)$, that is, for each $X \in \mathcal{A}_1$, and $Y \in \mathcal{A}_2$, $E_1(X) = F(Z_X)$, and $E_2(Y) = F(Z_Y)$ for some $Z_X, Z_Y \in \mathcal{A}$. Observables $E_1 : \mathcal{A}_1 \to \mathcal{L}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ are *functionally coexistent* if there is an observable $F : \mathcal{A} \to \mathcal{L}(\mathcal{H})$ and (measurable) functions $f_1 : \Omega \to \Omega_1$ and $f_2 : \Omega \to \Omega_2$ such that for each $X \in \mathcal{A}_1$, and $Y \in \mathcal{A}_2$, $E_1(X) = F(f_1^{-1}(X))$, and $E_2(Y) = F(f_2^{-1}(Y))$. Functionally coexistent observables are coexistent, but we do not know whether these notions coincide. On the other hand, functionally coexistent observables can be characterized in terms of biobservables and joint observables [1, 2].

In this paper we shall investigate the question under which conditions coexistent observables are functionally coexistent. Our results do not depend on the special properties of a Hilbert space. Therefore, we formulate and study this question in a more general frame of an effect algebra, treating, however, the important Hilbert space case separately in the final section of the paper.

A necessary and sufficient condition when for two real observables E, F on an effect algebra there is a measurable function f such that

$E = F \circ f^{-1}$ is given in [3, A.4.12 Theorem]. This condition is exactly what is needed to apply a procedure analogous to [4, Theorem 1.4]. This condition is, however, rather general and hard to be checked. In the present paper, we present some classes of observables, classified by properties of their ranges and the corresponding morphisms, which imply functional coexistence.

We start with a brief summary of the basic notions and definitions on effect algebras which shall be needed here. For more details and motivation, the reader is referred to [5, 6, 7, 8].

2. Basic definitions and notions

2.1. **General.** An *effect algebra* is a system $(L, 0, 1, \oplus)$ where L is a nonempty set, $0, 1$ are distinct elements of L and $\oplus$ is a partial binary operation on L that satisfies the following conditions (we write $a \perp b$ if $a \oplus b$ is defined):

- (E1) If $a \perp b$, then $b \perp a$ and $b \oplus a = a \oplus b$.
- (E2) If $a \perp b$ and $c \perp (a \oplus b)$, then $b \perp c$, $a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
- (E3) For every $a \in L$ there exists a unique $a' \in L$ such that $a \perp a'$ and $a \oplus a' = 1$.
- (E4) If $a \perp 1$, then $a = 0$.

In the sequel, whenever we write $a \oplus b$ we are implicitly assuming that $a \perp b$, that is, the sum $a \oplus b$ of a and b is defined in L.

We define $a \leq b$ if there is a $c \in L$ such that $a \oplus c = b$. It can be shown that the element c is unique, and that $(L, \leq, ')$ is an ordered set with $0 \leq a \leq 1$ for all $a \in L$, $a'' = a$ and $a \leq b$ implies $b' \leq a'$. This means, in particular, that de Morgan's laws hold in $(L, \leq, ')$: for any $a, b \in L$, $a \wedge b$ exists if and only if $a' \vee b'$ exists, in which case $(a \wedge b)' = a' \vee b'$. We note also that $a \perp b$ if and only if $a \leq b'$. If $a \leq b$, that is, $b = a \oplus c$ we also write $c = b \ominus a$ for the unique effect $c \in L$. We call the elements of L *effects*.

We say that an effect $a \in L$ is *sharp* if the greatest lower bound of a and a' exists in L (with respect to the order $\leq$ defined by $\oplus$) and equals with the least element 0 of L, that is, $a \wedge a' = 0$. An effect $a \in L$ is called *regular* if $a = 0, 1$ or if $a \not\leq a'$ and $a' \not\leq a$. Clearly, a sharp effect is regular, but a regular effect need not be sharp.

A simple example of an effect algebra is the unit interval $[0, 1] \subseteq \mathbb{R}$ of the real line $\mathbb{R}$. For $a, b \in [0, 1]$ we define $a \perp b$ if $a + b \leq 1$ and in this case $a \oplus b = a + b$. For $a \in [0, 1]$ we have $a' = 1 - a$.

A σ-algebra $\mathcal{A}$ of subsets of a nonempty set Ω constitutes another example of an effect algebra when $X \oplus Y$ is defined whenever $X \cap Y = \emptyset$

with $X \oplus Y = X \cup Y$. Clearly, $X' = \Omega \setminus X$ and the order defined by $\oplus$ on $\mathcal{A}$ coincides with the set inclusion. The order structure of the effect algebra $(\mathcal{A}, \oplus, \emptyset, \Omega)$ is that of a Boolean σ-algebra.

The set $\mathcal{E}(\mathcal{H})$ of the Hilbert space effects is a third important example of an effect algebra: for any $A, B \in \mathcal{E}(\mathcal{H})$, if $A + B \in \mathcal{E}(\mathcal{H})$, we define $A \oplus B = A + B$. Again, it is immediate that $A' = I - A$ and that the order induced on $\mathcal{E}(\mathcal{H})$ is the usual order of self adjoint operators which was used to define the set $\mathcal{E}(\mathcal{H})$. The effect algebra $(\mathcal{E}(\mathcal{H}), \oplus, O, I)$ does not form a lattice. An effect $A \in \mathcal{E}(\mathcal{H})$ is sharp exactly when it is a projection, and it is regular whenever its spectrum extends both below and above $\frac{1}{2}$.

A *subeffect algebra* of an effect algebra L is a subset K of L such that $0, 1 \in K$, $a \in K$ implies $a' \in K$ and $a, b \in K$, $a \perp b$, implies $a \oplus b \in K$. Notice that a subeffect algebra is an effect algebra in its own right. Moreover, the ordering in K is that inherited from L. Indeed, denote by $\leq_K$ the ordering in K, that is, $a \leq_K b$ if there is $c \in K$ such that $a \oplus c = b$. Clearly, if $a \leq_K b$ then $a \leq b$ in L. Conversely, assume that for some $a, b \in K$ we have $a \leq b$. Then there is $c \in L$ with $a \oplus c = b$. Now since $b' \oplus a \oplus c = 1$, we get $c = (b' \oplus a)' \in K$. Therefore $a \leq_K b$. The effect algebra $(\mathcal{A}, \oplus, \emptyset, \Omega)$ is, in a natural way, a subeffect algebra of $(2^\Omega, \oplus, \emptyset, \Omega)$ (with the same definition of $\oplus$). The set $\mathcal{P}(\mathcal{H})$ of all projection operators on a Hilbert space $\mathcal{H}$ is a subeffect algebra of $\mathcal{E}(\mathcal{H})$, the sharp elements of $\mathcal{E}(\mathcal{H})$. It is known to be a complete orthomodular lattice.

2.2. **Morphisms and observables.** If K and L are effect algebras, we say that $\phi : K \to L$ is *additive* if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus_K b) = \phi(a) \oplus_L \phi(b)$. If $\phi : K \to L$ is additive and $\phi(1_K) = 1_L$, then ϕ is a *morphism*. If $\phi : K \to L$ is a morphism and $\phi(a) \perp \phi(b)$ implies $a \perp b$, then ϕ is a *monomorphism*. A surjective monomorphism is an *isomorphism*. It is easy to see that ϕ is an isomorphism if and only is ϕ is bijective and ϕ^{-1} is a morphism. We note that a morphism $\phi : K \to L$ preserves the order and the 'complement'. Indeed, for any $a \in K$, $\phi(1_K) = \phi(a \oplus_K a') = \phi(a) \oplus_L \phi(a') = 1_L$ showing that $\phi(a') = \phi(a)'$. Also, if $a \leq_K b$, then $b = a \oplus_K c$ for a unique $c \in K$, so that $\phi(b) = \phi(a) \oplus_L \phi(c)$, which means that $\phi(a) \leq_L \phi(b)$. For $a, b \in K$, if $a \wedge_K b$ exists in K, then also $\phi(a \wedge_K b) \leq_L \phi(a), \phi(b)$, but $\phi(a \wedge_K b)$ need not be the greatest lower bound of $\phi(a), \phi(b)$ even if $\phi(a) \wedge_L \phi(b)$ would exist in L. A morphism ϕ is called a $\wedge$-*morphism* if $\phi(a) \wedge_L \phi(b)$ exists whenever $a \wedge_K b$ exists, and $\phi(a \wedge_K b) = \phi(a) \wedge_L \phi(b)$. A $\vee$-morphism is defined dually. By de Morgan laws a $\wedge$-morphism is also a $\vee$-morphism, and vice versa.

An effect algebra is called a *σ-effect algebra* if for any increasing sequence $a_1 \le a_2 \le \cdots$ of effects the supremum $\bigvee a_i$ exists in L. Equivalently, L is a σ-effect algebra if for any summable sequence $(a_i)_{i\in\mathbb{N}}$ [which is a sequence such that $\bigoplus_{i\le n} a_i := a_1 \oplus a_2 \oplus \cdots \oplus a_n$ exists for any $n \in \mathbb{N}$], the element $\bigoplus_{i\in\mathbb{N}} a_i := \bigvee_n \bigoplus_{i\le n} a_i$ exists in L. The above examples of effect algebras all are σ-effect algebras. A morphism $\phi : K \to L$, where K is a σ-effect algebra, is a *σ-morphism* if $\phi(\bigoplus_{i\in\mathbb{N}} a_i) = \bigoplus_{i\in\mathbb{N}} \phi(a_i)$ for any summable sequence $(a_i)_{i\in\mathbb{N}}$. Similarly, one defines the notions of $\sigma - \vee$ and $\sigma - \wedge$ -morphisms.

Let L be an effect algebra and $(\Omega, \mathcal{A})$ a measurable space. A σ-morphism $E : \mathcal{A} \to L$ is called an *observable*. In other words a map $E : \mathcal{A} \to L$ is an observable if

(O1) $E(\Omega) = 1_L$,

(O2) $E(\cup_{i\in\mathbb{N}} X_i) = \bigoplus_{i\in\mathbb{N}} E(X_i)$ for any disjoint sequence $(X_i) \subset \mathcal{A}$, meaning that the σ-sum $\bigoplus_{i\in\mathbb{N}} E(X_i)$ exists in L and equals with $E(\cup_{i\in\mathbb{N}} X_i)$.

The following observations are immediate:

Lemma 2.1. *Let $E : \mathcal{A} \to L$ be an observable. Then*

(a) *for any $X \in \mathcal{A}$, $E(X') = E(X)'$; in particular, $E(\emptyset) = 0_L$;*

(b) *if $X, Y \in \mathcal{A}$, $X \subseteq Y$, then $E(X) \le E(Y)$; in fact, $E(Y) = E(X) \oplus E(Y \setminus X)$, that is, $E(Y \setminus X) = E(Y) \ominus E(X)$;*

(c) *for each $X, Y \in \mathcal{A}$, $E(X \cup Y) = E(X \setminus (X \cap Y)) \oplus E(Y \setminus (X \cap Y)) \oplus E(X \cap Y)$, that is, $E(X \cup Y) \ominus E(X \cap Y) = (E(X) \ominus E(X \cap Y)) \oplus (E(Y) \ominus E(X \cap Y))$.*

We stress that in the effect algebra $\mathcal{A}$, if $(X_i)_{i\in\mathbb{N}} \subset \mathcal{A}$ is a disjoint sequence, then $\bigoplus_{i\in\mathbb{N}} X_i = \cup_{i\in\mathbb{N}} X_i = \bigvee_{i\in\mathbb{N}} X_i$, whereas in L the existence of $\bigoplus_{i\in\mathbb{N}} E(X_i)$ does not imply the existence of $\bigvee_{i\in\mathbb{N}} E(X_i)$ in L. Therefore, an observable $E : \mathcal{A} \to L$ need not be a $\sigma - \vee$ – homomorphism. Also, though $E(X \cap X') = 0_L$, the quantity $E(X) \wedge_L E(X)'$ need not exists at all, and if it does, it need not be equal to the zero effect 0_L. Similarly, $E(X \cap Y)$ may or may not equal to $E(X) \wedge_L E(Y)$, even if the latter quantity exists, at all. We call the space $(\Omega, \mathcal{A})$ the *value space* of the observable. We consider the case when $(\Omega, \mathcal{A}) \equiv (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, and we say that E is a real observable if it is defined on the real Borel space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Observables $E_1 : \mathcal{A}_1 \to L$ and $E_2 : \mathcal{A}_2 \to L$ are *coexistent* if there is an observable $F : \mathcal{A} \to L$ such that

$$\text{(C)} \qquad \mathrm{ran}\,(E_1) \cup \mathrm{ran}\,(E_2) \subseteq \mathrm{ran}\,(F),$$

and they are *functionally coexistent* if there is an observable F and measurable functions $f_1 : \Omega \to \Omega_1$ and $f_2 : \Omega \to \Omega_2$ such that

$$\text{(FC)} \qquad E_1 = F \circ f_1^{-1}, \quad E_2 = F \circ f_2^{-1}.$$

3. Two-valued observables

We say that an observable E is *two-valued* if its range is of the form $\mathrm{ran}\,(E) = \{0, a, a', 1\}$ for some effect $a \in L$.

Proposition 3.1. *Any two coexistent two-valued observables* $\{0, a, a', 1\}$ *and* $\{0, b, b', 1\}$ *are functionally coexistent.*

Proof. To demonstrate this simple fact, let $\{\alpha, \alpha'\}$ and $\{\beta, \beta'\}$ be two point value sets of the observables $\{0, a, a', 1\}$ and $\{0, b, b', 1\}$, respectively, and let F be an observable such that $F(X) = a$ and $F(Y) = b$. Consider the partition $\mathcal{R} = \{X \cap Y, X' \cap Y, X \cap Y', X' \cap Y'\}$ of the value space Ω of F into disjoint $\mathcal{A}$-sets, and let $1 \mapsto F(X \cap Y)$, $2 \mapsto F(X' \cap Y)$, $3 \mapsto F(X \cap Y')$, $4 \mapsto F(X' \cap Y')$, constitute a corresponding coarse-grained observable $F^{\mathcal{R}}$ of F. The maps $f_1 : 1, 3 \mapsto \alpha; 2, 4 \mapsto \alpha'$, and $f_2 : 1, 2 \mapsto \beta; 3, 4 \mapsto \beta'$, allow one to write $a = F^{\mathcal{R}}(f_1^{-1}(\alpha)) = F(X \cap Y) \oplus F(X \cap Y')$ and $b = F^{\mathcal{R}}(f_2^{-1}(\beta)) = F(X \cap Y) \oplus F(X' \cap Y)$, showing that the two-valued observables are, indeed, functionally coexistent. □

4. Sharp and regular observables

An observable E is called *sharp* if its range $\mathrm{ran}\,(E)$ consists of sharp effects and it is called *regular* if its range $\mathrm{ran}\,(E) := \{E(X) : X \in \mathcal{A}\}$ consists of regular effects. Evidently, a sharp observable is regular, but the converse is not true.

Theorem 4.1. *Let* $E : \mathcal{A} \to L$ *be a regular observable. Then* $\mathrm{ran}\,(E)$ *is a* σ*-subeffect algebra of* L*. Moreover, with respect to the partial order* $\leq$ *inherited from* L*,* $\mathrm{ran}\,(E)$ *is a Boolean algebra and* $E : \mathcal{A} \to \mathrm{ran}\,(E)$ *is a* $\sigma - \vee$ *(and* $\sigma - \wedge$*) -morphism.*

Proof. In the first part of the proof we follow [6]. Let $a, b \in \mathrm{ran}\,(E)$ and $a \perp b$. We have to prove that $a \oplus b \in \mathrm{ran}\,(E)$. Assume $a = E(X)$, $b = E(Y)$. Then $X = X \cap Y \cup (X \setminus (X \cap Y))$, $Y = X \cap Y \cup (Y \setminus (X \cap Y))$. Hence $E(X \cap Y) \leq E(X) = a$, $E(X \cap Y) \leq E(Y) = b$. Since $a \perp b$, that is, $b \leq a'$, it follows that $E(X \cap Y) \leq E(X \cap Y)'$, so that by the regularity assumption, $E(X \cap Y) = 0_L$. Therefore $a = E(X_1)$ and $b = E(Y_1)$, where $X_1 := X \setminus (X \cap Y)$, $Y_1 := Y \setminus (X \cap Y)$ are disjoint sets. So we get $a \oplus b = E(X_1) \oplus E(Y_1) = E(X_1 \cup Y_1) \in \mathrm{ran}\,(E)$. We note also that $X \cap Y_1 = \emptyset$ and $a \oplus b = E(X \cup Y_1)$. This observation will be used in the next paragraph.

Now assume that $(a_i)_{i \in \mathbb{N}}$ is a summable sequence in $\mathrm{ran}\,(E)$. Using the above argument, we find disjoint sets X_1, X_2 such that $a_1 = E(X_1)$, $a_2 = E(X_2)$. Now we proceed by induction. Assume that we have already found disjoint sets $X_1, \ldots, X_{n-1}$ such that $a_i = E(X_i)$, $i = 1, \ldots, n-1$. Then $a_1 \oplus \cdots \oplus a_{n-1} = E(X_1 \cup X_2 \cup \ldots \cup X_{n-1})$. By the summability assumption $(a_1 \oplus \cdots \oplus a_{n-1}) \perp a_n$. Therefore, there is a set $X_n \in \mathcal{A}$ such that $(X_1 \cup \ldots \cup X_{n-1}) \cap X_n = \emptyset$, and $a_n = E(X_n)$. Thus we find a sequence $X_i, i \in \mathbb{N}$, of disjoint sets such that $a_i = E(X_i)$, $i \in \mathbb{N}$. From the σ-additivity of E we obtain $E(\bigcup_i X_i) = \bigoplus_i E(X_i) = \bigoplus_i a_i$, which shows that ran (E) is closed under sums of summable sequences. It follows that ran (E) is a σ-subeffect algebra of L.

Let $E(X), E(Y) \in$ ran (E). We will prove that $E(X \cap Y) = E(X) \wedge_{\mathrm{ran}\,(E)} E(Y)$, that is, $E : \mathcal{A} \to$ ran (E) is a $\wedge$-morphism. Evidently, $E(X \cap Y) \leq E(X), E(Y)$. Assume that for some $Z \in \mathcal{A}$, $E(Z) \leq E(X), E(Y)$. We can write $Z = (Z \cap X \cap Y) \cup (Z \cap (X \cap Y)')$. Moreover,

$$\begin{aligned} E(Z \cap (X \cap Y)') &= E(Z \cap (X' \cup Y')) \\ &= E((Z \cap X' \cap Y) \cup (Z \cap X' \cap Y') \cup (Z \cap X \cap Y')) \\ &= E(Z \cap X' \cap Y) \oplus E((Z \cap X' \cap Y') \oplus E((Z \cap X \cap Y')) \\ &\leq E(Z) \leq E(X), E(Y). \end{aligned}$$

But we also have $E(Z \cap X' \cap Y) \leq E(X')$,$E(Z \cap X \cap Y') \leq E(Y')$, $E(Z \cap X' \cap Y') \leq E(X'), E(Y')$, so that the effects $E(Z \cap X' \cap Y)$, $E(Z \cap X' \cap Y')$, and $E(Z \cap X \cap Y')$ are irregular and thus equal 0_L. Therefore also $E(Z \cap (X \cap Y)') = 0_L$. Thus $E(Z) = E(Z \cap X \cap Y) \leq E(X \cap Y)$. This concludes the proof that $E(X \cap Y) = E(X) \wedge_{\mathrm{ran}\,(E)} E(Y)$. By de Morgan laws one gets the dual result: for any $X, Y \in \mathcal{A}$, $X \cap Y = \emptyset$, $E(X \cup Y) = E(X) \oplus E(Y) = E(X) \vee_{\mathrm{ran}\,(E)} E(Y)$. Also, if $(X_i) \subset \mathcal{A}$ is a disjoint sequence, then

$$\begin{aligned} E(\cup_{i=1}^{\infty} X_i) &= \bigoplus_{i=1}^{\infty} E(X_i) = \bigvee_{n=1}^{\infty} (\bigoplus_{i=1}^{n} E(X_i)) \\ &= \bigvee_{n=1}^{\infty} (\vee_{\mathrm{ran\,(E)}} \{E(X_1), \cdots, E(X_n)\}) \\ &= \vee_{\mathrm{ran\,(E)}} \{E(X_i) \,|\, i \in \mathbb{N}\}. \end{aligned}$$

To prove that ran (E) is a Boolean algebra, it remains to prove distributivity. This follows immediately from the fact that E is a $\wedge$-morphism and a $\vee$-morphism from a Boolean set. □

Corollary 4.2. *The range* ran (E) *of an observable E is a Boolean algebra (with the ordering inherited from L) if and only if E is regular.*

Proof. We have to prove the 'only if' part. Hence, assume that ran (E) is Boolean, and let $a = E(X)$ be an irregular element. Then $a \perp$

a, that is, $E(X) \leq E(X)'$, which in a Boolean algebra implies that $E(X) = a = 0$. □

The classical results of Sikorski [9] and Varadarajan [4] lead now to the following conclusion.

Corollary 4.3. *Any two observables whose ranges are contained in the range of a regular observable are functionally coexistent.*

We close this section with a further characterization of regular observables. We note that it was shown in [3, Appendix A] that property (ii) in the next theorem implies V-property (see next section). Our result is stronger.

Theorem 4.4. *Let $E : \mathcal{A} \to L$ be an observable. The following properties are equivalent:*

(i) *E is regular.*
(ii) *For any $X, Y \in \mathcal{A}$, $E(X) \leq E(Y)$ implies $E(Y) = E(X \cup Y)$.*

Proof. (i)⇒(ii). Assume that E is regular, and $E(X) \leq E(Y)$. Then $E(X \cup Y) = E(X) \vee_{\mathrm{ran}\,(E)} E(Y) = E(Y)$. (ii)⇒(i). Assume next that $E(X) \leq E(X')$. Then, by (ii), $E(X') = E(X \cup X') = 1_L$, so that $E(X) = 0_L$, which shows that E is regular. □

5. V-property and Strong observables

Let $E : \mathcal{A} \to L$ be an observable and Q a subset of $\mathrm{ran}\,(E)$. Following Chovanec and Kôpka [3, Appendix A] we say that E has the *V-property* on Q if the following condition is satisfied:

(V) for any $X, Y \in \mathcal{A}$, $X \subseteq Y$, and $c \in Q$, the inequalities $E(X) \leq c \leq E(Y)$ imply that there is a $Z \in \mathcal{A}$ such that $X \subseteq Z \subseteq Y$ and $c = E(Z)$.

The purpose of introducing this notion in [3] was the following theorem, which can be proved by a suitable adaptation of the well known theorem of Varadarajan [4, Theorem 1.4]. For completeness, we sketch here the proof.

Theorem 5.1. *Let E, F be real observables on an effect algebra L such that $\mathrm{ran}\,(E) \subseteq \mathrm{ran}\,(F)$, and let F have the V-property on $\mathrm{ran}\,(E)$. Then there is a Borel function $f : \mathbb{R} \to \mathbb{R}$ such that $E(X) = F \circ f^{-1}(X), X \in \mathcal{B}(\mathbb{R})$.*

Proof. By property V, if $X, Y \in \mathcal{B}(\mathbb{R})$, $X \subseteq Y$ and $F(X) \leq c \leq F(Y)$ for some $c \in \mathrm{ran}\,(E)$, then there is a $Z \in \mathcal{B}(\mathbb{R})$ such that $F(Z) = c$ and $X \subseteq Z \subseteq Y$. Let $r_1, r_2, \ldots$ be any distinct enumeration of rational numbers and let $D_i := \{t \in \mathbb{R} \mid t \leq r_i\}$. Evidently, $E(D_i) \leq E(D_j)$

whenever $r_i \leq r_j$. We shall now construct sets $Z_1, Z_2, \ldots$ in $\mathcal{B}(\mathbb{R})$ such that

(a) $F(Z_i) = E(D_i)$, $i = 1, 2, 3, \ldots$.

(b) $Z_i \subseteq Z_j$ whenever $r_i \leq r_j$.

Let $Z_1 \in \mathcal{B}(\mathbb{R})$ be such that $F(Z_1) = E(D_1)$. Suppose $Z_1, Z_2, \ldots, Z_n \in \mathcal{B}(\mathbb{R})$ have been constructed such that (i) $F(Z_i) = E(D_i)$ for $i = 1, 2, \ldots, n$, (ii) $Z_i \subseteq Z_j$ whenever $r_i \leq r_j$, $1 \leq i, j \leq n$. We shall construct Z_{n+1} as follows. Let $(i_1, i_2, \ldots, i_n)$ be the permutation of $(1, 2, \ldots, n)$ such that $r_{i_1} \leq r_{i_2} \leq \cdots \leq r_{i_n}$. Then there exists a unique k such that $r_{i_k} \leq r_{n+1} \leq r_{i_{k+1}}$ (we define $r_{i_0} = -\infty, r_{i_{n+1}} = +\infty$), and by property V, we can select Z_{n+1} such that $Z_{i_k} \subseteq Z_{n+1} \subseteq Z_{i_{k+1}}$. The collection $(Z_1, Z_2, \ldots, Z_{n+1})$ then has the same properties relative to $r_1, r_2, \ldots, r_{n+1}$ as $(Z_1, Z_2, \ldots, Z_n)$ had relative to $r_1, r_2, \ldots, r_n$. It follows by induction that there exists a sequence $Z_1, Z_2, \ldots$ in $\mathcal{B}(\mathbb{R})$ with properties (a) and (b). Since $F(\bigcap Z_i) \leq E((-\infty, r_i])$ for all r_i, $F(\bigcap Z_i) = 0_L$. Now replacing Z_k by $Z_k \setminus \bigcap_j Z_j$ if necessary, we may assume that $\bigcap_i Z_i = \emptyset$.

Now the proof follows exactly the pattern of [4, Theorem 1.4]. $\square$

For example, let $E : \{1, 2, 3, 4\} \to [0, 1]$ be an observable defined by $E(i) = 1/4$, $i = 1, 2, 3, 4$, and $F : \{1, 2, 3\} \to [0, 1]$ an observable defined by $F(1) = 1/2$, $F(2) = F(3) = 1/4$. Then $\operatorname{ran}(E) = \operatorname{ran}(F)$ as sets, and E has the V-property on the range $\operatorname{ran}(F)$ of F, but F does not have the V-property on the range $\operatorname{ran}(E)$ of E. Indeed, $\{2\} \subseteq \{1, 2\}$ and $F(2) = 1/4 \leq 1/2 \leq 3/4 = F(1, 2)$, but there is no X, $\{2\} \subseteq X \subseteq \{1, 2\}$ with $F(X) = 1/2$. Define $f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$ by $f(1) = 1 = f(2)$, $f(3) = 3$, $f(4) = 4$. Then $F = E \circ f^{-1}$. There is no function g such that $E = F \circ g^{-1}$.

We can give another characterization of V-property using congruences and strong morphisms of effect algebras as introduced in [10] and [11]. We start with recalling some definitions.

Let L be an effect algebra. A binary relation $\sim$ on L is called a *congruence* [10] if

(C1) $\sim$ is an equivalence relation;

(C2) $a \sim b$, $a_1 \sim b_1$, $a \perp b$, $a_1 \perp b_1$ imply $a \oplus b \sim a_1 \oplus b_1$;

(C3) $a \perp b$ and $b \sim c$ imply that there is $d \in L$ such that $d \sim a$ and $d \perp c$.

If $\sim$ is a congruence we let $[a]$ denote the equivalence class of $a \in L$. Define $[a] \perp [b]$ if there are $a_1 \in [a], b_1 \in [b]$ such that $a_1 \perp b_1$, and then define $[a] \oplus [b] = [a_1 \oplus b_1]$. It has been proved in [11] that the quotient space $P/\!\sim\; := \{[a] \,|\, a \in P\}$, with the operation $\oplus$ defined above and with $[0], [1]$ as zero and unit, is an effect algebra.

Let K and L be effect algebras. We say that a morphism $h : K \to L$ is *strong* if it has the following property:

(S) for $a, b \in K$, if $h(a) \perp h(b)$, then there is $a_1 \in K$ such that $a_1 \perp b$ and $h(a_1) = h(a)$.

For the proof of the following result, see [10, 11].

Lemma 5.2. *Let K, L be effect algebras, $h : K \to L$ a morphism. The following are equivalent:*

(i) *h is a strong morphism;*
(ii) *the relation $a \sim b$, defined by $h(a) = h(b)$, is a congruence.*

Theorem 5.3. *For any observable $E : \mathcal{A} \to L$ the following statements are equivalent:*

(i) *E has the V-property on its range* $\mathrm{ran}\,(E)$*;*
(ii) *E is a strong morphism.*

Proof. Let E be a strong morphism. Assume $X, Y, Z \in \mathcal{A}$, $X \subseteq Y$ and $E(X) \leq E(Z) \leq E(Y)$. Since $E(Z) \leq E(Y)$, that is, $E(Z) \perp E(Y')$, by the strong property of E, there is a $Z_1 \perp Y'$, that is $Z_1 \subset Y$, with $E(Z_1) = E(Z)$. It follows that

$$E(Y) \ominus E(Z) = E(Y \setminus Z_1) \in \mathrm{ran}\,(E).$$

Moreover,

$$E(Y \setminus Z_1) = E(Y) \ominus E(Z) \leq E(Y) \ominus E(X) = E(Y \setminus X).$$

Therefore, by the strong property, there is $V_1 \subseteq Y \setminus X$ such that $E(Y \setminus Z_1) = E(V_1)$. From $E(Y) \ominus E(Z) = E(V_1)$ we get

$$E(Y \setminus V_1) = E(Y) \ominus E(V_1) = E(Z).$$

Now $V_1 \subseteq Y \setminus X$ implies $X \subseteq Y \setminus V_1 \subseteq Y$. This proves the V-property on $\mathrm{ran}\,(E)$.

Conversely, assume that E has the V-property on $\mathrm{ran}\,(E)$. Let $E(X) \leq E(Y)$, then $E(\emptyset) \leq E(X) \leq E(Y)$ and $\emptyset \subseteq Y$, and $E(X) \in \mathrm{ran}\,(E)$. By property V, there is $X_1 \subseteq Y$ such that $E(X_1) = E(X)$, and the strong property follows. □

We say that an observable E is *strong* if E is a strong morphism.

Corollary 5.4. *Any two real observables whose ranges are contained in the range of a strong observable are functionally coexistent.*

6. The Riesz decomposition property

Consider an effect algebra $(L, \oplus, 0, 1)$. We say that L has the *Riesz decomposition property* [12] if the following is true in L:

(R) for any $a_1, a_2, b_1, b_2 \in L$, if $a_1 \oplus a_2 = b_1 \oplus b_2$, there are elements $w_{ij}, i = 1, 2$, in L such that $a_1 = w_{11} \oplus w_{12}$, $a_2 = w_{21} \oplus w_{22}$, $b_1 = w_{11} \oplus w_{21}$ and $b_2 = w_{12} \oplus w_{22}$.

The Riesz decomposition property can be extended by induction to any $m, n \in \mathbb{N}$, so that if $a_i, i = 1, 2, \ldots, m$, and $b_j, j = 1, 2, \ldots, n$, are such that

$$a_1 \oplus \cdots \oplus a_n = b_1 \oplus \cdots \oplus b_n,$$

then there are w_{ij}, $i \leq m, j \leq n$, such that

$$a_i = \oplus_j w_{ij}, \;\; b_j = \oplus_i w_{ij}.$$

Theorem 6.1. *Let E be a strong observable on an effect algebra L (equivalently, E has the V-property on* $\mathrm{ran}\,(E)$*). Then* $\mathrm{ran}\,(E)$ *is a subeffect algebra of L, which has the Riesz decomposition property.*

Proof. To show that $\mathrm{ran}\,(E)$ is a subeffect algebra of L, let $a, b \in \mathrm{ran}\,(E)$ be such that $a \oplus b$ is defined in L. Since E is strong, we can find $X, Y \in \mathcal{A}$ such that $X \cap Y = \emptyset$ and $a = E(X), b = E(Y)$. Then $a \oplus b = E(X \cup Y) \in \mathrm{ran}\,(E)$. With lemma 2.1 this proves that $\mathrm{ran}\,(E)$ is a subeffect algebra of L.

Now we prove the Riesz decomposition property. Assume that the effects $a_1, a_2, b_1, b_2 \in \mathrm{ran}\,(E)$ are such that $a_1 \oplus a_2 = b_1 \oplus b_2$. Since E is strong, we find a disjoint pair Y_1, Y_2 in $\mathcal{A}$ such that $b_1 = E(Y_1)$, $b_2 = E(Y_2)$. Now $a_1 \oplus a_2 = b_1 \oplus b_2 = E(Y_1 \cup Y_2)$. From $a_1 \leq E(Y_1 \cup Y_2)$, using once more the strong property, we can find X_1 such that $a_1 = E(X_1)$ and $X_1 \subseteq Y_1 \cup Y_2$. Put $X_2 := (Y_1 \cup Y_2) \ominus X_1$. Then we have $X_1 \cup X_2 = Y_1 \cup Y_2$, $E(X_1) \oplus E(X_2) = E(Y_1) \oplus E(Y_2)$. Put $w_{11} = E(X_1 \cap Y_1)$, $w_{12} = E(X_1 \setminus (X_1 \cap Y_1))$, $w_{21} = E(Y_1 \setminus (X_1 \cap Y_1))$. Now we have

$$\begin{aligned} E(X_1) \oplus w_{21} &= E(X_1 \cup (Y_1 \setminus (X_1 \cap Y_1))) \\ &= E(Y_1 \cup (X_1 \setminus (X_1 \cap Y_1))) \\ &= E(Y_1) \oplus w_{12}. \end{aligned}$$

Since $X_1 \subseteq X_1 \cup X_2$, $Y_1 \subseteq Y_1 \cup Y_2$, and $X_1 \cup X_2 = Y_1 \cup Y_2$, we have $X_1 \cup Y_1 \subseteq Y_1 \cup Y_2$, so that $E(X_1) \oplus w_{21} = E(Y_1) \oplus w_{12} \leq E(Y_1 \cup Y_2)$. Put $w_{22} = E(Y_1 \cup Y_2) \ominus (E(X_1) \oplus w_{21}) = E(Y_1 \cup Y_2) \ominus (E(Y_1) \oplus w_{12})$. Then

$$\begin{aligned} E(X_1) \oplus E(X_2) &= w_{11} \oplus w_{12} \oplus w_{21} \oplus w_{22} \\ &= E(Y_1) \oplus E(Y_2), \end{aligned}$$

hence $E(X_2) = w_{21} \oplus w_{22}$, $E(Y_2) = w_{12} \oplus w_{22}$. $\square$

We say that an observable $E : \mathcal{A} \to L$ is *simple* if there is a finite subset Ω_0 of Ω such that $E(\Omega_0) = 1$.

Proposition 6.2. *Any two simple observables E, F on an effect algebra with the Riesz decomposition property are functionally coexistent.*

Proof. Let $\{\lambda_1, \dots, \lambda_m\}$, $\{\mu_1, \dots, \mu_n\}$ be finite sets such that $E(\{\lambda_i\}) = a_i$, $F(\{\mu_j\}) = b_j$, and $\oplus_{i \le m} a_i = 1_L = \oplus_{j \le m} b_j$. Then there are w_{ij} with $a_i = \oplus_j w_{ij}$, $b_j = \oplus_i w_{ij}$. Clearly, $\oplus_{ij} w_{ij} = 1_L$, and we may define a simple observable G with value space $\{\omega_{ij}\}$ such that $G(\omega_{ij}) = w_{ij}$. Now define functions $e : \{\omega_{ij}\} \to \{\lambda_i\}$, $f : \{\omega_{ij}\} \to \{\mu_j\}$ such that $e(\omega_{ij}) = \lambda_i$, $f(\omega_{ij}) = \mu_j$. Then $E = G \circ e^{-1}$, $F = G \circ f^{-1}$. $\square$

Remark 6.3. In Sect. 2.5. of [1] we claimed that any two simple observables are functionally coexistent. In the above proposition this claim is proved under the additional assumption on the Riesz decomposition property. We do not have any counter example to disprove the more general statement of [1]. However, as pointed out by A. Dvurečenskij, the construction of the function f_1, resp. f_2, in [1] may fail since the involved sets X_i, resp. Y_j, need not be disjoint.

7. Regular vs. strong observables

A comparison between regular and strong observables can be derived from the following propositions.

Proposition 7.1. *Let $E : \mathcal{A} \to L$ be an observable. The following statements are equivalent:* (i) *$E : \mathcal{A} \to L$ is regular,* (ii) *$X \sim Y$ if $E(X) = E(Y)$, $X, Y \in \mathcal{A}$, coincides with the congruence on $\mathcal{A}$ generated by the ideal $I_E := \{X \in \mathcal{A} \mid E(X) = 0\}$.*

Proof. (i) $\Rightarrow$ (ii). By Theorem 4.1, $\mathrm{ran}\,(E)$ is a subeffect algebra of L, which is a Boolean algebra with respect to $\vee$ and $\wedge$ taken in $\mathrm{ran}\,(E)$. Moreover, the mapping $E : \mathcal{A} \to \mathrm{ran}\,(E)$ is a Boolean algebra (σ-) morphism. Therefore I_E is an ideal in $\mathcal{A}$, and the quotient $\mathcal{A}/I_E$ is isomorphic with $\mathrm{ran}\,(E)$ as Boolean algebras. Indeed, we recall that the Boolean algebra congruences coincide with the effect algebra congruences (cf. [10]), and $X \sim_{I_E} Y$ iff $(X \setminus X \cap Y) \cup (Y \setminus X \cap Y) \in I_E$. Now $E(X) = E(Y)$ implies $(X \setminus X \cap Y) \cup (Y \setminus X \cap Y) \in I_E$, that is, $X \sim_{I_E} Y$. Conversely, if $X \sim_{I_E} Y$, then $E(X) = E(X \cap Y) = E(Y)$. Therefore $\sim \equiv \sim_{I_E}$.

(ii) $\Rightarrow$ (i). (ii) implies that $\mathrm{ran}\,(E)$ is a Boolean algebra, hence by Corollary 4.2, E is regular. $\square$

Corollary 7.2. *Every regular observable is strong. The converse does not hold.*

Proof. The statement that regular implies strong can be derived from Proposition 7.1 and Lemma 5.2. Alternatively, we can use Theorem 5.3 and show directly that condition (ii) of Theorem 4.4 implies the V-property.

As a counterexample, consider the observable $E : \{1,2,3,4\} \to [0,1]$ defined by $E(i) = 1/4$, $i = 1,2,3,4$. It can be easily checked that E is a strong morphism. □

8. The Hilbert space case. Projection valued observables

We recall that in the Hilbert space frame an observable $E : \mathcal{A} \to \mathcal{E}(\mathcal{H})$ is regular if for each $X \in \mathcal{A}$, such that $O \neq E(X) \neq I$,

$$E(X) \not\leq \frac{1}{2}I,\ \frac{1}{2} \not\leq E(X).$$

With this specification all the results from the preceding sections can be applied. The sharp elements of $\mathcal{E}(\mathcal{H})$ coincide with the projections and they form a subeffect algebra $\mathcal{P}(\mathcal{H})$ of $\mathcal{E}(\mathcal{H})$, which is an orthomodular lattice. Moreover, the infimum of any two projections exists in $\mathcal{E}(\mathcal{H})$, and is again a projection. Thus the lattice operations in $\mathcal{P}(\mathcal{H})$ coincide with those in $\mathcal{E}(\mathcal{H})$.

An observable $E : \mathcal{A} \to \mathcal{E}(\mathcal{H})$ is sharp if it is projection valued, that is, if $E(X)^2 = E(X)$ for all $X \in \mathcal{A}$. It is well known that this is the case exactly when E is multiplicative: $E(X \cap Y) = E(X)E(Y)$ for all $X, Y \in \mathcal{A}$. For a Hilbert space observable $E : \mathcal{A} \to \mathcal{E}(\mathcal{H})$ Lemma 2.1 gets the familiar form:

- for any $X \in \mathcal{A}$, $E(X') = I - E(X)$,
- if $X, Y \in \mathcal{A}$, and $X \subset Y$, then $E(Y \setminus X) = E(Y) - E(X)$,
- for all $X, Y \in \mathcal{A}$, $F(X \cup Y) + F(X \cap Y) = F(X) + F(Y)$.

Moreover, for any $A \in \mathcal{E}(\mathcal{H}), P \in \mathcal{P}(\mathcal{H})$, if $A \leq P$, then $A = AP = PA$.

The just quoted results lead immediately to the following known result:

Lemma 8.1. *For any observable $E : \mathcal{A} \to \mathcal{E}(\mathcal{H})$, if $E(X)^2 = E(X)$ for some $X \in \mathcal{A}$, then $E(X)E(Y) = E(Y)E(X)$ for all $Y \in \mathcal{A}$.*

Proof. Assume that $E(X)^2 = E(X)$ for some $X \in \mathcal{A}$. For any $Y \in \mathcal{A}$, $X \cap Y \subseteq Y$, so that $E(Y) = E(Y \setminus (X \cap Y)) + E(X \cap Y)$, and $E(X) + E(Y) - E(X \cap Y) = E(X) + E(Y \setminus (X \cap Y)) = E(X \cup Y) \leq I$.

Therefore, the effects $E(X \cap Y)$ and $E(Y \setminus (X \cap Y))$ are below the projections $E(X)$ and $I - E(X)$, respectively, so that

$$\begin{aligned} E(X \cap Y) &= E(X)E(X \cap Y)E(X), \\ E(Y \setminus (Y \cap X)) &= (I - E(X))E(Y \setminus (Y \cap X))(I - E(X). \end{aligned}$$

Therefore, $E(Y) = E(Y \setminus (X \cap Y)) + E(X \cap Y) = E(X)E(X \cap Y)E(X) + (I - E(X))E(Y \setminus (Y \cap X))(I - E(X)$, which gives through multiplication by $E(X)$ that $E(X)E(Y) = E(Y)E(X)$. □

As an immediate corollary to this lemma one obtains the following:

Corollary 8.2. *Let E_1 and E_2 be two coexistent observables. If one of them is projection valued, then they are functionally coexistent.*

Proof. Assume that E_1 is projection valued. Since E_1 and E_2 are coexistent, Lemma 8.1 implies that E_1 and E_2 are commuting: $E_1(X)E_2(Y) = E_2(Y)E_1(X)$ for all $X \in \mathcal{A}_1, Y \in \mathcal{A}_2$. Thus the map $\mathcal{A}_1 \times \mathcal{A}_2 \ni (X, Y) \mapsto E_1(X)E_2(Y) \in \mathcal{E}(\mathcal{H})$ determines a biobservable of E_1 and E_2, which is to say that E_1 and E_2 are functionally coexistent. □

Corollary 8.3. *Let $E_1 : \mathcal{A}_1 \to \mathcal{E}(\mathcal{H})$ and $E_2 : \mathcal{A}_2 \to \mathcal{E}(\mathcal{H})$ be coexistent projection valued observables and let $F : \mathcal{A} \to \mathcal{E}(\mathcal{H})$ be an observable such that $ran(E_1) \cup ran(E_2) \subseteq ran(F)$, so that, for any $X \in \mathcal{A}_1$, $E_1(X) = F(Z_X)$, for some $Z_X \in \mathcal{A}$, and for any $Y \in \mathcal{A}_2$, $E_2(Y) = F(Z_Y)$ for some $Z_Y \in \mathcal{A}$. Then*

$$\begin{aligned}
(1) \qquad F(Z_X \cap Z_Y) &= E_1(X) \wedge_{\mathcal{E}(\mathcal{H})} E_2(Y) \\
&= E_1(X) \wedge_{\mathcal{P}(\mathcal{H})} E_2(Y) \\
&= E_1(X)E_2(Y), \\
(2) \qquad F(Z_X \cup Z_Y) &= E_1(X) \vee_{\mathcal{E}(\mathcal{H})} E_2(Y) \\
&= E_1(X) \vee_{\mathcal{P}(\mathcal{H})} E_2(Y) \\
&= E_1(X) + E_2(Y) - E_1(X)E_2(Y), \\
(3) \qquad E_1(X') &= F(Z_{X'}) = F(Z'_X), \\
(4) \qquad E_2(Y') &= F(Z_{Y'}) = F(Z'_Y), \\
(5) \qquad F(Z'_{X'} \cup Z'_{Y'}) &= F(Z_X \cup Z_Y).
\end{aligned}$$

Proof. For any two projections $P, R \in \mathcal{P}(\mathcal{H}) \subset \mathcal{E}(\mathcal{H})$ their greatest lower bound in $\mathcal{E}(\mathcal{H})$ exists and equals with their greatest lower bound in $\mathcal{P}(\mathcal{H})$: $P \wedge_{\mathcal{E}(\mathcal{H})} R = P \wedge_{\mathcal{P}(\mathcal{H})} R$. Since $E_1(X)$ and $E_2(Y)$ are commuting projections, one now has $E_1(X)E_2(Y) = E_1(X) \wedge_{\mathcal{P}(\mathcal{H})} E_2(Y) = E_1(X) \wedge_{\mathcal{E}(\mathcal{H})} E_2(Y)$, so that $F(Z_X \cap Z_Y) \leq E_1(X)E_2(Y)$. But one also has $F(Z'_X \cap Z_Y) \leq F(Z'_X) = F(Z_X)^\perp = E_1(X)^\perp = E_1(X')$ as well as $F(Z'_X \cap Z_Y) \leq F(Z_Y) = E_2(Y)$, so that $F(Z'_X \cap Z_Y) \leq$

$E_1(X)^\perp E_2(Y)$. Similarly, $F(Z_X \cap Z'_Y) \le E_1(X)E_2(Y)^\perp$, and $F(Z'_X \cap Z'_Y) \le E_1(X)^\perp E_2(Y)^\perp$. But then $I = F((Z_X \cap Z_Y) \cup (Z'_X \cap Z_Y) \cup (Z_X \cap Z'_Y) \cup (Z'_X \cap Z'_Y)) = F(Z_X \cap Z_Y) + F(Z'_X \cap Z_Y) + F(Z_X \cap Z'_Y) + F(Z'_X \cap Z'_Y) \le E_1(X)E_2(Y) + E_1(X)^\perp E_2(Y) + E_1(X)E_2(Y)^\perp + E_1(X)^\perp E_2(Y)^\perp = I$, which shows that Eq. (1) holds. In passing, we also saw that (3) and (4) are true. To demonstrate (2), we note that $F(Z_X \cup Z_Y) = F(Z_X) + F(Z_Y) - F(Z_X \cap Z_Y) = E_1(X) + E_2(Y) - E_1(X)E_2(Y) = E_1(X) \vee E_2(Y) = F(Z_X) \vee F(Z_Y)$. Finally, $F(Z'_{X'} \cup Z'_{Y'}) = F((Z_{X'} \cap Z_{Y'})') = F(Z_{X'} \cap Z_{Y'})^\perp = (E_1(X') \wedge E_2(Y'))^\perp = E_1(X) \vee E_2(Y) = F(Z_X \cup Z_Y)$. □

Corollary 8.4. *If E_1, E_2 are real observables satisfying the conditions of Corollary 8.3, the Boolean σ-algebra $\mathcal{B}$ generated by $ran(E_1) \cup ran(E_2)$ is contained in $ran(F)$, and there is a sharp observable G such that $ran(G) = \mathcal{B}$ (so that $E_1 = G \circ f_1^{-1}$ and $E_2 = G \circ f_2^{-1}$ for some measurable functions f_1, f_2).*

Proof. According to Neumark's theorem, there is a Hilbert space $\tilde{\mathcal{H}}$ containing $\mathcal{H}$ as a subspace, a projection $M : \tilde{\mathcal{H}} \to \mathcal{H}$ and a projection valued observable $\tilde{F}$ on $\tilde{\mathcal{H}}$ such that for any $Z \in \mathcal{A}$, $F(Z) = M\tilde{F}(Z)M$. Now for any $X \in \mathcal{A}_1$, $Y \in \mathcal{A}_2$, $E_1(X) = F(Z_X) = M\tilde{F}(Z_X)M$, $E_2(Y) = F(Z_Y) = M\tilde{F}(Z_Y)M$. For any projection $\tilde{A}$ on $\tilde{\mathcal{H}}$, $M\tilde{A}M$ is a projection iff $\tilde{A}M = M\tilde{A}$. So for any X, Y, $\tilde{F}(Z_X), \tilde{F}(Z_Y)$ commute with M, and therefore also the projections from the Boolean sub-σ-algebra $\tilde{\mathcal{B}}$ of $\mathcal{P}(\tilde{\mathcal{H}})$ commute with M. Moreover, $\tilde{\mathcal{B}}$ is contained in the Boolean σ-algebra ran $\tilde{F}$. Therefore $M\tilde{B}M$ is a Boolean σ-algebra of projections on $\mathcal{H}$, which contains ran $(E_1) \cup$ ran (E_2). The rest follows by [4]. □

8.1. **Examples.**

1. Consider a simple observable $E : i \mapsto E_i \in \mathcal{E}(\mathcal{H})$, $i = 1, \cdots, m$, which is commutative, that is, $E_iE_j = E_jE_i$. According to [13] there is an $n \times m$ stochastic matrix (λ_{ij}), that is $0 \le \lambda_{ij} \le 1$ for all i, j and $\sum_{j-1}^n \lambda_{ij} = 1$ for each $i = 1, \ldots, n$, and there are pairwise orthogonal projections $P_1, \ldots, P_n$ such that $\sum_{i=1}^n P_i = I$, and E is given by $j \mapsto E_j := \sum_{i=1}^n \lambda_{ij} P_i$. In [14] this representation was used to give an example of a non-regular observable.

Consider the case $m = n = 2$. It has been shown in [14], that E is non-regular iff $\lambda_{11} \le \lambda_{21}$ and $\lambda_{12} \le \lambda_{21}$, in which case $E_1 \le E_2 = I - E_1$. We note that in this case, E is also not strong, unless the above inequalities are equalities, i.e., $E_1 = E_2$.

Consider next the case $n = 2, m = 3$. It has been shown in [14], that if E_1, E_2, E_3 are disjoint atoms in ran (E), the range of E cannot be

Boolean. It follows that E cannot be regular. Notice that if $\lambda_{ij} = 1/3$, i.e., $E_1 = E_2 = E_3$, then E is non-regular, but it is strong.

2. Let ϕ, ψ be a pair of unit vectors in $\mathbb{C}^2$. Consider the projection operators $P[\phi], P[\psi]$, and $P_r := P[1/\sqrt{2}(\psi + i^r \phi)]$, with $r = 0, 1, 2, 3$. Let $\lambda_i, i = 1, 2, 3$, be any positive numbers which sum up to 1. Then $i \mapsto E_i$, with $E_1 = \lambda_1 P[\phi]$, $E_2 = \lambda_1 P[\psi]$, $E_3 = \lambda_2 P_0$, $E_4 = \lambda_2 P_2$, $E_5 = \lambda_3 P_1$, $E_6 = \lambda_3 P_3$, defines an observable E, with a 64-element noncommutative range [14]. If all the λ_i's are nonzero, this observable is non-regular and non-strong. Indeed, at least one of $\lambda_i' s$ must be less than $1/3$. Let $\lambda_i \leq 1/3$, and i be such that the projections in E_i and E_j are orthogonal. Then $E_i + E_{i,j} \leq I$, but there is no subset Δ of $\{1, 2, 3, 4, 5, 6\}$, $\Delta \cap \{i, j\} = \emptyset$ with $E(\Delta) = E_i$. Nevertheless, it is easy to check that that any observable F with $\mathrm{ran}\,(F) \subseteq \mathrm{ran}\,(E)$ is a function of E.

9. Concluding remarks

There are following classes of observables (represented as positive normalized operator measures $E : \mathcal{A} \to \mathcal{L}(\mathcal{H})$) which differ by properties of the corresponding σ-morphism from a Boolean σ-algebra $\mathcal{A}$ to the effect algebra of Hilbert space effects $\mathcal{E}(\mathcal{H})$:

1. *Sharp observables.* The range of such an observable consists of projections (sharp effects), and is a sub-σ-algebra of $\mathcal{P}(\mathcal{H})$, and a sub-effect algebra of $\mathcal{E}(\mathcal{H})$. The corresponding morphism preserves countable suprema and infima, and suprema and infima with respect to the range coincide with suprema and infima in $\mathcal{E}(\mathcal{H})$. Two coexistent observables one of which is sharp are always functionally coexistent. Also, any two observables whose ranges are contained in a range of a sharp observable, are functionally coexistent.

2. *Regular observables.* The range of a regular observable consists of regular effects, and is a sub-effect algebra of $\mathcal{E}(\mathcal{H})$. The corresponding morphism preserves countable suprema and infima with respect to the range, which may differ from suprema and infima in $\mathcal{E}(\mathcal{H})$, even when they would exist. Moreover, the range of an observable is a Boolean algebra with respect to the ordering inherited from $\mathcal{E}(\mathcal{H})$ iff the observable is regular. Any two observables whose ranges are contained in the range of a regular observable, are functionally coexistent.

3. *Strong observables.* The range of such an observable is a subeffect algebra of $\mathcal{E}(\mathcal{H})$. The corresponding morphism has the following property: if $E(X) + E(Y) \leq I$ for some $X, Y \in \mathcal{A}$ then there is $X_1 \in \mathcal{A}$ such that $X_1 \cap Y = \emptyset$ and $E(X) = E(X_1)$. Suprema and infima may be not preserved, and the range need not be Boolean. But any two real observables contained in the range of a strong observable are functionally coexistent.

4. *The Riesz decomposition property.* The range of an observable E satisfies the Riesz decomposition property if property (R) from § 6 is satisfied with L replaced by $\operatorname{ran}(E)$. In this case, any two simple observables with ranges contained in $\operatorname{ran}(E)$ are functionally coexistent. In particular, the range of a strong observable satisfies the Riesz decomposition property.

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