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SUPERPOSITION AND CONSTRUCTIONS OF GRAPHS WITHOUT NOWHERE-ZERO k-FLOWS

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Abstract. Using multi-terminal networks we build general methods for constructing graphs without nowhere-zero group- and integer-valued flows. In this way we unify known constructions of snarks (nontrivial cubic graphs without 3-edge-colorings, or equivalently, without nowhere-zero 4-flows) and provide new ones in the same process. Our methods also imply new results about nowhere-zero flows in graphs. For example we can show that the 3- and 5-flow conjectures of Tutte are equivalent to seemingly weaker problems.

1. Introduction

Nowhere-zero flows in graphs have been introduced by Tutte [T1, T2, T3]. Primarily he showed that a planar graph is k-face-colorable if and only if it admits a nowhere-zero k-flow (its edges can be oriented and assigned values $\pm 1, \dots, \pm(k-1)$ so that the sum of the incoming values is equal to the sum of the outcoming ones for every vertex of the graph). Tutte also proved the classical equivalence result which says that a graph admits a nowhere-zero k-flow if and only if it admits a flow whose values are the nonzero elements of a finite Abelian group of order k. Another important result in the topic was obtained by Jaeger [J2] and Kilpatrick [Ki] showing that every bridgeless graph admits a nowhere-zero 8-flow. The value 8 was improved to 6 by Seymour [S1].

There are three celebrated unsolved conjectures dealing with nowhere-zero flows in bridgeless graphs, all due to Tutte. The first is the 5-flow conjecture of [T1], that every such graph admits a nowhere-zero 5-flow. The 4-flow conjecture of [T3] suggests that if the graph does not contain a subgraph contractible to the Petersen graph, then it has a nowhere-zero 4-flow. Finally the 3-flow conjecture states that if the graph does not contain a 3-edge cut, then it has a nowhere-zero 3-flow.

Graphs which do not admit nowhere-zero k-flows will be called k-snarks. Note that it is an easy problem to recognize 2- and k-snarks for $k \geq 6$, because they are the graphs having a vertex of odd valency and the graphs having a bridge, respectively. On the other hand from results of Tutte [T1], Holyer [Ho] and Garey et al. [GJS] it follows that the problems to recognize 3- and 4-snarks are co-NP-complete. By [K6], the same holds for $k = 5$ if the 5-flow conjecture is false.

Until now only little has been known about constructions of k-snarks in general. The only exceptions are snarks, which are nontrivial cubic 4-snarks (or, equivalently,

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3-regular graphs without a 3-edge-coloring). By nontrivial we mean cyclically 4-edge-connected and with girth at least 5. Snarks are studied very intensively and several methods have been developed for their constructions. The main reason for interest in them is that among snarks must be the smallest counterexamples to the 5-flow conjecture and the cycle double cover conjecture (every bridgeless graph has a family of circuits which together cover each edge twice). Furthermore, by Tait [Ta], the four color theorem is equivalent to the statement that there exists no planar snark. But construction of snarks is not an easy task. For instance the first nontrivial infinite family of them was constructed in 1975 by Isaacs [I1], though the first snark, the Petersen graph (see Fig. 1.1), was known late in the 19th century. More details about the history of snarks are given in the beginning of Section 13.

In this paper we build general methods for constructing k-snarks. These methods are based on multi-terminal networks and have several advantages in comparison with the classical constructions of snarks. Primarily that dealing with k-flows we can better understand the 0–1 laws which underlie many constructions of snarks. This enables us to unify known constructions of snarks and provide new ones in the same process. Another advantage is that our techniques are not restricted only to snarks but can be directly used on constructing k-snarks in general and therefore lead to new results about nowhere-zero flows. For instance we show that the 3- and 5-flow conjectures of Tutte are equivalent to seemingly weaker (and also stronger) statements and obtain results about potential counterexamples to these conjectures. Similarly we can reformulate the four color theorem.

The main method of the paper is called superposition and is based on the following two ideas. If G is a k-snark, then replacing one of its edges by another k-snark we get a new k-snark. The second principle is that if we replace a vertex v of G by an arbitrary graph H and join the edges originally incident to v with vertices of H, we obtain a new k-snark. Further two basic methods are immersion and superjoin introduced in Sections 5 and 10, respectively. In Section 13 we demonstrate versatility and power of our techniques and construct several families of snarks with special properties.

2. Graphs, networks and flows

For a formal definition of graphs see, e. g., Bondy and Murty [BM]. In this paper we deal with finite unoriented graphs with multiple edges and loops. A graph $G = (V, E)$ consists of a set of vertices $V(G) = V$ and a set of edges $E(G) = E$.

By a *multi-terminal network*, briefly a *network*, we mean a pair (G, U) where G is a graph and $U = \langle u_1, \dots, u_n \rangle$ is an ordered set of pairwise different vertices of G. If no confusions can occur, we denote by U also the set $\{u_1, \dots, u_n\}$. (We apply this convention writing formulas $u \in U$, $U \cap W$, or $W \setminus U$ for $W \subseteq V(G)$.) The vertices from U and $V(G) \setminus U$ are called the *outer* and *inner* vertices of the network (G, U). We allow $n = 0$, i. e., $U = \emptyset$.

Any edge e of the graph G determines two opposite arcs arising from it after endowing e with two distinct orientations. (Roughly speaking, each edge of G is duplicated and the two resulting edges are directed oppositely. Only a loop determines two arcs directed to the same vertex, but this will cause no problems.) All arcs obtained in this way are called *arcs* of G and the set of them is called the *arc set* of G and denoted by $D(G)$. Clearly, $|D(G)| = 2|E(G)|$. If x is an arc of G, then denote by $-x$ the second arc arising from the same edge. Clearly $-(-x) = x$ and $x \neq -x$ for every arc x of G. If $X \subseteq D(G)$, then denote by $-X = \{x \in D(G); -x \in X\}$. By an *orientation* of G we mean every $X \subseteq D(G)$ so that $-X \cup X = D(G)$ and $-X \cap X = \emptyset$.

If $W \subseteq V(G)$, then the set of the arcs of $D(G)$ directed from W to $V(G) \setminus W$ is denoted by $\omega_G^+(W)$. Similarly denote $\omega_G^-(W) = -\omega_G^+(W)$. Clearly $|\omega_G^+(W)| = |\omega_G^-(W)|$ for every $W \subseteq V(G)$. We write $\omega_G^+(v)$, $\omega_G^-(v)$ instead of $\omega_G^+(\{v\})$, $\omega_G^-(\{v\})$, respectively. If v is a vertex of G adjacent to r loops, then the value $|\omega_G^+(v)| + 2r = |\omega_G^-(v)| + 2r$ is called the *valency* of v.

In this paper, every Abelian group is additive and has order at least two. If G is a graph and A is an Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(-x) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that

$$\partial\varphi(v) = \sum_{x \in \omega_G^+(v)} \varphi(x) \qquad (v \in V(G)) \tag{2.1}$$

is called the *boundary* of φ (note that a similar, but formally different notion was used in Jaeger et al. [JLPT]). If H is a subgraph of G, then $\varphi_{|H}$ denotes the restriction of φ to $D(H)$. Clearly $\varphi_{|H}$ is an A-chain in H if φ is an A-chain in G.

If (G, U) is a network, then an A-chain φ in G is called an *A-flow* in (G, U) if $\partial\varphi(v) = 0$ for every inner vertex v of G. Moreover, the set of edges associated with the arcs of G having nonzero values in φ is called the *support* of φ. An A-flow φ in (G, U) is called *nowhere-zero* if its support is equal to $E(G)$.

Remark 2.1. Let $X \subseteq D(G)$ be an orientation of a graph G. Then a mapping $\psi : X \to A$ can be extended into an A-flow φ in (G, U) if and only if the following equality is satisfied for every inner vertex v of (G, U):

$$\sum_{x \in \omega_G^+(v) \cap X} \psi(x) = \sum_{x \in \omega_G^-(v) \cap X} \psi(x). \tag{2.2}$$

By a (nowhere-zero) A-flow in a graph G we mean a (nowhere-zero) A-flow in the network $(G, \emptyset)$. (A graph G is usually identified with the network $(G, \emptyset)$ in this paper.) By Remark 2.1, our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in [J4], [S2], [T4], [Y], [Z2]. The only difference is that instead of a fixed (but arbitrary) orientation of a graph G we use the set $D(G)$ as a domain for a flow. On the other hand if we want to describe an A-flow in (G, U), it is enough to give its values on an orientation of G so that (2.2) is satisfied for every inner vertex v of (G, U).

Proposition 2.2. *Let (G, U) be a network, $W \subseteq V(G)$, and φ be an A-flow in (G, U). Then*

$$\sum_{x \in \omega_G^+(W)} \varphi(x) = \sum_{v \in U \cap W} \partial\varphi(v).$$

Proof. Since $\varphi(-x) = -\varphi(x)$ for every $x \in D(G)$ and $\partial\varphi(v) = 0$ for every $v \in W \setminus U$, we have

$$\sum_{x \in \omega_G^+(W)} \varphi(x) = \sum_{v \in W} \sum_{x \in \omega_G^+(v)} \varphi(x) = \sum_{v \in W} \partial\varphi(v) = \sum_{v \in U \cap W} \partial\varphi(v). \qquad \square$$

Proposition 2.3. *Let (G,U) be a network and φ be an A-flow in (G,U). Then*

$$\sum_{u\in U}\partial\varphi(u)=0.$$

Proof. Follows directly from Proposition 2.2 after setting $W=V(G)$. $\square$

In order to generalize the classical equivalence results of Tutte we need to introduce the following notions. If k is an integer ≥ 2, then by a *(nowhere-zero) unlimited k-flow* φ in a network (G,U) we mean a (nowhere-zero) $\mathbb{Z}$-flow in (G,U) such that $|\varphi(x)|<k$ for every $x\in D(G)$. Furthermore, if also $|\partial\varphi(u)|<k$ holds for every $u\in U$, we speak about a *(nowhere-zero) limited k-flow* in (G,U). Note that both these notions coincides with the usual definition of (nowhere-zero) k-flows in graphs (see, e. g., Jaeger [J4]).

With every A-flow in a network (G,U), where $U=\langle u_1,\dots,u_n\rangle$, is associated a *characteristic vector* $\chi(\varphi)=\langle z_1,\dots,z_n\rangle$ so that $z_i=0$ if $\partial\varphi(u_i)=0$ and $z_i=1$ otherwise. The *A-characteristic set* $\chi_A(G,U)$ of the network (G,U) is the set of all characteristic vectors $\chi(\varphi)$ where φ is a nowhere-zero A-flow in (G,U). The *k-characteristic set* $\chi_k(G,U)$ of (G,U) is the set of all characteristic vectors $\chi(\varphi)$ where φ is a nowhere-zero limited k-flow in (G,U).

For a network (G,U), $U=\langle u_1,\dots,u_n\rangle$ and a binary vector $\mathbf{z}=\langle z_1,\dots,z_n\rangle$, construct the graph $(G,U)^{\mathbf{z}}$ as follows: add a new vertex v_0 to G and join it with every vertex $u_i\in U$ such that $z_i=1$ (the valency of v_0 in $(G,U)^{\mathbf{z}}$ is equal to the number of nonzero coordinates from $\mathbf{z}$).

Proposition 2.4. *Let (G,U) be a network, $U=\langle u_1,\dots,u_n\rangle$, and $\mathbf{z}=\langle z_1,\dots,z_n\rangle$ be a binary vector. Then $\mathbf{z}\in\chi_A(G,U)$ (resp. $\mathbf{z}\in\chi_k(G,U)$) if and only if $(G,U)^{\mathbf{z}}$ admits a nowhere-zero A-flow (resp. a nowhere-zero limited k-flow).*

Proof. If φ is a nowhere-zero limited k-flow in $(G,U)^{\mathbf{z}}$, then $\varphi_{|G}$ is a nowhere-zero limited k-flow in (G,U) and $\chi(\varphi_{|G})=\mathbf{z}$. Thus $\mathbf{z}\in\chi_k(G,U)$. On the other hand if $\mathbf{z}=(z_1,\dots,z_n)\in\chi_k(G,U)$, then there exists a nowhere-zero limited k-flow ψ in (G,U) so that $\chi(\psi)=\mathbf{z}$. Thus $|\partial\psi(u)|<k$ for every $u\in U$ and, by Proposition 2.3, $\sum_{u\in U}\partial\psi(u)=0$. Then ψ can be extended into a nowhere-zero limited k-flow ψ' in $(G,U)^{\mathbf{z}}$ after setting $\psi'(x_i)=-\psi'(-x_i)=-\partial\psi(u_i)$ where x_i denotes the arc directed from u_i to v_0 (for every i with $z_i\neq 0$).

The statement can be proved analogously for A-flows. $\square$

Theorem 2.5. *Let (G,U) be a network. Then the following statements are pairwise equivalent:*

(1) *(G,U) has a nowhere-zero $\mathbb{Z}_k$-flow;*
(2) *(G,U) has a nowhere-zero A-flow for any Abelian group A of order k;*
(3) *(G,U) has a nowhere-zero limited k-flow.*
(4) *(G,U) has a nowhere-zero unlimited k-flow.*

Proof. Items (1), (2) and (3) are equivalent by Proposition 2.4 and the results of Tutte [T1, T2] (who proved this statement for graphs, i. e., if $U=\emptyset$ – see also Zhang [Z2] or [K10]). Clearly, (3) implies (4). If (G,U) has a nowhere-zero unlimited k-flow, then taking its values $\mod k$ we get a nowhere-zero $\mathbb{Z}_k$-flow. Thus (4) implies (1). $\square$

Theorem 2.6. *If (G, U) is a network, then $\chi_k(G, U) = \chi_A(G, U)$ for every Abelian group A of order k.*

Proof. Follows directly from Theorem 2.5 and Proposition 2.4. □

A *characteristic vector* $\chi'(\varphi) = \langle z_1, \ldots, z_n \rangle$ of an unlimited k-flow φ in (G, U) is the vector $\mathbf{z} = \langle z_1, \ldots, z_n \rangle$ so that $z_i = 0$ if $\partial\varphi(u_i) \equiv 0 \mod k$ and $z_i = 1$ otherwise. Let $\chi'_k(G, U)$ be the set of all vectors $\chi'(\varphi)$ where φ is a nowhere-zero unlimited k-flow in (G, U). We claim that $\chi_k(G, U) = \chi'_k(G, U)$. Clearly $\chi_k(G, U) \subseteq \chi'_k(G, U)$. If $\mathbf{z} \in \chi'_k(G, U)$ and φ is a nowhere-zero unlimited k-flow in (G, U) so that $\chi'(\varphi) = \mathbf{z}$, then taking the values of $\varphi \mod k$ we get a nowhere-zero $\mathbb{Z}_k$-flow ψ so that $\chi(\psi) = \chi'(\varphi)$, whence, by Theorem 2.6, $\mathbf{z} \in \chi_k(G, U)$, which proves the claim.

We have shown that the study of nowhere-zero limited or unlimited k-flows is, in certain sense, equivalent to the study of nowhere-zero A-flows where A is an Abelian group of order k. But flows with values from finite groups are easier to handle than integral flows. With respect to this fact, we define a (*nowhere-zero*) *k-flow* and *k-chain* in a network (G, U) to be every (nowhere-zero) A-flow and A-chain in (G, U), respectively, where A is an Abelian group of order k. Similarly we shall use notation $\chi_k(G, U)$ instead of $\chi_A(G, U)$ (which is correct by Theorem 2.6). In fact we need integral flows only in the proofs of the next two statements.

Proposition 2.7. *If a network (G, U) admits a nowhere-zero k-flow, then it admits a nowhere-zero $(k + 1)$-flow.*

Proof. Every nowhere-zero (un)limited k-flow is a nowhere-zero (un)limited $(k + 1)$-flow. Thus the statement follows from Theorem 2.5. □

Proposition 2.8. *If (G, U) is a network, then $\chi_k(G, U) \subseteq \chi_{k+1}(G, U)$.*

Proof. Follows from Propositions 2.4 and 2.7. □

From now on integral flows will not be used.

3. k-Snarks

Definition 3.1. By a *k-snark* we mean every network without a nowhere-zero k-flow (k is an integer ≥ 2). We say that a graph G is a *k-snark* if $(G, \emptyset)$ is a k-snark.

By Proposition 2.8, every k-snark is also a k'-snark for every integer $2 \leq k' \leq k$. Furthermore, if $(G, \langle u_1, \ldots, u_n \rangle)$ is a k-snark, then so is $(G, \langle u_{i_1}, \ldots, u_{i_m} \rangle)$ for every $\{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$.

The following results are well known (see, e. g., Jaeger [J4]).

Proposition 3.2. *A graph is a 2-snark if and only if it contains a vertex of odd valency.*

Proposition 3.3. *If G is a cubic graph, then:*

(a) *G is a 3-snark if and only if it is non-bipartite;*
(b) *G is a 4-snark if and only if it is not 3-edge-colorable.*

Cyclically 4-edge-connected cubic (3-regular) graphs without a 3-edge-coloring and with girth at least 5 are called *snarks* (see, e. g., [Ga, CW, K3, K4, WW, W2]). By the previous proposition, snarks have no nowhere-zero 4-flow, and this fact has led us to defining k-snarks as networks without nowhere-zero k-flows. Clearly, snarks form a proper subclass of 4-snarks.

Proposition 3.4. *A graph containing a bridge is a k-snark for every $k \geq 2$.*

Proof. Let e be a bridge in G and W be the vertex set of one component produced after removing e. Then $\omega_G^+(W)$ consists of a single arc x (arising from e) and, by Proposition 2.2, $\varphi(x) = 0$ for each k-flow in G, $k \geq 2$. □

By *subdivision* of an edge we mean the insertion of a new 2-valent (inner) vertex into the edge. Two networks are *homeomorphic* if they arise from the same network after applying finitely many subdivisions of edges. The following statement is trivial.

Proposition 3.5. *Let (G, U) and (G', U) be two homeomorphic networks. Then $\chi_k(G, U) = \chi_k(G', U)$ for every $k \geq 2$. In particular, (G, U) is a k-snark if and only if (G', U) is.*

By Seymour [S1] (see also Fan [F1]), there are no bridgeless k-snarks for any $k \geq 6$. The 5-flow conjecture of Tutte suggests that there are no bridgeless 5-snarks.

4. Superposition

Let (G, U), $U = \langle u_1, \dots, u_n\rangle$, and (G', U'), $U' = \langle u_1', \dots, u_n'\rangle$, be networks. Suppose that G' has a (not necessarily induced) subgraph H so that after deleting all edges from H and identifying all vertices from H to one new vertex w we get the graph G. Moreover, let $u_i' = u_i$ if $u_i \neq w$ and $u_i' \in V(H)$ if $u_i = w$ (this implies that $|V(H) \cap U'| \leq 1$). Then (G', U') is called a *w-superposition* (or a *vertex superposition*) of G and we write $(G', U') = \mathcal{S}(G, U)$. H is also denoted by $\mathcal{S}_w$. Now $D(G)$ may be chosen to be a subset of $D(G')$. Thus if φ is a k-flow in (G', U'), then its restriction to $D(G)$, denoted by $\varphi_{\mathcal{S}}$, is a k-chain in (G, U).

Proposition 4.1. *If φ is a (nowhere-zero) k-flow in (G', U'), then $\varphi_{\mathcal{S}}$ is a (nowhere-zero) k-flow in (G, U) and $\partial\varphi_{\mathcal{S}}(u_i) = \partial\varphi(u_i')$ for $i = 1, \dots, n$, i. e., $\chi(\varphi_{\mathcal{S}}) = \chi(\varphi)$.*

Proof. For every vertex v of G, $v \neq w$, $\omega_G^+(v) = \omega_{G'}^+(v)$ and $\partial\varphi_{\mathcal{S}}(v) = \partial\varphi(v)$. Clearly $\omega_G^+(w) = \omega_{G'}^+(V(H))$ and using Proposition 2.2 we get $\partial\varphi_{\mathcal{S}}(w) = \sum_{x \in \omega_G^+(w)} \varphi_{\mathcal{S}}(x) = \sum_{x \in \omega_{G'}^+(V(H))} \varphi(x) = \sum_{v \in U' \cap V(H)} \partial\varphi(v)$. Thus $\partial\varphi_{\mathcal{S}}(u_i) = \partial\varphi(u_i')$ $(i = 1, \dots, n)$ and $\partial\varphi_{\mathcal{S}}(v) = 0$ for every inner vertex v of (G, U). This implies the statement. □

The flow $\varphi_{\mathcal{S}}$ is called the flow *induced* from φ by $\mathcal{S}$.

For example, the networks $(G_1, \langle u_1, u_2\rangle)$ and $(G_1', \langle u_1, u_2'\rangle)$, indicated in Fig. 4.1, are w- and u_2-superpositions of $(P, \langle u_1, u_2\rangle)$, respectively.

Suppose we get a network (G', U'), $U' = \langle u_1', \dots, u_n'\rangle$, from a network (G, U), $U = \langle u_1, \dots, u_n\rangle$, by the following process. Take a network $(H, \langle v_1, v_2\rangle)$, delete from G an edge $e = (w_1, w_2)$ and identify the sets of vertices $\{w_1, v_1\}$ and $\{w_2, v_2\}$ to new vertices w_1' and w_2', respectively. Furthermore, let $u_i' = u_i$ if $u_i \neq w_1, w_2$, and $u_i' = w_1'$ (or w_2') if $u_i = w_1$ (or w_2). Then (G', U') is called an *e-superposition* (or an *edge superposition*) of G and we write $(G', U') = \mathcal{S}(G, U)$. The network $(H, \langle v_1, v_2\rangle)$ is also denoted by $\mathcal{S}_e$. Now $D(G - e)$ and $D(H)$ are subsets of $D(G')$. Suppose that x_1 and x_2 are the arcs arising from e and directed out of w_1 and w_2, respectively $(x_1 = -x_2)$. Let φ be a k-flow in (G', U'). Then denote by φ_H its restriction to $D(H)$.

Clearly, φ_H is a k-flow in $(H, \langle v_1, v_2 \rangle)$ and, by Proposition 2.3, $\varphi_H(v_1) = -\varphi_H(v_2)$. Then define a k-chain $\varphi_{\mathcal{S}}$ in (G, U) so that

$$\begin{aligned} &\varphi_{\mathcal{S}}(x_j) = \partial \varphi_H(v_j), \qquad (j = 1, 2), \\ &\varphi_{\mathcal{S}}(x) = \varphi(x) \qquad \text{for } x \in D(G - e). \end{aligned} \tag{4.1}$$

We always assume that $v_1 \neq v_2$. If e is a loop (i. e., $w_1 = w_2$) then v_1, v_2 and w_1 are identified to one vertex $w'_1 = w'_2$. But $\varphi_{\mathcal{S}}$ is well defined also in this case.

Proposition 4.2. *If φ is a k-flow in (G', U'), then $\varphi_{\mathcal{S}}$ is a k-flow in (G, U) and $\partial\varphi_{\mathcal{S}}(u_i) = \partial\varphi(u'_i)$ for $i = 1, \dots, n$, i. e., $\chi(\varphi_{\mathcal{S}}) = \chi(\varphi)$.*

Proof. For every $v \in V(G) \setminus \{w_1, w_2\}$, $\omega^+_G(v) = \omega^+_{G'}(v)$ and $\partial\varphi_{\mathcal{S}}(v) = \partial\varphi(v)$. If $w_1 \neq w_2$, then $\omega^+_{G'}(w'_j) = \omega^+_{G-e}(w_j) \cup \omega^+_H(v_j)$, and, by (4.1) and (2.1), $\partial\varphi_{\mathcal{S}}(w_j) = \partial\varphi(w'_j)$, $j = 1, 2$. If $w_1 = w_2$, then $w'_1 = w'_2$ and $\omega^+_{G'}(w'_1) = \omega^+_{G-e}(w_1) \cup \omega^+_H(v_1) \cup \omega^+_H(v_2)$, and, by (4.1) and (2.1), $\partial\varphi_{\mathcal{S}}(w_1) = \partial\varphi(w'_1)$. This implies the statement. $\square$

The flow $\varphi_{\mathcal{S}}$ is called the flow *induced* from φ by $\mathcal{S}$.

For example $(G_2, \langle u_1, w'_2 \rangle)$ is an e-superposition of $(P, \langle u_1, u_2 \rangle)$ (see Fig. 4.2).

A network (G', U'), $U' = \langle u'_1, \dots, u'_n \rangle$, is called a *superposition* of a network (G, U), $U = \langle u_1, \dots, u_n \rangle$, if there exist networks (G_0, U_0), $\dots$,(G_r, U_r), $(G_0, U_0) = (G, U)$, $(G_r, U_r) = (G', U')$ such that $(G_i, U_i) = \mathcal{S}_i(G_{i-1}, U_{i-1})$ and $\mathcal{S}_i$ is either a vertex or an edge superposition for $i = 1, \dots, r$. We write $(G', U') = \mathcal{S}(G, U)$ where $\mathcal{S} = \mathcal{S}_r \circ \cdots \circ \mathcal{S}_2 \circ \mathcal{S}_1$ is also called the *composition* of $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r$. If φ is a k-flow in (G', U'), then by the flow *induced* from φ by $\mathcal{S}$ we mean the k-flow $\varphi_{\mathcal{S}}$ in (G, U) so that $\varphi_{\mathcal{S}} = (\dots(\varphi_{\mathcal{S}_r})_{\mathcal{S}_{r-1}} \dots)_{\mathcal{S}_1}$. By Propositions 4.1 and 4.2, $\varphi_{\mathcal{S}}$ is well defined and $\partial\varphi_{\mathcal{S}}(u_i) = \partial\varphi(u'_i)$ for $i = 1, \dots, n$, i. e., $\chi(\varphi_{\mathcal{S}}) = \chi(\varphi)$.

Moreover, let $(G'', U'') = \mathcal{S}'(G', U')$ where $\mathcal{S}' = \mathcal{S}'_t \circ \cdots \circ \mathcal{S}'_2 \circ \mathcal{S}'_1$. Then $(G'', U'') = \mathcal{S}''(G, U)$ where $\mathcal{S}'' = \mathcal{S}'_t \circ \cdots \circ \mathcal{S}'_2 \circ \mathcal{S}'_1 \circ \mathcal{S}_r \circ \cdots \circ \mathcal{S}_2 \circ \mathcal{S}_1$. We also write $\mathcal{S}'' = \mathcal{S}' \circ \mathcal{S}$ and call $\mathcal{S}''$ the *composition* of $\mathcal{S}$ and $\mathcal{S}'$.

Definition 4.3. A superposition $\mathcal{S}(G, U)$ is called *A-strong* if $\varphi_{\mathcal{S}}$ is a nowhere-zero A-flow in (G, U) for every nowhere-zero A-flow φ in $\mathcal{S}(G, U)$. A superposition is called *k-strong* if it is A'-strong for every Abelian group A' of order $\leq k$.

In view of Theorems 2.5 and 2.6, one would expect that if A is an Abelian group of order k, then the notions of k-strong and A-strong superpositions coincide. In Examples 4.10 and 4.11 we show that this is not true.

The following statement is a cornerstone of our constructions.

Theorem 4.4. *Let (G', U') be an A-strong (k-strong) superposition of (G, U) where A is an Abelian group of order $k \geq 2$. Then $\chi_k(G', U') \subseteq \chi_k(G, U)$. In particular, if (G, U) is a k-snark, then so is (G', U').*

Proof. Let $(G', U') = \mathcal{S}(G, U)$ be A-strong and φ be a nowhere-zero A-flow in (G', U'). Then $\varphi_{\mathcal{S}}$ is a nowhere-zero A-flow in (G, U) and $\chi(\varphi_{\mathcal{S}}) = \chi(\varphi)$. $\square$

It is clear that any composition of two A-strong (k-strong) superpositions is an A-strong (k-strong) superposition. By Proposition 4.1, a vertex superposition is k-strong for every $k \geq 2$. Now we study edge superpositions.

Proposition 4.5. *Every A-strong edge superposition is $|A|$-strong.*

Proof. Let $(G', U') = \mathcal{S}(G, U)$ be an e-superposition so that $\mathcal{S}_e = (H, \langle v_1, v_2\rangle)$. Suppose that it is A-strong but not A'-strong where $|A'| \leq |A|$. Then there exists a nowhere-zero A'-flow in (G', U') so that $\varphi_{\mathcal{S}}$ is not nowhere-zero. Let $\varphi_H, x_j, w_j, w'_j$ $(j = 1, 2)$ have the same meaning as in the text preceding (4.1). Then $\varphi_{\mathcal{S}}$ can have zero values only on the arcs x_1, x_2, i. e., $\partial\varphi_H(v_1) = \partial\varphi_H(v_2) = 0$ and φ_H is a nowhere-zero A'-flow in H. Thus $(G - e, U)$ and H admit nowhere-zero A'-flows and, by Theorem 2.5 and Proposition 2.7, also nowhere-zero A-flows. But the latter flows induce a nowhere-zero A-flow ψ in (G', U') so that $\psi_{\mathcal{S}}(x_1) = -\psi_{\mathcal{S}}(x_2) = 0$, which contradicts the assumptions. □

Proposition 4.6. *Let $\mathcal{S}(G, U)$ be an e-superposition so that $\mathcal{S}_e = (H, \langle v_1, v_2\rangle)$ and H is a k-snark. Then $\mathcal{S}(G, U)$ is k-strong.*

Proof. If $\mathcal{S}(G, U)$ is not k-strong, then there exists a nowhere-zero A'-flow φ in (G', U') so that $|A'| \leq k$ and $\varphi_{\mathcal{S}}$ is not nowhere-zero. Now similarly as in the proof of Proposition 4.5, we get a nowhere-zero A'-flow in H – a contradiction. □

The e-superposition indicated in Fig. 4.2 is 4-strong, because it "replaces" an edge by a snark. Similarly the w- and u_2-superpositions indicated in Fig. 4.1 are k-strong for any $k \geq 2$ by Proposition 4.1. Thus Theorem 4.1 and Propositions 4.1, 4.6 give a technique producing infinite classes of bridgeless k-snarks if we have at lest one bridgeless k-snark. Examples of such constructions are given is Sections 13.1 – 13.4, and 13.6. Furthermore, as shown in the next examples, this approach also generalizes some known constructions.

Example 4.7. *Dot Product.* Take two snarks G_1 and G_2 and delete two adjacent vertices v_1, v_2 from G_1 and two edges (x, y), (z, t) from G_2. Then joining the vertices of degree two by four new edges as indicated in Fig. 4.3 we get a new graph $G_1 \cdot G_2$, which is known to be a snark. It is called a *dot product* of G_1 and G_2. This construction was introduced independently by Isaacs [I1] and Adelson-Velskij and Titov [AVT]. We show that it can be also obtained using the techniques of superposition. First we take the graph G'_2 depicted in Fig. 4.4. It is a 4-snark, by Proposition 3.5. Then, by Proposition 4.6, G' is a 4-strong e-superposition of G_1. Applying twice vertex superpositions on G' we obtain G'', a graph homeomorphic with $G_1 \cdot G_2$.

Example 4.8. Let H_1 be a k-snark and H_2 be a graph. Suppose that v_1 and v_2 are vertices of H_1 and H_2, respectively, having the same valency. Then consider the graph H_3 indicated in Fig. 4.5. H_3 is also a k-snark, because it is homeomorphic with the graph H_4 arising as v_1-superposition of H_1. This operation is well-known and it has often been used to construct cyclically 3-edge-connected cubic 4-snarks (see, e. g., [WW, CW, Z2]).

Not every k-strong superposition can be obtained as a composition of vertex and edge superpositions satisfying the properties of Proposition 4.6. We give one example.

Example 4.9. Take a circuit C of length 5 in the Petersen graph P and a superposition P' of P replacing the vertices from C by $P - (u'_1, u'_2)$ and the edges from C by two parallel edges as indicated in Fig. 4.6. The superposition is 4-strong. This follows from Proposition 2.3 and the fact that $\langle 1,1,0\rangle \notin \chi_4(P-(u_1',u_2'),\langle u_1',u_2',u_3'\rangle)$ (if this is false, we get a nowhere-zero 4-flow in P).

Example 4.10. Let G and G' be the graphs with orientations X and X', respectively, indicated in Fig. 4.7. Then $G' = \mathcal{S}'(G)$ for $\mathcal{S}' = \mathcal{S}_3' \circ \mathcal{S}_2' \circ \mathcal{S}_1'$ where $\mathcal{S}_1'$ is an edge superposition replacing the edge associated with a by three parallel edges and $\mathcal{S}_2'$ and $\mathcal{S}_3'$ are vertex superpositions replacing the vertices u and v by three isolated vertices u_1,u_2,u_3 and v_1,v_2,v_3, respectively. Clearly, if φ is a k-flow in G', then $\varphi_{\mathcal{S}'}(a) = \varphi(a_1)+\varphi(a_2)+\varphi(a_3)$ and $\varphi_{\mathcal{S}'}(x) = \varphi(x_1)$ for $x = b,c,d,i$. We claim that $\mathcal{S}'(G)$ is $\mathbb{Z}_2\times\mathbb{Z}_2$-strong but not $\mathbb{Z}_4$-strong. Let φ be a nowhere-zero $\mathbb{Z}_2\times\mathbb{Z}_2$-flow in G'. Then $\varphi(a_1)=\varphi(a_2)$, whence $\varphi_{\mathcal{S}'}(a) = \varphi(a_3)\neq 0$. Thus $\varphi_{\mathcal{S}'}$ is nowhere-zero. On the other hand let ψ be a nowhere-zero $\mathbb{Z}_4$-flow on G' so that $\psi(a_1)=\psi(c_1)=1$ and $\psi(d_1)=2$. Then $\psi_{\mathcal{S}'}(a) = 1+1+2 = 0$.

Example 4.11. Let G and G'' be the graphs with orientations X and X'', respectively, indicated in Fig. 4.7. Then $G'' = \mathcal{S}''(G)$ for $\mathcal{S}'' = \mathcal{S}_3''\circ\mathcal{S}_2''\circ\mathcal{S}_1''$ where $\mathcal{S}_1''$ is an edge superposition which replaces the edge associated with a by two parallel edges and $\mathcal{S}_2''$ and $\mathcal{S}_3''$ are similar as in Example 4.10. Clearly, if φ is a k-flow in G'', then $\varphi_{\mathcal{S}''}(a) = \varphi(a_4)+\varphi(a_5)$ and $\varphi_{\mathcal{S}''}(x) = \varphi(x_4)$ for $x=b,c,d,i$. We claim that $\mathcal{S}''$ is A-strong if and only if the order of A is odd. Let A be of odd order and φ be a nowhere-zero A-flow in G''. Then $\varphi_{\mathcal{S}''}(a) = \varphi(a_4)+\varphi(a_5) = 2\varphi(a_4)\neq 0$. Thus $\mathcal{S}''(G)$ is A-strong. If the order of A is even, then there exists $\alpha\in A$ satisfying $2\alpha = 0$ and a nowhere-zero A-flow ψ in G'' so that $\psi(a_4)=\psi(d_4)=\alpha$. Thus $\psi_{\mathcal{S}''}(a) = 2\alpha = 0$ and $\mathcal{S}''(G)$ is not A-strong.

5. Operations on networks

Suppose that (G,U) is a network, $U=\langle u_1,\dots,u_n\rangle$, and r is an integer, $0\le r\le n/2$. Then the *r-join* (or simply a *join*) of (G,U) is the network $(G_1,\langle u_{2r+1},\dots,u_n\rangle)$ where G_1 is the graph obtained from G after identifying the sets of vertices $\{u_1,u_2\}$, $\dots$, $\{u_{2r-1},u_{2r}\}$ to new vertices v_1, $\dots$, v_r, respectively. Furthermore, if $n=2r$, then we speak about the *complete join* of (G,U). Note that the 0-join of (G,U) is just (G,U).

Suppose that (G,U) and (G',U') are disjoint networks, $U=\langle u_1,\dots,u_n\rangle$, $U' = \langle u_1',\dots,u_m'\rangle$, and r is an integer, $0\le r\le n,m$. Then the *r-join* (or simply a *join*) of (G,U) and (G',U') is the r-join of the network $(G\cup G',\langle u_1,u_1',\dots,u_r,u_r',u_{r+1},\dots,u_n,u_{r+1}',\dots,u_m\rangle)$. If $n=m=r$, then we speak about the *complete join* of (G,U) and (G',U').

Definition 5.1. Suppose that (G,U), (G',U') are disjoint networks, $U=\langle u_1,\dots,u_n\rangle$, and $U'=\langle u_1',\dots,u_m'\rangle$. By an *immersion* of (G',U') into (G,U) we mean a network (G'',U) arising by the following process: choose pairwise different vertices $v_1,\dots,v_m$ from G and identify the sets of vertices $\{v_1,u_1'\}$, $\dots$,$\{v_m,u_m'\}$ to vertices which will be denoted by v_1, $\dots$, v_m, respectively (in other words, every vertex u_i' is deleted and the incident edges are joined with v_i, $i=1,\dots,m$).

For instance, let $(G,\langle u_1,u_2,u_3\rangle)$ and $(G',\langle u_1',u_2'\rangle)$ be the networks from Fig. 5.1. Then $(G'',\langle u_1,u_2,u_3\rangle)$ is an immersion of $(G',\langle u_1',u_2'\rangle)$ into $(G,\langle u_1,u_2,u_3\rangle)$.

6. Proper and improper networks

Definition 6.1. An outer vertex u_i of the network (G, U), $U = \langle u_1, \dots, u_n \rangle$, is called *$k$-proper* (*$k$-improper*) if for every $\langle z_1, \dots, z_n \rangle \in \chi_k(G, U)$, we have $z_i = 1$ ($z_i = 0$). A network is called *k-proper* (*k-improper*) if all of its outer vertices are k-proper (k-improper). A network (G, U) is called *weakly k-proper* (*weakly k-improper*) if every vector from $\chi_k(G, U)$ has at least one coordinate equal one (zero).

For example, if G is a graph and $u_1, \dots, u_n$ are vertices of G of valency 1, then the network $(G, \langle u_1, \dots, u_n \rangle)$ is k-proper for every $k \geq 2$. By Proposition 2.3, a network $(H, \langle u \rangle)$ is k-improper for every graph H, $u \in V(H)$, and $k \geq 2$.

Proposition 6.2. *If G is a k-snark and $u_1, \dots, u_n$ are pairwise different vertices of G, then $(G, \langle u_1, \dots, u_n \rangle)$ is weakly k-proper.*

Proof. Follows from the fact that if $(G, \langle u_1, \dots, u_n \rangle)$ has a nowhere-zero k-flow φ with $\chi(\varphi) = \langle 0, \dots, 0 \rangle$, then φ is a nowhere-zero k-flow in G. □

Proposition 6.3. *Let G be a k-snark and H be the graph arising from G after deleting edges (u, u_1), … , (u, u_n) where $u, u_1, \dots, u_n$ are pairwise different vertices of G. Then $(H, \langle u_1, \dots, u_n \rangle)$ is weakly k-improper.*

Proof. Suppose that $(H, \langle u_1, \dots, u_n \rangle)$ has a nowhere-zero k-flow φ so that $\chi(\varphi) = \langle 1, \dots, 1 \rangle$. Then, by Proposition 2.4, the graph $H' = (H, \langle u_1, \dots, u_n \rangle)^{\langle 1, \dots, 1 \rangle}$ has a nowhere-zero k-flow. But H' is a u-superposition of G, thus, by Proposition 4.1, it is a k-snark – a contradiction. □

Proposition 6.4. *If a graph G is a k-snark, then $(G, \langle u_1, u_2 \rangle)$ is k-proper for every two different vertices u_1 and u_2 of G. Furthermore, if u_1 and u_2 are joined by an edge e, then $(G - e, \langle u_1, u_2 \rangle)$ is k-improper.*

Proof. By Proposition 2.3, no network can have a vector in its k-characteristic set with just one nonzero coordinate. Thus the statement follows from Propositions 6.2, 6.3 and 3.5. □

Clearly, every k-proper or k-improper network with just two outer vertices can be constructed using Proposition 6.4. Similarly (G, U) is weakly k-proper or weakly k-improper if and only if G or $(G, U)^{\langle 1, \dots, 1 \rangle}$ are k-snarks, respectively.

If $(G, \langle u_1, \dots, u_n \rangle)$ is a k-improper network, then so is $(G, \langle u_{i_1}, \dots, u_{i_m} \rangle)$ for every $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$. Moreover, if H is the graph arising from G after identifying the vertices u_1 and u_2 to a new vertex u'_1, then the network $(H, \langle u'_1, u_3, \dots, u_n \rangle)$ is k-improper. The following statement contains several useful observations. Its proof is left to the reader.

Proposition 6.5. *A join of one or two k-proper (k-improper) networks is a k-proper (k-improper) network.*

The complete join of a k-proper and a weakly k-improper (a k-improper and a weakly k-proper) network is a k-snark.

An r-join ($r \geq 1$) of a k-proper and a k-improper network is a k-snark.

A join of one k-snark or two k-snarks is a k-snark.

Example 6.6. The dot product from Example 4.7 can be also expressed as an application of the techniques from this section. By Proposition 6.4, $(G_2', \langle u_1', u_2' \rangle)$ and $(G_1 - (v_1, v_2), \langle v_1, v_2 \rangle)$ are 4-proper and 4-improper, respectively (see Fig. 4.3 and 4.4). Their complete join is G', which is a 4-snark.

The following theorem presents another cornerstone of our techniques.

Theorem 6.7. *Let (G'', U) be an immersion of (G', U') into (G, U) and (G', U') be k-improper. Then $\chi_k(G'', U) \subseteq \chi_k(G, U)$. In particular, if (G, U) is a k-snark, then so is (G'', U).*

Proof. Let φ be a nowhere-zero k-flow in (G'', U). Since (G', U') is k-improper, then $\varphi_{|G}$ is a nowhere-zero k-flow in (G, U), and, furthermore, $\chi(\varphi_{|G}) = \chi(\varphi)$ holds. $\square$

Example 6.8. We again describe the dot product (see Fig. 4.3 and 4.4). Clearly G' is an immersion of the 4-improper network $(G_1 - (v_1, v_2), \langle v_1, v_2 \rangle)$ into the 4-snark G_2', thus, by Theorem 6.7, G' is a 4-snark.

7. Networks with large reluctance

Definition 7.1. Let (G, U), $U = \langle u_1, \dots, u_n \rangle$, be a network. We say that a set $W \subseteq V(G)$ *k-relieves* (G, U) if W contains elements $w_1, \dots, w_m$ such that $(G, \langle u_1, \dots, u_n, w_1, \dots, w_m \rangle)$ admits a nowhere-zero k-flow. The minimal cardinality of a set k-relieving (G, U) is called the *k-reluctance* of (G, U) and denoted by $\rho_k(G, U)$.

This parameter expresses how far a k-snark is from admitting a nowhere zero k-flow. Some other parameters of this kind are discussed in [K8].

By Proposition 2.7, $\rho_k(G, U) \geq \rho_{k+1}(G, U)$. A network (G, U) is a k-snark if and only if $\rho_k(G, U) \geq 1$ and, by Proposition 2.3, a graph G is a k-snark if and only if $\rho_k(G) \geq 2$. The trivial bound $\rho_k(G, U) \leq |V(G) \setminus U|$ is valid for every $k \geq 2$.

Proposition 7.2. *Let $\rho_k(G) \geq r$ and $w_1, \dots, w_r$ be pairwise different vertices of G. Then $(G, \langle w_1, \dots, w_r \rangle)$ is a k-proper network.*

Proof. If there exists $i \in \{1, \dots, r\}$ and a nowhere-zero k-flow φ in $(G, \langle w_1, \dots, w_r \rangle)$ so that $\partial\varphi(w_i) = 0$, then the set $\{w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_r\}$ k-relieves G, which contradicts the assumptions. $\square$

Graphs and networks with arbitrarily large reluctance can be constructed using the following statements.

Proposition 7.3. *Let $(G_1, \langle u_{1,1}, \dots, u_{1,n_1} \rangle)$, $\dots$, $(G_r, \langle u_{r,1}, \dots, u_{r,n_r} \rangle)$ be pairwise disjoint k-snarks, $G = G_1 \cup \dots \cup G_r$, and $U = \langle u_{1,1}, \dots, u_{1,n_1}, \dots, u_{r,1}, \dots, u_{r,n_r} \rangle$. Then $\rho_k(G, U) \geq r$.*

Proof. If W k-relieves (G, U), then W has an inner vertex of $(G_i, \langle u_{i,1}, \dots, u_{i,n_i} \rangle)$ for every $i = 1, \dots, r$, i. e., $|W| \geq r$. Thus $\rho_k(G, U) \geq r$. $\square$

Proposition 7.4. *If (G', U') is a join of (G, U), then $\rho_k(G', U') \geq \rho_k(G, U)$.*

Proof. Let W' be the smallest subset of $V(G')$ that k-relieves (G', U'), and W be the set of the vertices from W' that are inner in (G, U). Then W k-relieves (G, U). Thus $\rho_k(G', U') \geq \rho_k(G, U)$. $\square$

Proposition 7.5. *Two homeomorphic networks have the same k-reluctance for every $k \geq 2$.*

Proof. Let (G, U) be a network and G' be the graph obtained from G after subdividing an edge $e = (v_1, v_2)$ by a new vertex v. Every set that k-relieves (G, U) also k-relieves (G', U), thus $\rho_k(G, U) \geq \rho_k(G', U)$. Let W' be the smallest subset of $V(G')$ that k-relieves (G', U). Take $W \subseteq V(G)$ so that $W = (W' \setminus \{v\}) \cup \{v_1\}$ if $v \in W'$ and $W = W'$ otherwise. Then W k-relieves (G, U). Therefore $\rho_k(G, U) = \rho_k(G', U)$. This implies the statement because two homeomorphic networks arise from the same network after applying a finite number of subdivisions. $\square$

Proposition 7.6. *If (G', U') is a vertex superposition of (G, U), then $\rho_k(G', U') \geq \rho_k(G, U)$.*

Proof. Let $(G', U') = \mathcal{S}(G, U)$ be a v-superposition, $v \in V(G)$. Take a minimal W' that k-relieves (G', U'). Set $W \setminus \{v\} = W' \setminus V(\mathcal{S}_v)$ and $v \in W$ iff $W' \cap V(\mathcal{S}_v) \neq \emptyset$. Then $|W| \leq |W'|$ and W k-relieves (G, U), concluding the proof. $\square$

In the following example we apply methods from this section to constructing proper networks with a large number of outer vertices.

Example 7.7. Since the Petersen graph is a 4-snark, then, by Proposition 6.4, $(P, \langle u_1, u_2 \rangle)$ and $(P - (u'_1, u'_2), \langle u'_1, u'_2 \rangle)$ are 4-proper and 4-improper, respectively (see Fig. 7.1). Applying 1-join on them we get a 4-snark $(H, \langle u_2, u'_2 \rangle)$, by Proposition 6.5. Applying joins on four copies of $(H, \langle u_2, u'_2 \rangle)$, we get a 4-snark H' with 4-reluctance 4 (see Fig. 7.2). Then, by Proposition 7.2, $(H', \langle w_1, w_2, w_3, w_4 \rangle)$ is 4-proper.

8. Even and odd networks

In this section we generalize some ideas from Goldberg [Gol].

Definition 8.1. A network (G, U) is called *k-even* (*k-odd*) if every vector from $\chi_k(G, U)$ has an even (odd) number of nonzero coordinates.

Proposition 8.2. *Every weakly k-improper network with precisely three outer vertices is k-even.*

Proof. Let (G, U) be weakly k-improper, $U = \langle u_1, u_2, u_3 \rangle$, and $\mathbf{z} \in \chi_k(G, U)$. Then $\mathbf{z}$ cannot have three nonzero coordinates. By Proposition 2.3, it cannot have just one nonzero coordinate. Thus (G, U) is k-even. $\square$

The following statement contains several useful observations. Its proof is left to the reader.

Proposition 8.3. *Every k-proper network with an even (odd) number of outer vertices is k-even (k-odd).*

Every k-improper network is k-even.

A join of a k-even (k-odd) network is k-even (k-odd). In particular, the complete join of a k-odd network is a k-snark.

A join of two k-even (k-odd) networks is a k-even network.

A join of a k-even and a k-odd network is a k-odd network. In particular, the complete join of a k-even and a k-odd network is a k-snark.

Example 8.4. Let v_1, v_2, v_3 be the vertices of valency two in $P-v$ (see Fig. 8.1). Thus by Propositions 6.3 and 8.2, $(P-v, \langle v_1, v_2, v_3\rangle)$ is 4-even and so is $(G_3, \langle u_1, u_2, u_3\rangle)$ by Proposition 8.3. Thus the graph G_4 from Fig. 8.2 is a 4-snark.

9. Snarks constructed from Boolean formulas

In this section we generalize ideas from Holyer [Ho] and constructively show that a negation of a tautology corresponds to a family of k-snarks.

We introduce some technical notions which will be used primarily in this section. A network $(G, \langle u_1, u_2, u_3\rangle)$ is called *k-inverting* if it is k-even and the vertex u_3 is k-proper (in other words, $\chi_k(G, \langle u_1, u_2, u_3\rangle) \subseteq \{\langle 1,0,1\rangle, \langle 0,1,1\rangle\}$).

Proposition 9.1. *Let r be an odd (even) positive integer, (G, U) be a k-even (k-odd) network with $r+2$ outer vertices, and (G', U') be a k-proper network with $r+1$ outer vertices. Then the r-join of (G, U) and (G', U') is k-inverting.*

Proof. Follows directly from the definitions. $\square$

A network $(G, \langle u_1, u_2, u_3\rangle)$ is called *k-oriented* if each of its k-characteristic vectors is either $\langle 0,0,0\rangle$, or $\langle 1,1,0\rangle$, or $\langle 1,1,1\rangle$.

Proposition 9.2. *Let $(G, \langle u_1, u_2, u_3\rangle)$ and $(G', \langle u_1', u_2', u_3'\rangle)$ be two distinct k-inverting networks and H the graph obtained after identifying the sets of vertices $\{u_2, u_2'\}$ and $\{u_3, u_3'\}$ to new vertices v_2 and v_3, respectively. Then the network $(H, \langle u_1, u_1', v_3\rangle)$ is k-oriented.*

Proof. Let φ be a nowhere-zero k-flow in $(H, \langle u_1, u_1', v_3\rangle)$. If $\partial\varphi(u_1) = \partial\varphi_{|G}(u_1) \neq 0$, then $\partial\varphi_{|G}(u_2) = -\partial\varphi_{|G'}(u_2') = 0$ and $\partial\varphi_{|G'}(u_1') = \partial\varphi(u_1') \neq 0$ (if we write $\partial\varphi_{|G}$, we always mean $\partial(\varphi_{|G})$).

If $\partial\varphi(u_1) = \partial\varphi_{|G}(u_1) = 0$, then $\partial\varphi_{|G}(u_2) = -\partial\varphi_{|G'}(u_2') \neq 0$ and $\partial\varphi_{|G'}(u_1') = \partial\varphi(u_1') = 0$. Furthermore, by Proposition 2.3, $\partial\varphi_{|G}(u_3) = -\partial\varphi_{|G}(u_2) = \partial\varphi_{|G'}(u_2') = -\partial\varphi_{|G'}(u_3')$, whence $\partial\varphi(v_3) = 0$.

Thus $\chi_k(H, \langle u_1, u_1', v_3\rangle) \subseteq \{\langle 0,0,0\rangle, \langle 1,1,0\rangle, \langle 1,1,1\rangle\}$. $\square$

Example 9.3. Take the networks $(P-v, \langle v_1, v_2, v_3\rangle)$ and $(P, \langle u_1, u_2\rangle)$ from Fig. 8.1 and 7.1, respectively, and apply 1-join on them. We get a 4-inverting network $(G_5, \langle v_2, v_3, u_2\rangle)$. From two copies of this network we get a 4-oriented network $(G_6, \langle v_2, v_2', w_3\rangle)$ (see Fig. 9.1).

Definition 9.4. A network (G, U) is called *k-stable* if every vector from $\chi_k(G, U)$ has all coordinates equal.

Proposition 9.5. *Let $(G_1, \langle u_{1,1}, u_{1,2}, u_{1,3}\rangle)$, ... , $(G_n, \langle u_{n,1}, u_{n,2}, u_{n,3}\rangle)$ be k-oriented networks and G the graph obtained from $G_1, \dots, G_n$ after identifying the sets of vertices $\{u_{1,3}, u_{2,2}\}$, $\{u_{2,3}, u_{3,2}\}$, ... , $\{u_{n-1,3}, u_{n,2}\}$, $\{u_{n,3}, u_{1,2}\}$ to new vertices v_1, v_2, ... ,v_{n-1}, v_n, respectively. Then the network $(G, \langle u_{1,1}, u_{2,1}, \dots, u_{n,1}\rangle)$ is k-stable.*

Proof. Let φ be a nowhere-zero k-flow in the network $(G, \langle u_{1,1}, u_{2,1}, \dots, u_{n,1}\rangle)$ and assume that there exists an index i so that $\partial\varphi(u_{i,1}) = 0$. Without loss of generality we can assume that $i = 1$. Then $\partial\varphi(u_{1,1}) = \partial\varphi_{|G_1}(u_{1,1}) = \partial\varphi_{|G_1}(u_{1,2}) = \partial\varphi_{|G_1}(u_{1,3}) = 0$. Therefore $\partial\varphi_{|G_2}(u_{2,2}) = -\partial\varphi_{|G_1}(u_{1,3}) = 0$ and $\partial\varphi_{|G_2}(u_{2,2}) = \partial\varphi_{|G_2}(u_{2,3}) =$

$\partial\varphi_{|G_2}(u_{2,1}) = \partial\varphi(u_{2,1}) = 0$. Similarly, using induction, we can show that $\partial\varphi(u_{j,1}) = 0$ for every $j = 1, \ldots, n$. This implies the statement. $\square$

A network $(G, \langle u_1, \ldots, u_n\rangle)$ is called *k-satisfaction testing* if u_n is k-proper.

Proposition 9.6. *Let (G, U) be a network and (G', U') a k-proper network with $r + 1$ outer vertices, $r \geq 1$. Then the r-join of (G, U) and (G', U') is k-satisfaction testing.*

Proof. Follows directly from the definitions. $\square$

Construction 9.7. Suppose that C is a Boolean formula given in a conjunctive normal form, i. e., C is a set of clauses $\{C_1, \ldots, C_n\}$ in variables $y_1, y_2, \ldots, y_m$ and each clause C_i consists of literals $l_{i,1}, l_{i,2}, \ldots, l_{i,n_i}$ where a literal $l_{i,j}$ is either a variable y_s or its negation $\overline{y}_s$. If we have a truth assignment to the variables of C, then a clause is satisfied if at least one of its literals has value "true". C is satisfiable if there exists a truth assignment which satisfies all the clauses in C. Let C be not satisfiable.

Take a k-satisfaction testing network $(G_i, \langle u_{i,1}, \ldots, u_{i,n_i}, u_{i,n_i+1}\rangle)$ for each clause C_i of C. The outer vertices $u_{i,1}, \ldots, u_{i,n_i}$ are called the *outputs* of G_i.

Let y_s be a variable in C which appears in just p clauses of C. Then take a k-stable network $(H_s, \langle v_{s,1}, \ldots, v_{s,q}\rangle)$, $q \geq p$. The outer vertices $v_{s,1}, \ldots, v_{s,p}$ are called the *outputs* of H_s.

Assume that the operation of negation of a variable appears t times in C. Then, for $r = 1, \ldots, t$, let $(F_r, \langle w_{r,1}, w_{r,2}, w_{r,3}\rangle)$ be a k-inverting network. The vertices $w_{r,1}$ and $w_{r,2}$ are called the *outputs* of F_r.

Now we construct a new network (G, U) from these networks. The construction runs as follows. Suppose the literal $l_{i,j}$ in clause C_i is y_s. Then identify one output from G_i with one output from H_s to a new inner vertex (of (G, U)). If $l_{i,j}$ is $\overline{y}_s$, then take one F_r and identify one output from G_i with one output of this F_r to a new inner vertex and identify one output from H_s with the second output of F_r to another new inner vertex.

Repeating this process for all clauses and its literals we get (G, U). Its outer vertices are the outer vertices of all networks used in the construction which have not been outputs. We claim that (G, U) is a k-snark. On the contrary suppose there exists a nowhere-zero k-flow φ in (G, U). Then take a truth assignment to the variables of C so that a variable y_s has value "true" if $\partial\varphi_{|H_s}(v_{s,1}) \neq 0$, and value "false" otherwise. By Proposition 2.3, for each clause C_i, there exists at least one output $u_{i,j}$ of G_i so that $\partial\varphi_{|G_i}(u_{i,j}) \neq 0$. But then, in our assignment, each clause is satisfied, that means C is satisfiable. Thus (G, U) is a k-snark.

In some special cases we can deduce also a converse, namely that C is not satisfiable if (G, U) is a k-snark (see [Ho], [K6]). Construction 9.7 gives nontrivial k-snarks for $k = 3, 4$, and if the 5-flow conjecture is false, then also for $k = 5$ (see [K6]).

Formula $y_1 \wedge \overline{y}_1$ is a negation of tautology and has literals $C_1 = y_1$ and $C_2 = \overline{y}_1$. Taking one copy of the 4-inverting network $(G_5, \langle v_2, v_3, u_2\rangle)$ from Example 9.3, two copies of the 4-satisfaction testing network $(K_2, \langle w_1, w_2\rangle)$, one copy of the 4-stable network $(K_3, \langle u_1, u_2\rangle)$, and applying Construction 9.7 we get a 4-snark $(G_7, \langle w_2, w_2', u_2\rangle)$ indicated in Fig. 9.2.

10. Superjoin of multipoles

Superjoin is a generalization of join and it will be a useful tool for formulating some more complicated statements about joins. It has similar features to superposition, but it will be defined on graphs with semiedges. Fiol [Fi] called such graphs multipoles.

Formally, by a *multipole* we mean a triple $M = (V, E, S)$ such that (V, E) is a graph and S is a finite set of *semiedges* of M. Each semiedge is incident to one vertex (see Fig. 10.1). Note that $E \cap S = \emptyset$. V and E are called vertex and edge sets of M, respectively. The *compactification* of M is the graph G_M obtained from M after adding a new vertex v_M and replacing all semiedges of M by edges incident to v_M (the valency of v_M is equal to the number of semiedges in M – see Fig. 10.1). A valency of a vertex of M is its valency in G_M. Every semiedge incident to a vertex v determines one arc directed out of v. Thus the *arc set* of M, denoted by $D(M)$, is equal to $D(G_M) \setminus \omega^+_{G_M}(v_M)$. Similarly as for graphs define $\omega^+_M(v) = \omega^+_{G_M}(v)$ and $\omega^-_M(v) = \omega^-_{G_M}(v) \setminus \omega^+_{G_M}(v_M)$ for every vertex v of M.

If A is an Abelian group, then $\varphi : D(M) \to A$ is called a (*nowhere-zero*) A-*flow* in M if there exists a (nowhere-zero) A-flow ψ in G_M so that φ is the restriction of ψ to $D(M)$. Then, by Theorem 2.5, M has a nowhere-zero $\mathbb{Z}_k$-flow if and only if it has a nowhere-zero A-flow for any Abelian group of order k. Therefore, similarly as for graphs and networks, by the term (*nowhere-zero*) k-*flow* we mean any (nowhere-zero) A-flow where A is an Abelian group of order k. By Proposition 2.7, if a multipole admits a nowhere-zero k-flow, then it admits a nowhere-zero $(k+1)$-flow. Finally, a multipole is called a k-*snark* if it does not admit a nowhere-zero k-flow. Clearly, a multipole is a k-snark if and only if its compactification is a k-snark.

Let M be a multipole. Then by a *superjoin* of M we mean a network $\mathcal{J}(M)$ arising by the following process: Replace each vertex v of M by a network $\mathcal{J}(v) = (G_v, \langle u_{y_1}, \ldots, u_{y_{n_v}} \rangle)$ where $\{y_1, \ldots, y_{n_v}\}$ is the set of the arcs leaving v (it is the union of $\omega^+_M(v)$ and the set of the arcs arising from the loops incident to v, i. e., n_v is the valency of v in M). $\mathcal{J}(v)$ and $\mathcal{J}(v')$ are disjoint if $v \neq v'$. For every edge $e = (v_1, v_2)$ of M take the arcs x_1 and x_2 arising from e and directed out of v_1 and v_2, respectively, and identify the two outer vertices u_{x_1} and u_{x_2} from $\mathcal{J}(v_1)$ and $\mathcal{J}(v_2)$, respectively, to a new inner vertex v_e (of $\mathcal{J}(M)$). We get a network $\mathcal{J}(M) = (G, \langle u_{t_1}, \ldots, u_{t_n} \rangle)$ where $t_1, \ldots, t_n$ are the arcs arising from the semiedges of M.

Let φ be a k-flow in $\mathcal{J}(M)$. For every vertex v of M, $D(G_v)$ is a subset of $D(G)$ and let φ_v denotes the restriction of φ to $D(G_v)$. Then φ_v is a k-flow in the network $\mathcal{J}(v) = (G_v, \langle u_{y_1}, \ldots, u_{y_{n_v}} \rangle)$, and, by Proposition 2.3, $\sum_{i=1}^{n_v} \partial\varphi_v(u_{y_i}) = 0$. Furthermore, if $e = (v_1, v_2)$ is an edge of M and x_1, x_2 are the arcs arising from e and directed out of v_1, v_2, respectively, then $\partial\varphi_{v_1}(u_{x_1}) = -\partial\varphi_{v_2}(u_{x_2})$. Thus we can define a flow $\varphi_{\mathcal{J}}$ in M so that if x is an arc of M directed out of a vertex v, then

$$\varphi_{\mathcal{J}}(x) = \partial\varphi_v(u_x), \tag{10.1}$$

where G_v and u_x have the same meaning as above.

Note that any r-join of a network is in fact a superjoin of a multipole having one vertex with r loops and an appropriate number of semiedges and any r-join of two networks is a superjoin of a multipole having two vertices connected by r parallel edges and with an appropriate number of semiedges.

Proposition 10.1. *If a multipole M is a k-snark and $\mathcal{J}(M)$ is a superjoin of M such that $\mathcal{J}(v)$ is k-proper for every vertex v of M, then $\mathcal{J}(M)$ is a k-snark.*

Proof. If φ is a nowhere-zero k-flow in $\mathcal{J}(M)$, then $\varphi_{\mathcal{J}}$ is a nowhere-zero k-flow in M, because every $\mathcal{J}(v)$ is k-proper. Since such a flow does not exist, $\mathcal{J}(M)$ is a k-snark. $\square$

Now we describe a method for constructing k-improper networks with a large number of outer vertices.

Proposition 10.2. *Suppose that M is a multipole and $W \subseteq V(M)$ such that there is no edge joining two different vertices from W, every vertex from W is incident to precisely one semiedge, and no semiedge is incident to a vertex from $V(G) \setminus W$. Let $\mathcal{J}(M)$ be a superjoin of M such that*

- *$\mathcal{J}(v)$ is k-proper if $v \in V(M) \setminus W$,*
- *$\mathcal{J}(v)$ is k-even (k-odd) if $v \in W$ and the valency of v is odd (even).*

Then $\mathcal{J}(M)$ is k-improper.

Proof. If $v \in W$ is incident to r loops, then the following changes are applied on the network $\mathcal{J}(v)$ by constructing $\mathcal{J}(M)$ (the notation is the same as in the definition of superjoin): r pairs of outer vertices of $\mathcal{J}(v)$ are identified to r new inner vertices and $n_v - (2r + 1)$ outer vertices of $\mathcal{J}(v)$ are identified with k-proper outer vertices of some of the networks $\mathcal{J}(w)$, where $w \in V(G) \setminus W$. Thus the last outer vertex of $\mathcal{J}(v)$, which is also an outer vertex of $\mathcal{J}(M)$, must be k-improper in $\mathcal{J}(M)$. $\square$

Note that setting " ... *the valency of v is even (odd)*" in the last but one line of Proposition 10.2, we get that the resulting network $\mathcal{J}(M)$ must be k-proper.

Example 10.3. Let M be the multipole from Fig. 10.2 and W be the set of its vertices of valency three. Take a superjoin $\mathcal{J}(M)$ so that $\mathcal{J}(v)$ is a copy of $(P - v, \langle v_1, v_2, v_3\rangle)$ (from Fig. 8.1) if $v \in W$ and a copy of $(P, \langle u_1, u_2\rangle)$ (from Fig. 7.1) otherwise. Then the network $\mathcal{J}(M) = (G_8, \langle w_1, w_2, w_3, w_4\rangle)$ from Fig. 10.2 is 4-improper.

Now we can reformulate some results about joins. For instance, the following two statements follow directly from Proposition 8.3 (and can be also proved similarly as Proposition 10.2).

Proposition 10.4. *Suppose that $\mathcal{J}(G)$ is a superjoin of a graph G such that $\mathcal{J}(v)$ is k-odd for an odd number of vertices of G and otherwise it is k-even. Then $\mathcal{J}(G)$ is a k-snark.*

Proposition 10.5. *Suppose that $\mathcal{J}(M)$ is a superjoin of a multipole M such that $\mathcal{J}(v)$ is k-odd for an odd (even) number of vertices of M and otherwise it is k-even. Then $\mathcal{J}(M)$ is k-odd (k-even).*

11. Snarks constructed from cubic digraphs and supersnarks

By a *cubic digraph* we mean an orientation X of a cubic graph G such that $|\omega_G^+(v) \cap X|$ is equal to 1 or 2 for every vertex v of G. Furthermore, by a *directed 2-factor* of X we mean every $Y \subseteq X$ such that $|\omega_G^+(v) \cap Y| = |\omega_G^-(v) \cap Y| = 1$ for every vertex v of G.

Note that there exist cubic digraphs without a directed 2-factor. For example, let X be an orientation of a cubic graph G indicated in Fig. 11.1 satisfying $\omega_G^+(W_1) \subseteq X$. If Y is a directed 2-factor of X, then $\omega_G^+(W_1) \cap Y = \emptyset$. Therefore, using a simple parity argument we can check that X has no directed 2-factor.

Construction 11.1. Let G be a cubic graph and X be a cubic digraph of G having no directed 2-factor. Take a superjoin $\mathcal{J}(G)$ so that, for every vertex v of G, $\mathcal{J}(v) = (G_v, \langle u_{v,1}, u_{v,2}, u_{v,3}\rangle)$ is a k-inverting network (see the definition in the beginning of Section 9) and the k-proper outer vertex $u_{v,3}$ is associated with the arc $x \in \omega_G^+(v)$ with the property that $\{x\}$ is equal either to $\omega_G^+(v) \cap X$ or to $\omega_G^-(v) \cap X$. If φ is a nowhere-zero k-flow in $\mathcal{J}(G)$, then the support of $\varphi_{\mathcal{J}}$ is a 2-factor of G which induces a directed 2-factor Y of X. Since such Y does not exist, $\mathcal{J}(G)$ is a k-snark.

In this paper, by a *k-supersnark* we mean a k-regular graph whose edges cannot be colored by k colors. For example, every snark is a 3-supersnark. (Note that k-supersnarks are in literature sometimes called k-snarks, but for the purpose of this paper it is more suitable to use this term for graphs without nowhere-zero k-flows.) Meredith [M] constructed nontrivial k-supersnarks for each $k \geq 4$. By Kotzig [Kot], the line graph of a snark is a 4-supersnark. Now we present a reverse process and show that from every 4- and 3-supersnark we can get 4- and 3-snarks, respectively.

In the remainder of this section we deal with $\mathbb{Z}_3$- or $\mathbb{Z}_2 \times \mathbb{Z}_2$-flows because we use also properties of the structures of these groups.

A network (G, U) is called 4-*color* if it has four outer vertices and $\partial\varphi(u) \neq \partial\varphi(u')$ for every nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow in (G, U) and every two different outer vertices u, u'.

Proposition 11.2. *Every 4-odd network with four outer vertices is 4-color.*

Proof. Follows from Proposition 2.3 and the arithmetic in the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. □

Thus 4-color networks can be constructed using Propositions 10.5 (for example, taking a multipole M with 4 semiedges and all vertices of valency 3 and replacing one vertex of M by a 4-odd network and the rest of the vertices of M by 4-even networks). Using 4-color networks we can transform every 4-supersnark to a 4-snark applying the following statement.

Proposition 11.3. *If a graph G is a 4-supersnark and $\mathcal{J}(G)$ is a superjoin of G such that $\mathcal{J}(v)$ is 4-color for every $v \in V(G)$, then $\mathcal{J}(G)$ is a 4-snark.*

Proof. Let φ be a nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow in $\mathcal{J}(G)$. Then assign to each $e \in E(G)$ the value $\varphi_{\mathcal{J}}(x) = \varphi_{\mathcal{J}}(-x)$ (the identity $a = -a$ holds in $\mathbb{Z}_2 \times \mathbb{Z}_2$) where x and $-x$ are the opposite arcs arising from e. We get a 4-edge-coloring of G by the elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ – a contradiction. □

A network (G, U) is called 3-*color* if it has three outer vertices and $\partial\varphi(u) \neq \partial\varphi(u')$ for every nowhere-zero $\mathbb{Z}_3$-flow in (G, U) and every two different outer vertices u, u'.

Proposition 11.4. *Let $(G, \langle u_1, u_2, u_3\rangle)$ be a 3-stable network and $(G', \langle u_1', u_2'\rangle)$ be a 3-proper network. Suppose that G'' is the graph obtained from G and G' after identifying the sets of vertices $\{u_1, u_1'\}$ and $\{u_2, u_2'\}$ to new vertices u_1'' and u_2'', respectively. Then $(G'', \langle u_1'', u_2'', u_3\rangle)$ is a 3-color network.*

Proof. Follows from Proposition 2.3 and the arithmetic in the group $\mathbb{Z}_3$. □

Proposition 11.5. *Let G be a 3-supersnark and M_G be the multipole arising from G after subdividing every edge e from G by a new vertex w_e and attaching a new semiedge s_e to w_e. Take a superjoin $\mathcal{J}(M_G)$ so that $\mathcal{J}(v)$ is 3-color for every $v \in V(G)$ and $\mathcal{J}(w_e)$ is 3-stable for every $e \in E(G)$. Then $\mathcal{J}(M_G)$ is a 3-snark.*

Proof. Let $X = \cup\{\omega^+_{M_G}(w_e); e \in E(G)\}$ and $M_G = (V, E, S)$. For every $h \in E \cup S$, denote by x_h the arc arising from h and belonging to X. Suppose that φ is a nowhere-zero $\mathbb{Z}_3$-flow in $\mathcal{J}(M_G)$. Then color each $h \in E \cup S$ by $\varphi_{\mathcal{J}}(x_h)$. We get a coloring of the edges and semiedges of M_G by elements of $\mathbb{Z}_3$ so that the edges incident to a vertex $v \in V(G)$ are colored by pairwise different colors and the edges and semiedge incident to a vertex w_e, $e \in E(G)$, are colored by the same color. This corresponds to a 3-edge-coloring of G, which does not exist. Thus $\mathcal{J}(M_G)$ is a 3-snark. □

12. 3-snarks

In this section we deal with 3-snarks and the 3-flow conjecture. Note that $\mathbb{Z}_3$ is the single Abelian group of order 3. Thus every 3-flow is a $\mathbb{Z}_3$-flow. First of all we present a very simple example of a 3-stable network and extend item (a) of Proposition 3.3 for networks.

Lemma 12.1. *Let v be an inner vertex of valency three in a network with a nowhere-zero $\mathbb{Z}_3$-flow φ. Then φ has the same value on all arcs directed out of v.*

Proof. Follows from the arithmetic in the group $\mathbb{Z}_3$. □

Corollary 12.2. *Let u_1, u_2, u_3 be the vertices of valency two in $K_{3,2}$. Then $(K_{3,2}, \langle u_1, u_2, u_3\rangle)$ is a 3-stable network.*

Proof. Follows directly from Lemma 12.1. □

Corollary 12.3. *Let (G, U) be a network where all inner vertices have valency three. Then (G, U) is a 3-snark if and only if it has an odd circuit not containing an outer vertex.*

Proof. Let C be an odd circuit in (G, U) not containing an outer vertex and $x_1, \ldots, x_n$ be arcs of C so that they form an oriented circuit. Let φ be a nowhere-zero 3-flow in (G, U). Then, by Lemma 12.1, $\varphi(x_1) = -\varphi(x_2) = \varphi(x_3) = \cdots = \varphi(x_n) = -\varphi(x_1)$, what is not possible in $\mathbb{Z}_3 \setminus \{0\}$. Thus (G, U) is a 3-snark.

Suppose that every odd circuit from G contains an outer vertex. Then split each outer vertex of (G, U) to vertices of valency 1. We get a bipartite graph G' and let $u'_1, \ldots, u'_n$ be the vertices of G' of valency 1 (they arise from the outer vertices of (G, U)). Let V_1 and V_2 be the partition of the vertices of G' into two independent sets. Then take a nowhere-zero 3-flow φ in $(G', \langle u'_1, \ldots, u'_n\rangle)$ so that $\varphi(x) = 1$ if the arc x is directed form V_1 to V_2 and $\varphi(x) = 2$ otherwise. But φ is also a nowhere-zero 3-flow in (G, U), completing the proof. □

In Fig. 12.1 we have a 3-snark K_4, a 3-improper network $(G_9, \langle v_1, v_2\rangle)$, $G_9 = K_4 - (v_1, v_2)$, a 3-stable network $(K_{3,2}, \langle u_1, u_2, u_3\rangle)$, and a 3-color network $(G_{10}, \langle u_1, u_2, u_3\rangle)$, $G_{10} - (u_1, u_2) = K_{3,2}$. By Proposition 3.4, G_{11} is a k-snark for any $k \geq 2$, thus also a 3-snark and a 3-supersnark. Then, applying the construction from Proposition 11.5 for G_{11}, we get a 3-snark $(G_{12}, \langle w_1, w_2, w_3\rangle)$.

The 3-*flow conjecture* of Tutte suggests that every graph without edge cuts of cardinality 1 and 3 has a nowhere-zero 3-flow. An equivalent form of this conjecture is that there does not exist a 4-edge-connected 3-snark (see [J4]). This conjecture is true for planar graphs as follows from the Theorem of Grötzsch [Gro] (see also Thomassen [Th]) and the result of Tutte [T1] which says that a planar graph admits a nowhere-zero k-flow if and only if it is k-face-colorable. We show that the 3-flow conjecture can be strengthened in the following way.

Conjecture 12.4. Every bridgeless graph with at most three edge cuts of cardinality 3 has a nowhere-zero 3-flow.

An edge cut in a graph is called *trivial* if all its edges are incident to one vertex.

Theorem 12.5. *Conjecture* 12.4 *is equivalent to the* 3*-flow conjecture.*

Proof. Suppose that Conjecture 12.4 is not true and G is the smallest counterexample. Then every edge cut C of cardinality 3 must be trivial. (If not, take C so that $G - C$ has components H' and H''. Identifying the vertices of H' and H'' to vertices v' and v'' we get from G new graphs G' and G'', respectively. At least one of G', G'' is a smaller counterexample, otherwise we get a nowhere-zero 3-flow in G by Lemma 12.1.) Let v_1, v_2, and v_3 be vertices of G of valency 3. Let edges e_1, e_2 form a perfect matching in K_4. Apply 3-strong e_1- and e_2-superpositions $\mathcal{S}$ and $\mathcal{S}'$ (see Definition 4.3 and Proposition 4.6), respectively, on K_4 so that $\mathcal{S}_{e_1} = \mathcal{S}'_{e_2} = (G, \langle v_1, v_2 \rangle)$. We get a 3-snark G_1, that has two vertices v_3 and v_3' of valency 3. Now apply again 3-strong e_1- and e_2-superpositions $\mathcal{S}''$ and $\mathcal{S}'''$, respectively, on K_4 so that $\mathcal{S}''_{e_1} = \mathcal{S}'''_{e_2} = (G_1, \langle v_3, v_3' \rangle)$. We get a 3-snark G_2 without 1- or 3-edge cuts. The rest of the proof is trivial. □

By a theorem of Grünbaum [Gru], proved by Aksionov [A] and Borodin [Bo], Conjecture 12.4 is satisfied for planar graphs. This conjecture is, in certain sense, the strongest possible, because the following holds.

Theorem 12.6. *There exists an infinite family of (planar)* 3*-snarks having just four trivial edge cuts of cardinality* 3 *and no other edge cut of cardinality* ≤ 3.

Proof. Take the minimal class of graphs containing K_4 and closed under the operation of replacing an edge between two vertices of valency 3 by a copy of K_4. This class has the required property. □

Note that, by Jaeger [J2, J4], every 4-edge-connected graph has a nowhere-zero 4-flow. Using his methods (see [J4, Lemma 4.1 and Theorem 4.7]) we can show that every bridgeless graph with at most four edge cuts of cardinality ≤ 3 has a nowhere-zero 4-flow.

Now we study what happens if the 3-flow conjecture is false.

Theorem 12.7. *If the* 3*-flow conjecture is not true, then it is an* NP*-complete problem to determine whether a* 4*-edge-connected graph admits a nowhere-zero* 3*-flow.*

Proof. By Tutte [T1], a planar graph admits a nowhere-zero k-flow if and only if its dual is k-vertex-colorable. By Garey, Johnson and Stockmeyer [GJS], it is an NP-complete problem to decide whether a planar graph is 3-vertex-colorable. Thus, it is an NP-complete problem to decide whether a graph admits a nowhere-zero 3-flow. Suppose that the 3-flow conjecture is false. Then choose a 4-edge-connected 3-snark

G of the smallest possible order. Let u, v be two different vertices of G. Clearly, $(G, \langle u, v\rangle)$ is not a 3-snark (otherwise the graph G' obtained as 1-join of $(G, \langle u, v\rangle)$ is a 3-snark, and, since G' is 4-edge-connected, we get a contradiction). Thus, by Proposition 6.4, $\chi_k(G, \langle u, v\rangle) = \{\langle 1, 1\rangle\}$ and there exist nowhere-zero 3-flows φ, ψ in $(G, \langle u, v\rangle)$ so that $\partial\varphi(u) = -\partial\varphi(v) = -\partial\psi(u) = \partial\psi(v) = 1$. This holds for any two different vertices u, v of G. Suppose that H is a graph. Let $\mathcal{S}(H)$ be a 3-strong superposition of H which arises after replacing each edge of H by a copy of the 3-snark G. Then, by the previous facts, H is a 3-snark if and only if $\mathcal{S}(H)$ is. Furthermore $\mathcal{S}(H)$ is 4-edge-connected if H is connected. Thus we get a polynomial reduction to the problem of [GJS]. $\square$

Theorem 12.7 gives information about a possible complexity of the problem to decide whether a 4-edge-connected graph admits a nowhere-zero 3-flow without having a solution of the 3-flow conjecture. Let us note, that we can similarly show that if there had existed a 4-edge-connected planar 3-snark, then it would be an NP-complete problem to decide whether a 4-edge-connected planar graph admits a nowhere-zero 3-flow. But this problem is polynomial by [Gro], [Th].

Theorem 12.8. *If the 3-flow conjecture is not true, then there exists an integer $r > 0$ so that for every integer $k \geq 2$ there exists a 4-edge-connected graph G_k of order k satisfying $\rho_3(G_k) \geq \lfloor k/r \rfloor$.*

Proof. For every $n \geq 2$, take a network $(H_n, \langle w_1, w_2\rangle)$ where H_n is a 4-edge-connected graph of order n. Let G be a 4-edge-connected 3-snark G of order r. Then there exists a 2-edge-connected 3-snark $(G', \langle u_1, u_2\rangle)$ such that its 1-join is G. Let $k \geq 2$ and $p = \lceil k/r \rceil$, $q = k - \lfloor k/r \rfloor r$. Take a circuit C_p of length p and a vertex v_1 from C_p. Suppose that $\mathcal{J}(C_p)$ is a superjoin of C_p so that $\mathcal{J}(v)$ is $(G', \langle u_1, u_2\rangle)$ for every vertex v of C_p different from v_1 and $\mathcal{J}(v_1)$ is either $(H_{q+1}, \langle w_1, w_2\rangle)$ (if $q > 0$) or $(G', \langle u_1, u_2\rangle)$ (if $q = 0$). Then, by Propositions 7.3 and 7.4, $\rho_3(\mathcal{J}(C_p)) \geq \lfloor k/r \rfloor$. Moreover $\mathcal{J}(C_p)$ is 4-edge-connected and has k vertices, which implies the statement. $\square$

Let $\overline{\rho}_3(k)$ denote the maximal 3-reluctance of a 4-edge-connected graph of order k. If there is no 4-edge-connected 3-snark, then $\overline{\rho}_3(k)/k = 0$ for every $k > 0$. Otherwise, by Theorem 12.8, $\liminf_{k\to\infty} \overline{\rho}_3(k)/k \geq 1/r > 0$. Therefore the 3-flow conjecture is equivalent to the following problem (see [K8] for more details).

Conjecture 12.9. $\liminf_{k\to\infty} \overline{\rho}_3(k)/k = 0$.

13. Snarks

If we deal with nowhere-zero 4-flows in graphs, then it is convenient to use $\mathbb{Z}_2\times\mathbb{Z}_2$-flows, because the identity $a = -a$ holds in $\mathbb{Z}_2\times\mathbb{Z}_2$ and, therefore, every nowhere-zero $\mathbb{Z}_2\times\mathbb{Z}_2$-flow φ in a graph G can be considered as a mapping $\varphi : E(G) \to \mathbb{Z}_2\times\mathbb{Z}_2$ so that for every vertex v of G the sum of the values of the edges incident to v is zero. In particular, if we deal with a cubic graph G, then every nowhere-zero $\mathbb{Z}_2\times\mathbb{Z}_2$-flow in G corresponds to a 3-edge-coloring of G by the nonzero elements of the group $\mathbb{Z}_2\times\mathbb{Z}_2$ and vice versa.

The term snark was introduced by Gardner [Ga] borrowing this name from Lewis Caroll's ballad "The Hunting of the Snarks". The study of snarks began in the work of Tait [Ta], who in fact proved that the four-color problem is equivalent to the statement that there are no planar snarks. The interest in this topic has intensified since realizing that several significant conjectures (in particular the 5-flow conjecture and the cycle double cover conjecture) must have snarks as minimal counterexamples. From our everyday experience we might concede that snarks are not common among cubic graphs. A result of Robinson and Wormald [RW], stating that almost every cubic graph is hamiltonian, thus also 3-edge-colorable, supports this observation.

The first known snark is the Petersen graph (see Fig. 1.1) discussed by Petersen [Pe] in 1898. This graph appeared already in the paper of Kempe [Ke] in 1886. The next snark was found by Blanuša [Bl] in 1946 (see B_{18} in Fig. 13.5). In 1948 another snark, having 210 vertices, was discovered by Descartes [D]. The fourth snark was found by Szekeres [Sz] in 1973 and has 50 vertices. In 1975 Isaacs [I1] presented constructions of two infinite families of snarks. The first one is the family of *flower snarks* (as pointed out in [Gol], these graphs have been constructed independently by Grinberg). The second family is based on the construction of dot product which was also introduced independently by Adelson-Velskij and Titov [AVT]. Interesting infinite families of snarks were found by Goldberg [Gol] and Holyer [Ho]. Further variants can be found in [Si, CH, W1, Pr] and in the unpublished papers [I2, CS]. For surveys see [CMRS, CW, WW, W2]. New infinite families of snarks we constructed in [K1, K3, K4, K5].

The general methods from Sections 2–11 can be directly used for constructions of 4-snarks. From them we can easily obtain snarks after splitting vertices of valency greater then four to vertices of valencies two and three and subsequently suppressing the vertices of valency two. In this section we present several examples.

An edge cut of a graph is called *cyclic* if after deleting its edges we get at least two components having cycles. A graph is called *cyclically k-edge-connected* if it does not have a cyclic edge cut of cardinality smaller than k.

1. Cyclically 6-edge-connected snarks. In Fig. 13.1 are depicted flower snarks I_3, I_5, and I_7. Similarly we can construct graph a I_{2k+1} of order $4(2k+1)$ for every $k \geq 1$. Flower snarks $I_7, I_9, \ldots$ were the only cyclically 6-edge-connected snarks known previous to the introduction of superposition techniques.

Using superposition we can construct new infinite families of cyclically 6-edge-connected snarks. For instance the following construction was announced in [K4]. Let C be a circuit of length 6 in the Petersen graph P. Take a 4-strong superposition of P so that the edges of C are replaced by copies of I_5 and each vertex of valency 7 is replaced by vertices of valencies 3, 2, and 2 as indicated in Fig. 13.2. Suppressing the vertices of valency 2 we get a cyclically 6-edge-connected snark G_{118} of order 118 depicted in Fig. 13.3. We can similarly construct a snark G_{120} of order 120 depicted in Fig. 13.4 (it suffices to replace one vertex of C by K_4) and snarks with 122 or 124 vertices. Furthermore, using I_{2k+1} ($k \geq 3$) instead of I_5 we can obtain cyclically 6-edge-connected snarks of any even order ≥ 118. Thus the following statement holds.

Theorem 13.1. *For every even $n \geq 118$, there exists a cyclically 6-edge-connected snark with n vertices.*

In fact we can replace every edge of P by a copy of flower snark and then split the vertices of valency 9 to three vertices of valency 3. We again get a cyclically 6-edge-connected snark. Furthermore, we can use any snark or 2-edge-connected (cubic) 4-snark instead of P .

2. Simple 5-cut snarks. A cyclic k-edge cut is called *simple*, if after removing its edges we get at least one component consisting of a circuit. If a cubic graph G is cyclically 5-edge-connected and every cyclic 5-edge cut of G is simple, then we say that G is *simple 5-cut* cubic graph. For example, the Petersen graph and the flower snark I_5 are simple 5-cut snarks. These are the only such snarks known until recently. As pointed out by Škoviera [Sk], simple 5-cut snarks are indecomposable in the sense of the paper of Cameron, Chetwynd and Watkins [CCW].

If we use a circuit C of length 5 in the construction preceding Theorem 13.1, we get a simple 5-cut snark of order 100. Furthermore, if we use the Blanuša snark B_{18} instead of I_5, we get a simple 5-cut snark G_{90} of order 90 (see Fig. 13.5). Clearly, combining these constructions or using I_{2k+1} ($k \geq 3$) we can obtain simple 5-cut snark of any even order ≥ 90.

Theorem 13.2. *For every even $n \geq 90$, there exists a simple 5-cut snark with n vertices.*

3. Snarks with large girths. Jaeger and Swart [JS] conjectured that every snark has girth at most 6. This conjecture was very interesting because, by Goddyn [God] and Celmins [C] (see also [MCB], [Je], [Z2], [K9]), if it had been true, then it would imply positive solutions of the cycle double cover and the 5-flow conjectures. Using superposition we have constructed snarks with arbitrary large girths in [K3]. Now we sketch only the main ideas of the construction and refer to [K3] for more details.

Let G be a cyclically 5-edge-connected cubic graph with girth $c \geq 7$. Remove vertex sets of paths of lengths 2 and 4 from G obtaining graphs A and B, respectively, as indicated in Fig. 13.6. The graph J from Fig. 13.7 is a snark because it is a 4-strong superposition of P. Note that every circuit in J of length less than c contains either w or w'. Then the graph S depicted in Fig. 13.8 is a cyclically 5-edge-connected snark (it is a 4-strong superposition of P) and has girth at least c. Thus taking for G the bipartite sextet graphs from Biggs and Hoare [BH] and Weiss [Wei] we get the following result.

Theorem 13.3. *There exists an infinite family of cyclically 5-edge-connected snarks whose girth c and order n satisfy the inequality $c \geq (4/3 + o(1)) \log_2 n$.*

4. A 4-improper cubic network. By Proposition 6.4, $(P - (u_1', u_2'), \langle u_1', u_2' \rangle)$ is a 4-improper network (see Fig. 7.1). The network $(G_{102}, \langle u_1, u_2 \rangle)$ from Fig. 13.9 is also 4-improper, because it is homeomorphic with a 4-strong superposition of $(P - (u_1', u_2'), \langle u_1', u_2' \rangle)$.

Let H be a disjoint union of two copies of I_7. Applying immersion of three copies of $(G_{102}, \langle u_1, u_2 \rangle)$ into H we get the graph G_{356} indicated in Fig. 13.10. By Theorem 6.7, G_{356} is a 4-snark. Splitting the vertices of G_{356} of valency 6 to three vertices of valency 2 and suppressing them we get the cyclically 6-edge-connected snark G_{350} of order 350 from Fig. 13.10. In this way we can "glue together" two snarks to a new snark.

From two copies of G_{102} we get a network $(G_{207}, \langle u_1, u_2, u_3 \rangle)$ depicted in Fig. 13.11. Clearly, for every nowhere-zero $\mathbb{Z}_2 \times \mathbb{Z}_2$-flow φ in the network we have $(0,0) \neq \partial\varphi(u_1) \neq \partial\varphi(u_2) \neq (0,0)$, which, by Theorem 2.6 and Proposition 2.3, gives that $(G_{207}, \langle u_1, u_2, u_3 \rangle)$ is 4-proper. This will be used in the following construction.

5. Improving a construction of Goldberg. The 4-snark G_4 from Fig. 8.2, can be constructed using Proposition 10.4, because it is in fact a superjoin $\mathcal{J}(K_4)$ where for one vertex w of K_4, $\mathcal{J}(w)$ is equal to $K_{3,1}$ with the outer vertices of valency one and for every $v' \in V(K_4 - w)$, $\mathcal{J}(v')$ is equal to $(P - v, \langle v_1, v_2, v_3\rangle)$ from Fig. 8.1. Splitting the vertices of G_4 of valency 4 to vertices of valency 2 and suppressing them we get the snark G'_4 from Fig. 13.12. Note that this construction was in fact used in Goldberg [Gol].

Using G_{207} instead of $K_{3,1}$ we obtain the cyclically 6-edge-connected snark G''_4 of order 222 depicted in Fig. 13.13. But instead of K_4 we can use any 3-edge-connected cubic graph H of order n and applying this method (i. e., taking superjoin $\mathcal{J}(H)$ so that $\mathcal{J}(w)$ is either $K_{3,1}$ or G_{207} for a fixed vertex w of H and $\mathcal{J}(v')$ is $(P - v, \langle v_1, v_2, v_3\rangle)$ otherwise). We get cyclically 5- and 6-edge connected snarks H' and H'' of orders $6n+4$ and $6n+204$, respectively. In fact both H' and H'' in certain sense "copy" the structure of the graph H. Thus, after all, even the family of cyclically 6-edge-connected snarks proves to be surprisingly rich.

6. Avoiding the flower snarks. With respect to the constructions used until now, it seems that without flower snarks we would not be able to construct cyclically 6-edge-connected snarks. Now we show that this is not true. Replace in the Petersen graph P two edges by copies of P. We obtain the 4-snark G'_{26} from Fig. 13.14. Replacing the vertices of valency 5 by vertices of valencies 3 and 2 and suppressing the latter ones we can get the snark G_{26} from Fig. 13.14. Using $(G_{26}, \langle v_1, v_2\rangle)$ instead of $(I_5, \langle v_1, v_2\rangle)$ in the construction from Section 13.1, we get a new cyclically 6-edge-connected snark of order 154. It was constructed only from copies of the Petersen graph.

7. Snarks with large reluctance. Applying 1-join to the networks $(G_{102}, \langle u_1, u_2\rangle)$ and $(I_5, \langle v_1, v_2\rangle)$ we get the 4-snark $(G_{121}, \langle w_1, w_2\rangle)$ from Fig. 13.15.

Take a circuit C of length r and its superjoin $\mathcal{J}(C)$ so that for each $v \in V(C)$, $\mathcal{J}(v)$ is $(G_{121}, \langle w_1, w_2\rangle)$. By Propositions 7.3 and 7.4, we get a graph with 4-reluctance r. Then, by Propositions 7.5 and 7.6, splitting the vertices of valency 6 to three vertices of valency 2 and suppressing them we get a cyclically 6-edge-connected snark of order $118r$ and 4-reluctance at least r indicated in Fig. 13.16.

Theorem 13.4. *For every $r > 0$, there exists a cyclically 6-edge-connected snark of order $118r$ and 4-reluctance at least r.*

8. Snarks with large oddness. By *oddness* of a cubic graph G we mean the minimal number n so that the vertices of G can be covered by a family of vertex disjoint circuits containing n odd circuits (an isolated vertex is also an odd circuit). This parameter was introduced in [K2] and [HK]. Clearly, the oddness of a cubic graph is always an even number. Furthermore, a cubic graph has oddness zero if and only if it has a 3-edge-coloring (or, equivalently, a nowhere-zero 4-flow). Note that the 5-flow and cycle double cover conjectures are verified for snarks with oddness two (see [J1, J4, K2, HK]).

Lemma 13.5. *The oddness of a cubic graph is greater than or equal to its 4-reluctance.*

Proof. Take a family of vertex disjoint circuits covering a cubic graph G and $W \subseteq V(G)$ so that W contains precisely one vertex from every odd circuit of this family. Then we can easily check that W 4-relieves G. This proves the statement. $\square$

Theorem 13.6. *For every $r > 0$, there exists a cyclically 6-edge-connected snark of order $118(2r-1)$ and oddness at least $2r$.*

Proof. Follows from Theorem 13.4, Lemma 13.5 and the fact that oddness of a cubic graph is always even. $\square$

Finally let us note that snarks also have many other interesting properties. For instance no snark can be hamiltonian, and, as follows from [K2, HK], snarks with oddness at least four cannot have a Hamilton path. Therefore our constructions can be used also in studying nonhamiltonian graphs or graphs without Hamilton paths and related topics.

14. 5-snarks

As pointed out before, the *5-flow conjecture* of Tutte suggests that there are no bridgeless 5-snarks. If there exists a bridgeless 5-snark, then applying superposition recursively so that we replace one edge of a bridgeless 5-snark by another copy of a 5-snark, we get an infinite family of bridgeless 5-snarks.

By Celmins [C] (see also Jaeger [J4]) the smallest counterexample to the 5-flow conjecture must be a cyclically 5-edge connected snark with girth at least 7. Using the techniques of Section 13.1 we can show the following.

Theorem 14.1. *If the 5-flow conjecture is not true, then there exists an infinite family of cyclically 5-edge-connected cubic 5-snarks with girth at least 7.*

We skip a formal proof, because we prove a stronger statement.

Theorem 14.2. *If the 5-flow conjecture is not true, then there exist integers $r, k_0 > 0$ and $s \geq 0$ so that for every integer $k \geq k_0$ there exists a cyclically 5-edge-connected cubic graph G_k of order $2k$, with girth at least 7, and $\rho_5(G_k) \geq \lfloor k/r \rfloor - s$.*

Proof. If the 5-flow conjecture is false, then, by Celmins [C], there exists a cyclically 5-edge-connected cubic 5-snark G with girth at least 7. Let u_1, u_2 be vertices of G so that their distance is at least 3. Consider a circuit C in G and an edge $e = (v_1, v_2)$ of C. Let $C - e$ consists of the edges $e_1, \ldots, e_m$. For $i = 1, \ldots, m$, apply e_i-superpositions $\mathcal{S}^{(i)}$ on $G - e$ so that $\mathcal{S}^{(i)}_{e_i} = (G, \langle u_1, u_2 \rangle)$. Applying reductions similarly as in Section 13.4 we get a 5-improper network $(G', \langle w_1, w_2 \rangle)$. Symbolic representations of $(G, \langle u_1, u_2 \rangle)$, $(G - e, \langle v_1, v_2 \rangle)$ and $(G', \langle w_1, w_2 \rangle)$ are indicated in Fig. 14.1. The 1-join of $(G', \langle w_1, w_2 \rangle)$ and $(G, \langle u_1, u_2 \rangle)$ is a 5-snark $(G'', \langle w_2, u_2 \rangle)$. Assume that G'' has order $2r + 3$. Then $r > 0$, because the smallest cubic graph with girth 7 has order 24 (see [BM], [K3]) and is cyclically 7-edge-connected. Moreover, using the techniques from Section 13.1 we can check that there exists $n_0 > 0$ so that for every $n \geq n_0$ there exists a cyclically 5-edge-connected graph H_n of order $2n$ and with girth at least 7. Take $k \geq k_0 = r(n_0 + 1)$ and $p = \lfloor k/r \rfloor - n_0$, $q = k - pr$. Then $p > 0$ and $q \geq n_0$. Consider a circuit C_{p+1} of order $p + 1$ with a vertex v_0. Take a superjoin $\mathcal{J}(C_{p+1})$ so that $\mathcal{J}(v)$ is $(G'', \langle w_2, u_2 \rangle)$ if $v \neq v_0$ and $\mathcal{J}(v_0)$ is $(H_{q+1}, \langle v_1', v_2' \rangle)$ (where v_1', v_2' are vertices of H_{q+1} so that their distance is at least 3). Splitting the vertices of $\mathcal{J}(C_{p+1})$ of valency 6 to vertices of valency 2 and suppressing them we get a cyclically 5-edge-connected graph G_k of order $2k$ and with girth at least 7 (similarly as in Section 13.7). By Propositions 7.3–7.6, $\rho_5(G_k) \geq p$. Setting $s = n_0$ we get $p \geq \lfloor k/r \rfloor - s$, concluding the proof. $\square$

Let $\overline{\rho}_5(k)$ denote the maximal 5-reluctance of a cyclically 5-edge-connected cubic graph of order $2k$ and with girth at least 7. Then, similarly as in the end of Section 12, we can show that, by Theorem 14.2, the 5-flow conjecture is equivalent to the following problem (see [K8] for more details).

Conjecture 14.3. $\liminf_{k\to\infty} \overline{\rho}_5(k)/k = 0$.

In an accompanied paper [K6] we have extended the results from Section 9 and proved the following analogue of Theorem 12.7.

Theorem 14.4. *If the 5-flow conjecture is not true, then it is an* NP*-complete problem to decide whether a (cubic) graph admits a nowhere-zero 5-flow.*

15. Concluding remarks

Remark 15.1. If the four color theorem had been false, then there would exist a planar snark and, by Birkhoff [Bi], also a simple 5-cut cubic planar snark (see Section 13.2). Furthermore, using the same ideas as in Theorem 14.2 we can show that then there exist integers $r, k_0 > 0$ and $s \geq 0$ so that for every integer $k \geq k_0$ we have a simple 5-cut planar cubic graph G_k of order $2k$ satisfying $\rho_4(G_k) \geq \lfloor k/r \rfloor - s$. Thus, denoting by $\overline{\rho}_4(k)$ the maximal 4-reluctance of a simple 5-cut cubic planar graph of order $2k$ we get that the four color theorem is equivalent to the statement that $\liminf_{k\to\infty} \overline{\rho}_4(k)/k = 0$.

A network (G, U), $U = \langle u_1, \ldots, u_n \rangle$ is called *k-nontrivial* if there exists $\mathbf{z} = \langle z_1, \ldots, z_n \rangle$ so that $(G, U)^{\mathbf{z}}$ is a bridgeless k-snark. In this case $\mathbf{z} \notin \chi_k(G, U)$. But to find a k-nontrivial network and a bridgeless graph which is a k-snark are two equivalent problems. Thus the study of characteristic sets of networks is closely related to the problems about nowhere-zero flows in graphs.

Propositions 2.4, 2.8 and the theorem of Seymour [S1] imply that

$$\chi_2(G, U) \subseteq \chi_3(G, U) \subseteq \chi_4(G, U) \subseteq \chi_5(G, U) \subseteq \chi_6(G, U) = \chi_7(G, U) = \ldots$$

holds. Furthermore, the 5-flow conjecture is equivalent to the statement that also $\chi_5(G, U) = \chi_6(G, U)$ holds for every network (G, U). On the other hand the sequence of k-characteristic sets can increase for parameters $k = 2, 3, 4$. For instance we claim that if P is the Petersen graph and U is an ordered set containing all vertices of P, then

$$\chi_2(P, U) \subsetneq \chi_3(P, U) \subsetneq \chi_4(P, U) \subsetneq \chi_5(P, U) = \chi_6(P, U) = \ldots.$$

Really, $\chi_2(P, U)$ contains only the vector with all coordinates equal 1. On the other hand $\chi_3(P, U)$ contains also vectors with some zero coordinates, but no vector with just two nonzero coordinates (by Corollary 12.3), and such vectors are in $\chi_4(P, U)$. We can check that $\chi_5(P, U)$ has also the zero vector (which is not in $\chi_4(P, U)$) and that $\chi_5(P, U) = \chi_6(P, U)$. This proves the claim.

Methods discussed in this paper have two common features. Primarily, they are recursive, that means they create new k-snarks from known k-snarks. Furthermore they are based on properties of k-characteristic sets of networks, that means on 0–1 laws. Let us note that the majority of the constructions of snarks surveyed in beginning of Section 13 are also based on 0–1 laws.

If A is an Abelian group and (G, U) is a network, $U = \langle u_1, \ldots, u_n \rangle$, then denote by $\partial_A(G, U)$ the set of the vectors $\langle \partial\varphi(u_1), \ldots, \partial\varphi(u_n) \rangle$ where φ runs through all nowhere-zero A-flows in (G, U). It would be interesting to develop methods based on the properties of $\partial_A(G, U)$. In fact, the construction of flower snarks is based on similar ideas.

As pointed out in the first section, the problems to decide whether a graph admits nowhere-zero 3- or 4-flows are NP-complete. With respect to Conjectures 12.9 and 14.3, it would be interesting to study the complexity of the problems to approximate (or evaluate) k-reluctance for $k = 3$ and 4. For $k = 5$ we probably can get hypothetical results similar to those of [K6]. For $k = 2$ the problem is trivial, because $\rho_2(G, U)$ is equal to the number of the inner vertices of (G, U) with odd valency. Another problem is to study the relationship between oddness of graphs in general (see [K2], [HK]) and 4-reluctance (or some other parameters introduced in [K8]).

Suppose that $\mathcal{R}_k$ denotes the class of cubic graphs with oddness at least $2k$. By Robinson and Wormald [RW], almost every graph from $\mathcal{R}_0$ belongs to $\mathcal{R}_0 \setminus \mathcal{R}_1$. What can be said about $\mathcal{R}_k$? Does almost every graph from $\mathcal{R}_k$ belong to $\mathcal{R}_k \setminus \mathcal{R}_{k+1}$? We conjecture a positive answer to this question. Similar questions can be asked for oddness of graphs in general or for reluctance of graphs.

In Section 2 we introduced chains and flows as mappings with the domain $D(G)$. A similar approach we have used in [K7] for a flow model which generalizes several models from combinatorial optimization and also the classical flow model of Ford and Fulkerson.

Let Π be a property of networks and Abelian groups satisfying the following: if a network (G, U) (or a family $\mathfrak{F}$ of networks) and an Abelian group A have the property Π, then (G, U) (or the family $\mathfrak{F}$) and A' have the property Π for every Abelian group A' of order $\leq |A|$. In this case we say that Π is *hereditary*. For instance the classical equivalence results of Tutte in fact say that the property "a graph G admits a nowhere-zero A-flow" is hereditary. We have proved that also the properties "a network (G, U) admits a nowhere-zero A-flow", "$\mathbf{z} \notin \chi_A(G, U)$" and "$(G', U')$ is an A-strong edge superposition of (G, U)" are hereditary. These facts are certainly aesthetical assets of our methods and have great practical influence upon the notation, because we can use the terms "k-flow", "k-snark", "$\chi_k(G, U)$", etc. Unfortunately, the property "(G', U') is an A-strong superposition of (G, U)" is not hereditary, as follows from Examples 4.10 and 4.11.

The aim of this paper was to build a general theory on constructing k-snarks. Thus we avoided complicated constructions here. More applications of our techniques can be found in [K3, K5, K6, K8] and other papers which are in the stage of preparation.

Further details about nowhere-zero flows in graphs and related problems can be found in Jaeger [J3, J4, JLPT], Jackson [Jac], Raspaud [Ra], Seymour [S2], Tutte [T1, T2, T3, T4], Younger [Y], Diestel [Di], Fan [F2], Jensen and Toft [JT] and Zhang [Z1, Z2].

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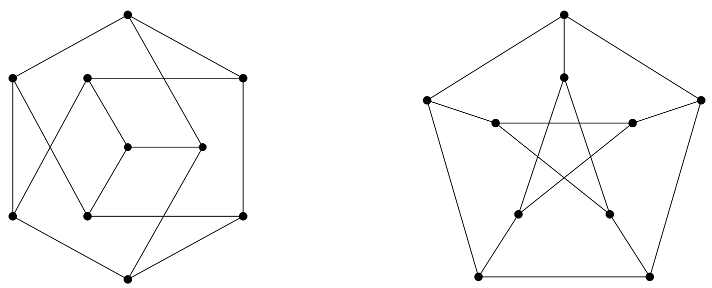


The Petersen graph

Fig. 1.1

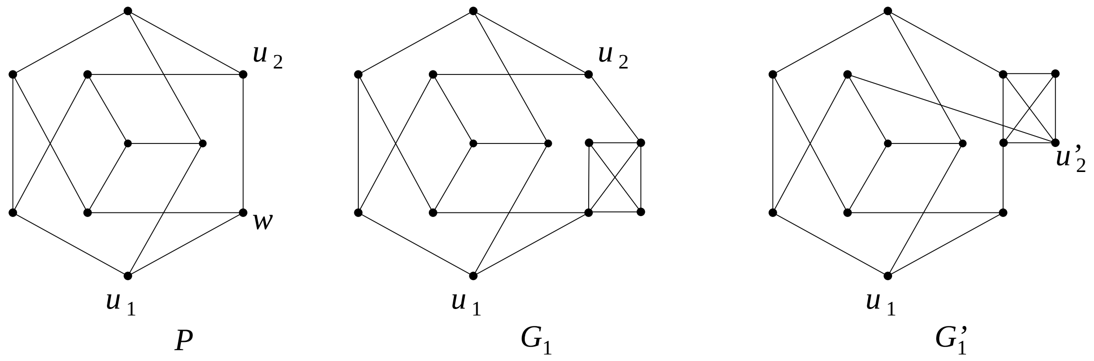
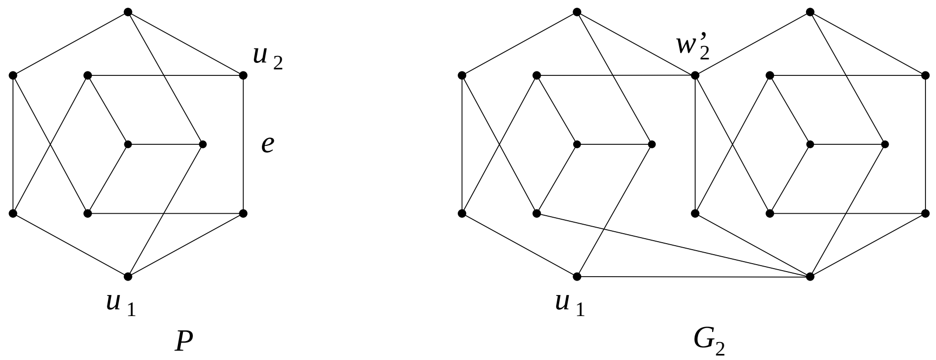


Fig. 4.1



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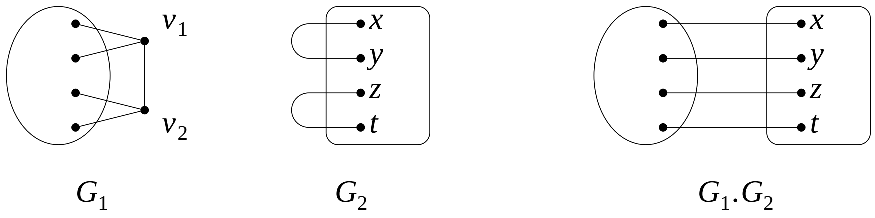

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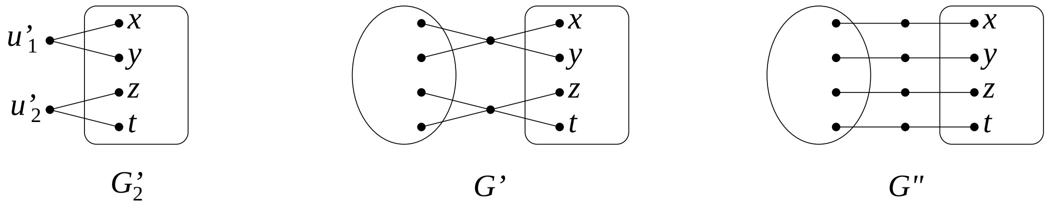
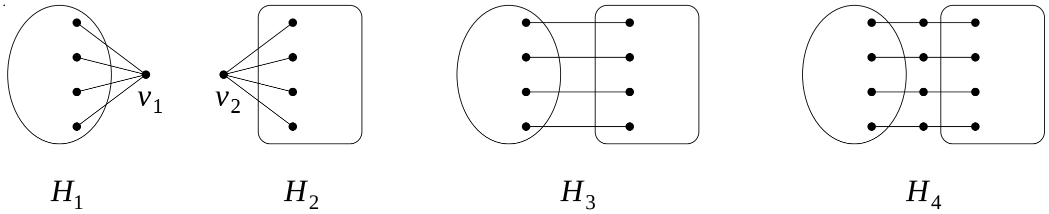


Fig. 4.4



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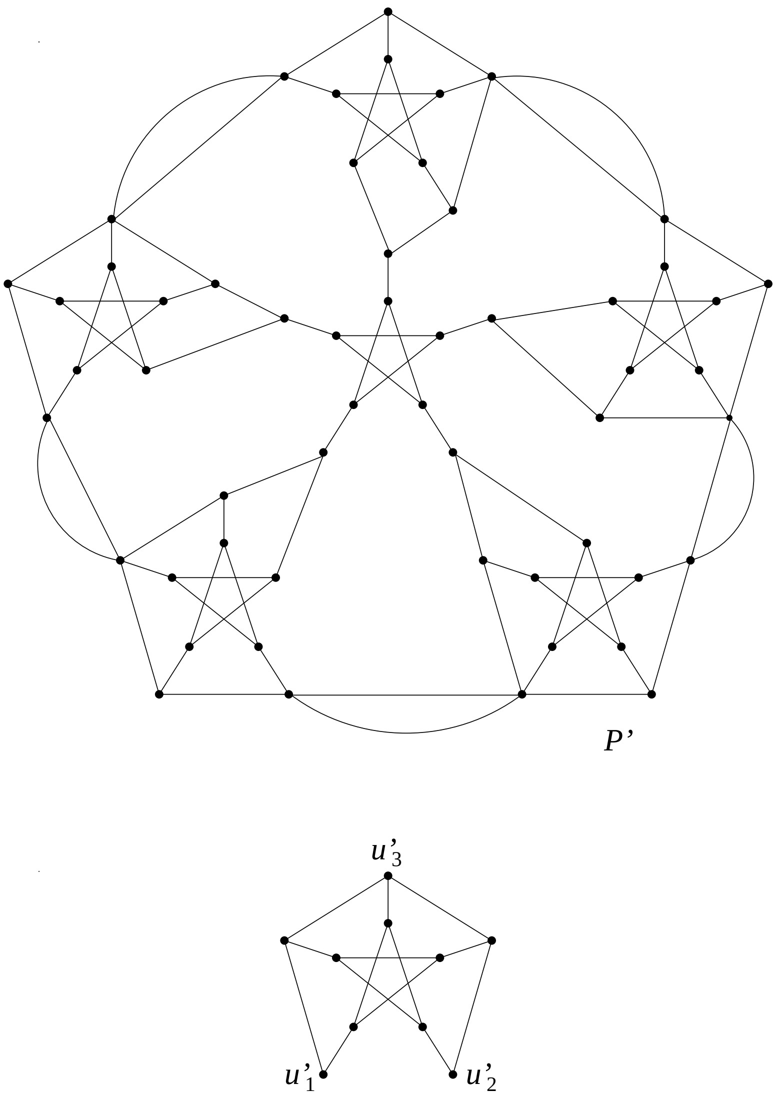

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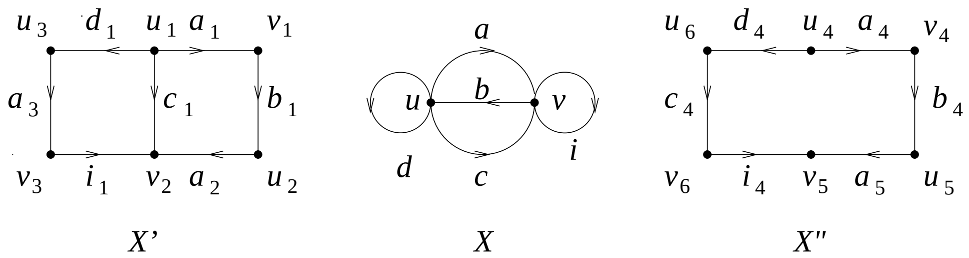
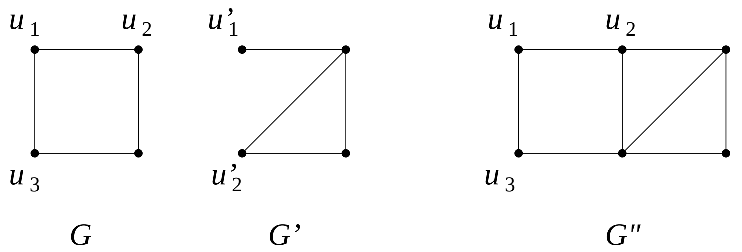


Fig. 4.7



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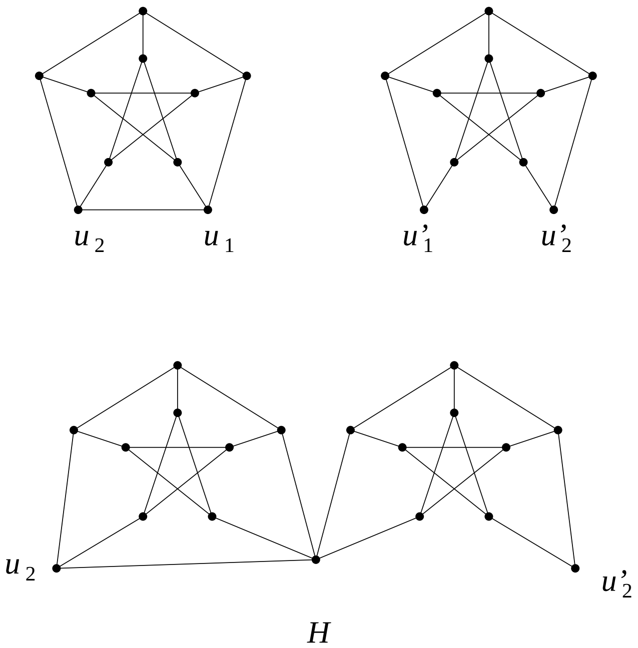

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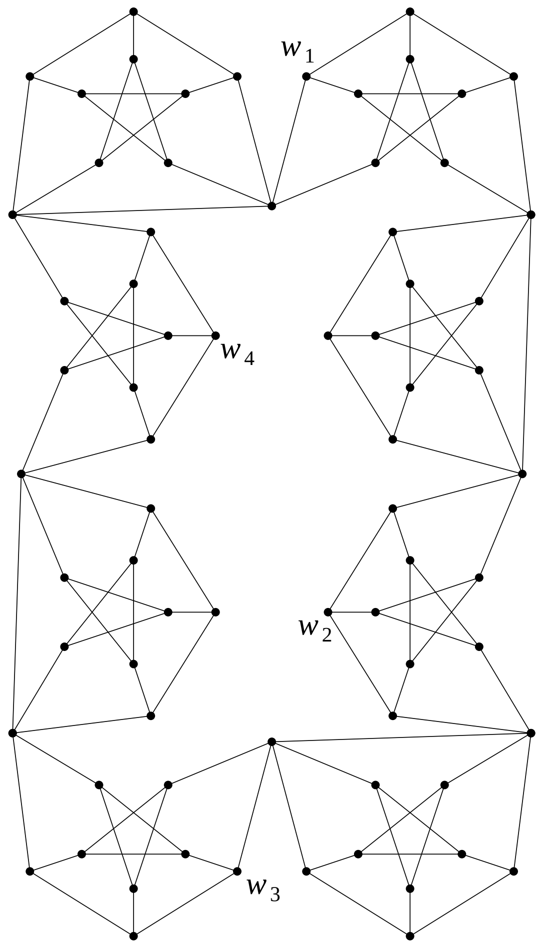

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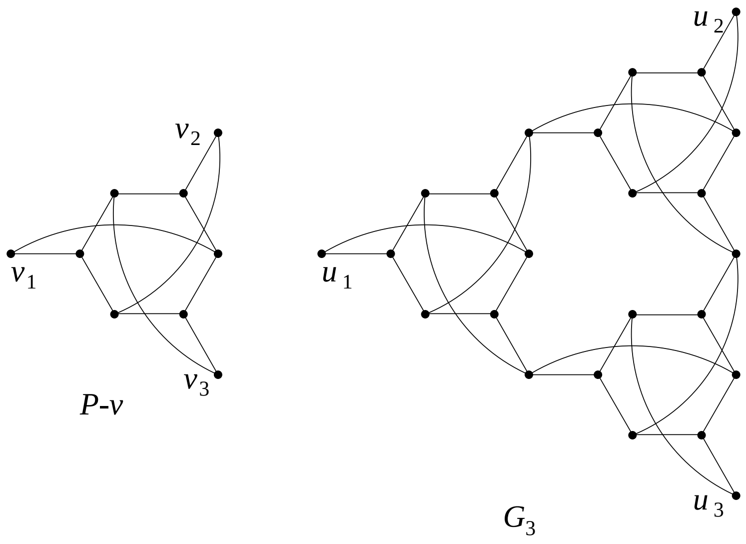

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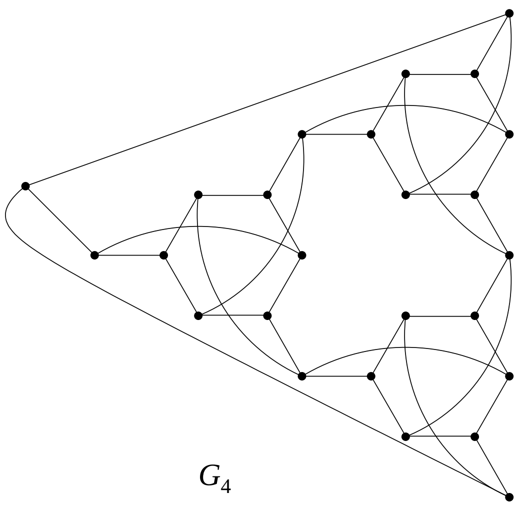

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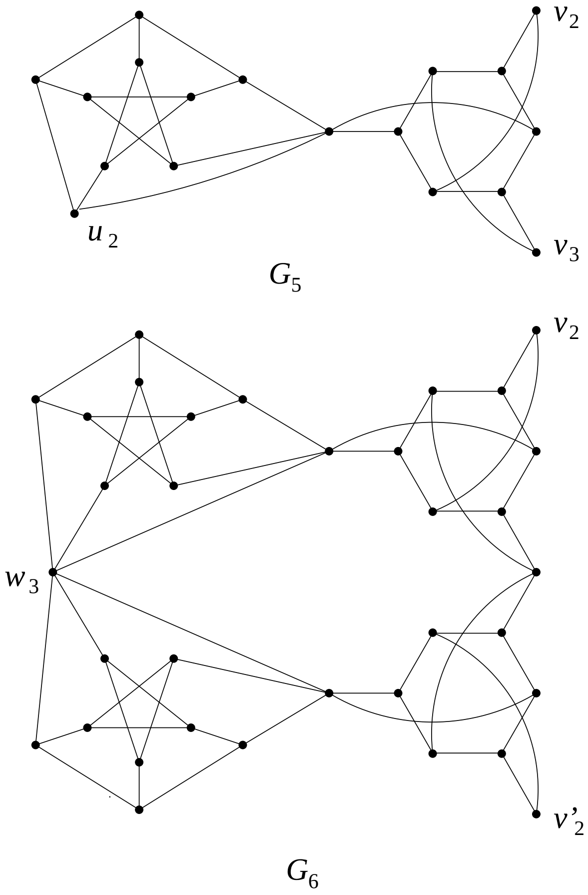

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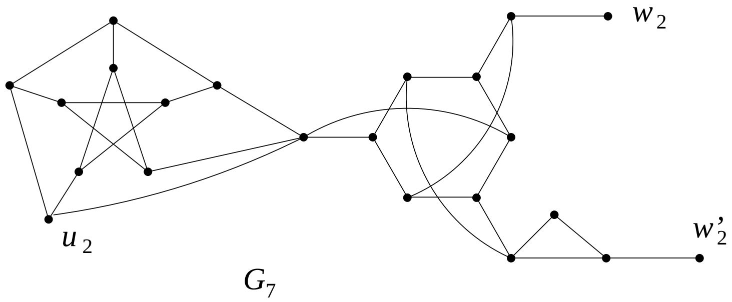

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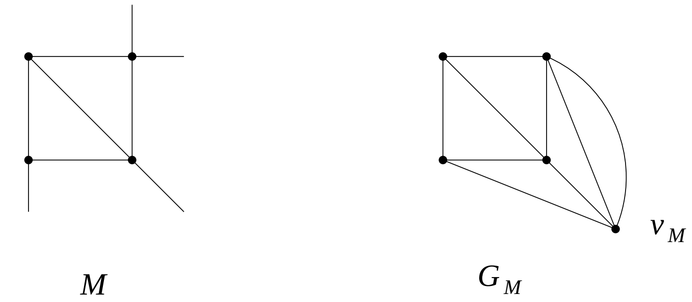
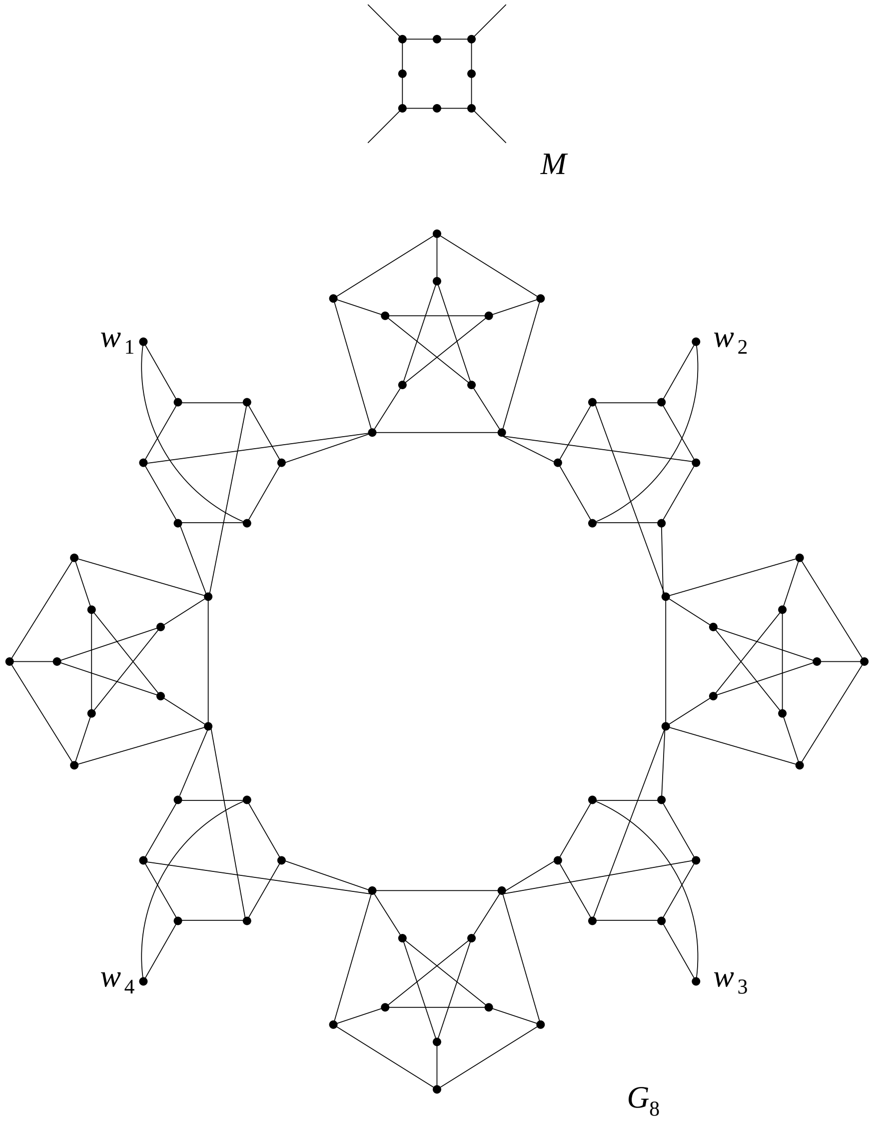


Fig. 10.1



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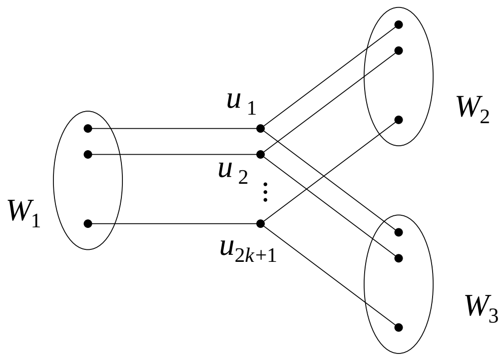

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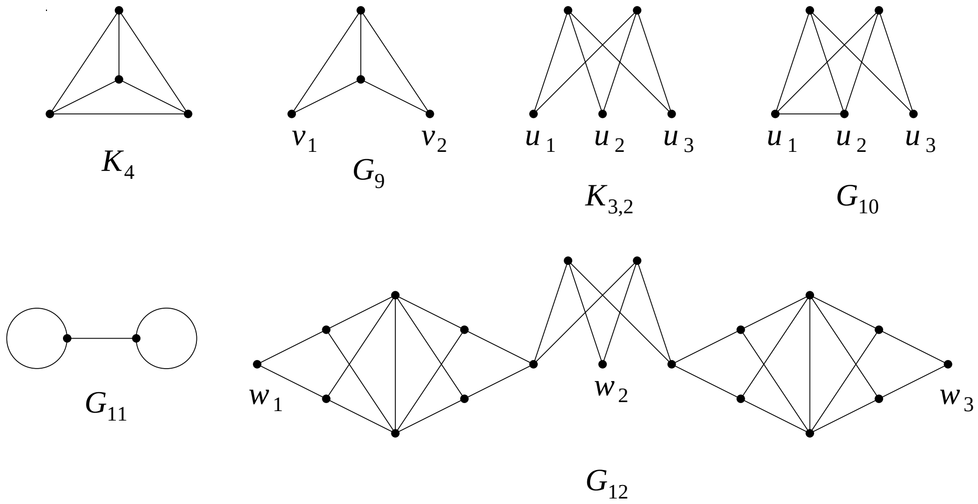

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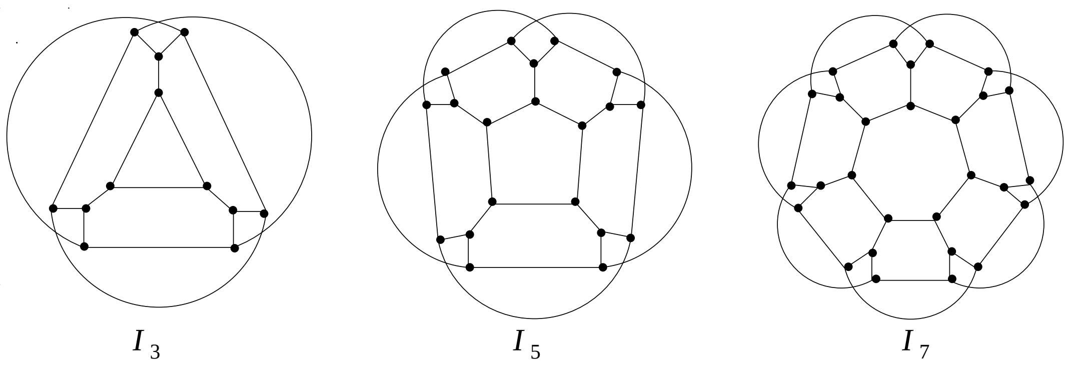
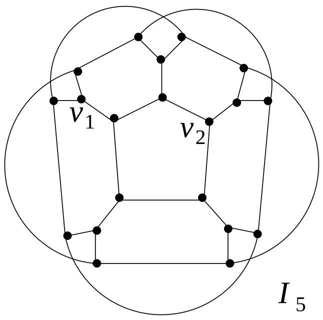
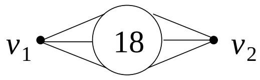


Fig. 13.1



I_5



a symbolic representation of I_5

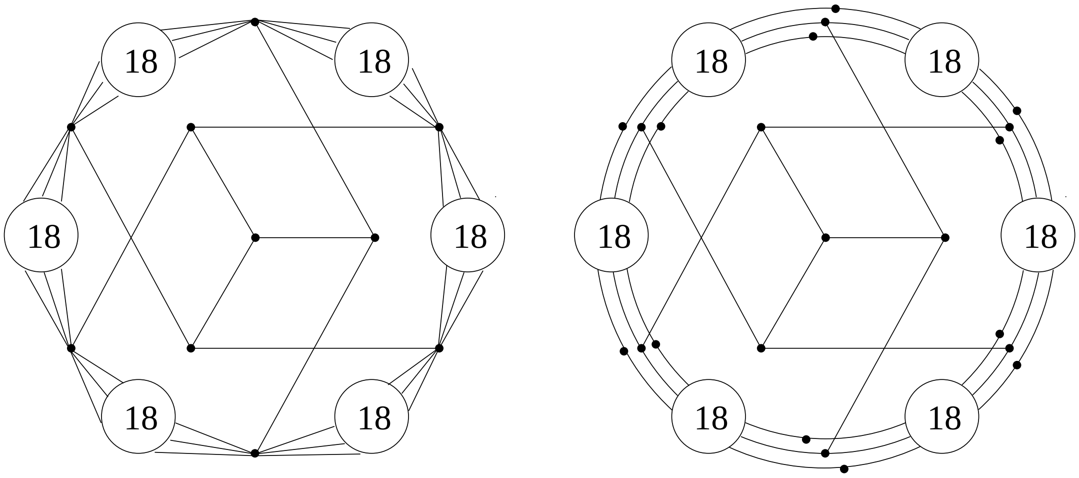
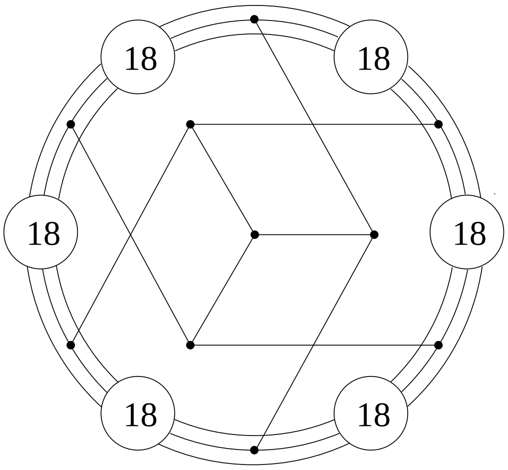
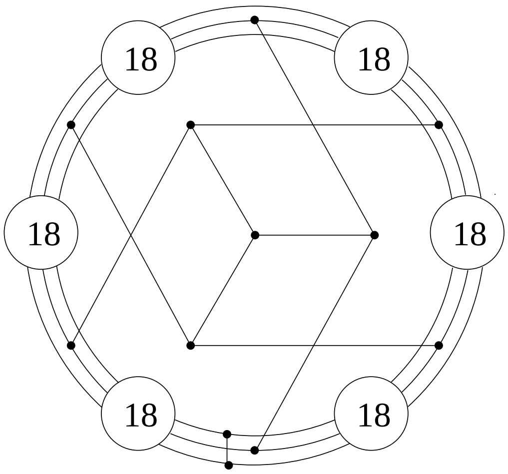


Fig. 13.2



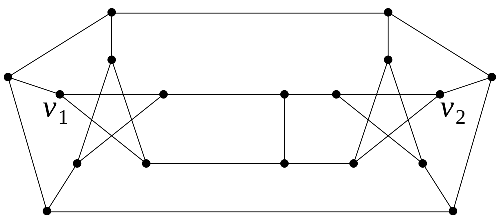
G_{118}

Fig. 13.3

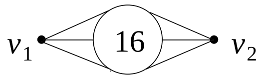


G_{120}

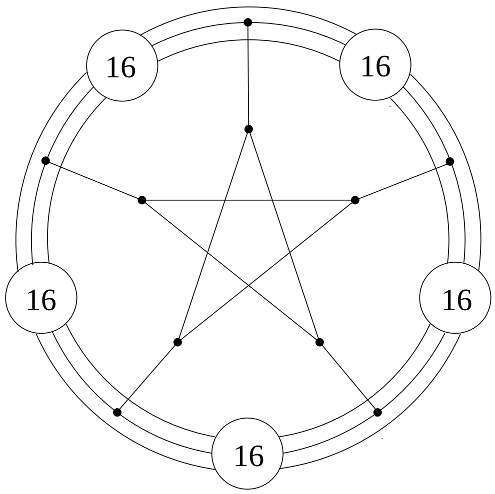
Fig. 13.4



B_{18}

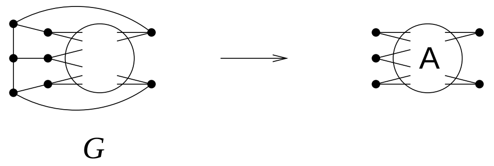


a symbolic representation of B_{18}

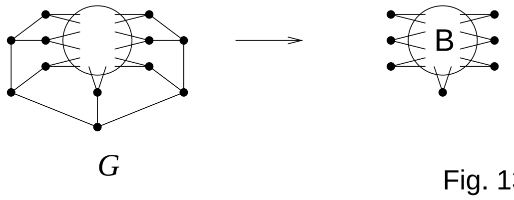


G_{90}

Fig. 13.5

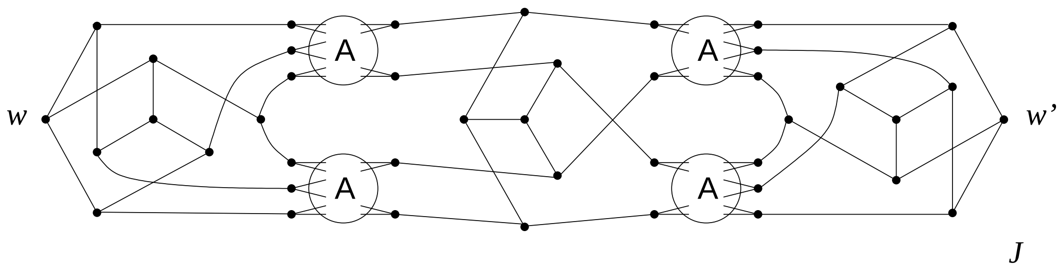


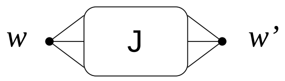
a symbolic representation of
A



a symbolic representation of
B

Fig. 13.6





a symbolic representation of
J

Fig. 13.7

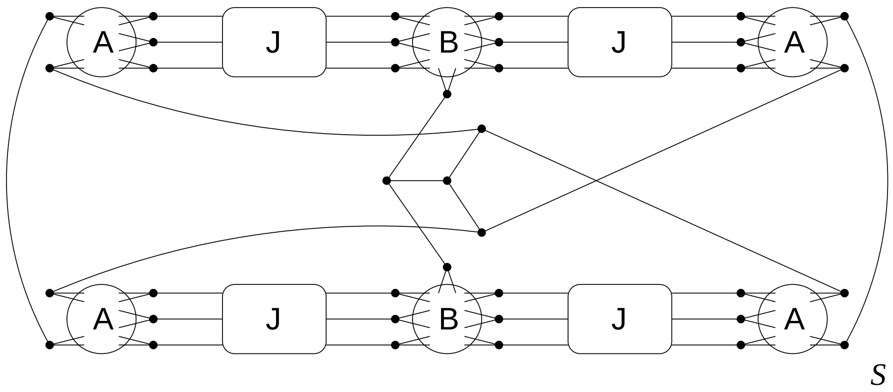
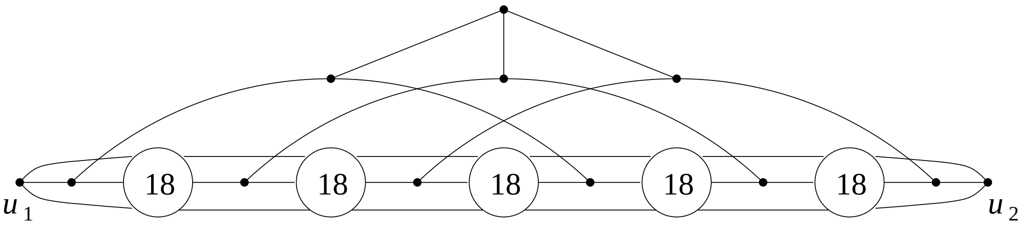
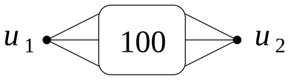


Fig. 13.8



G_{102}



a symbolic representation of G_{102}

Fig. 13.9

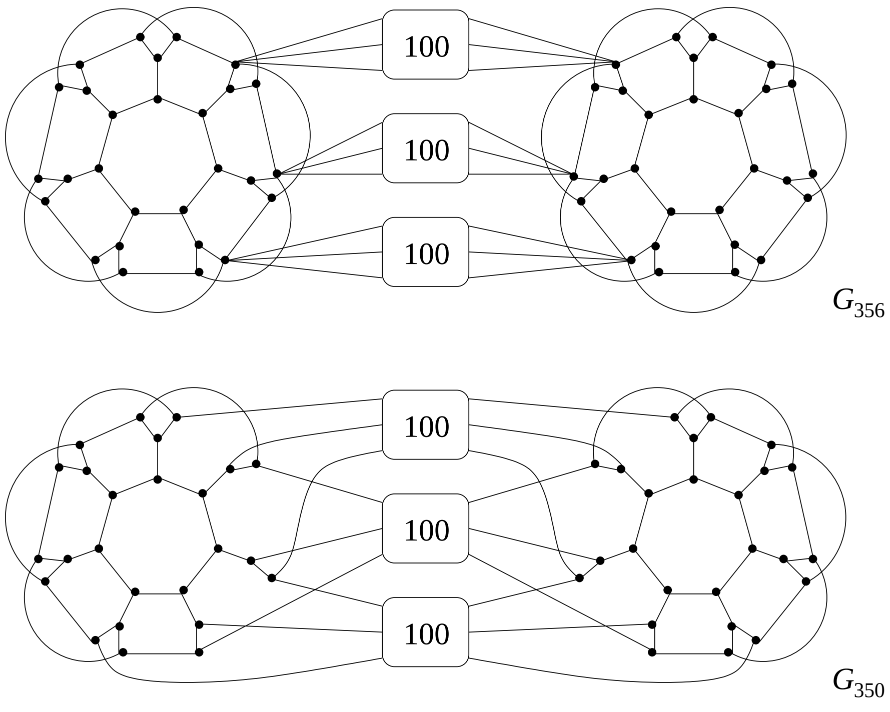


Fig. 13.10

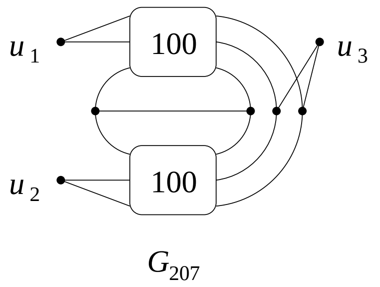
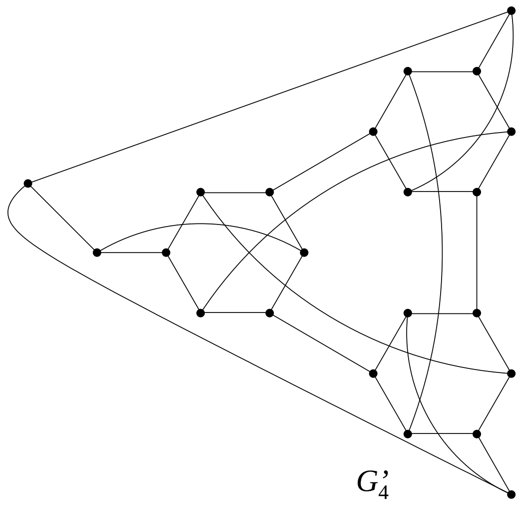


Fig. 13.11



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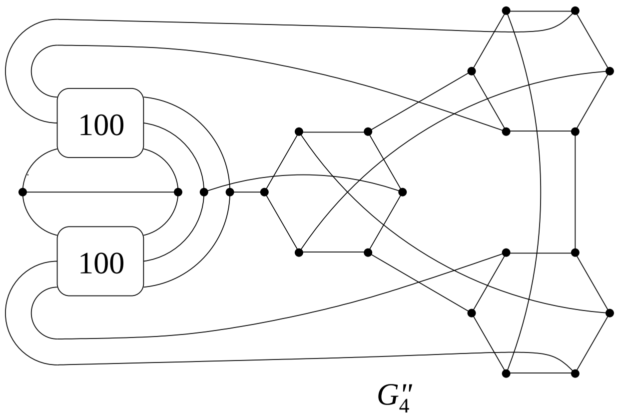

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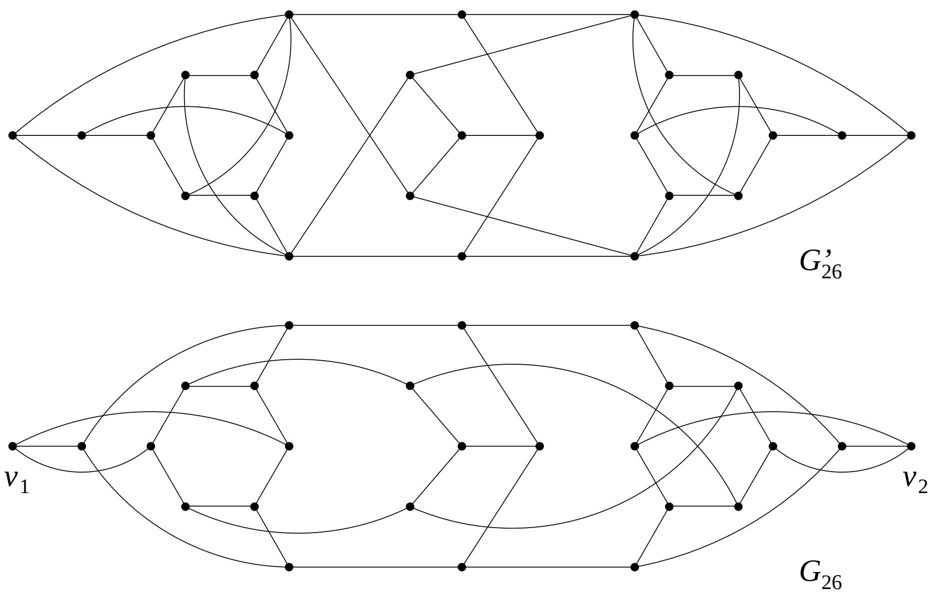

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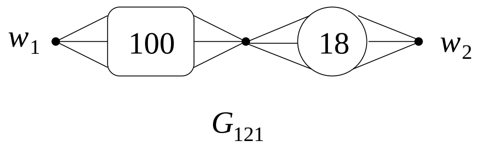

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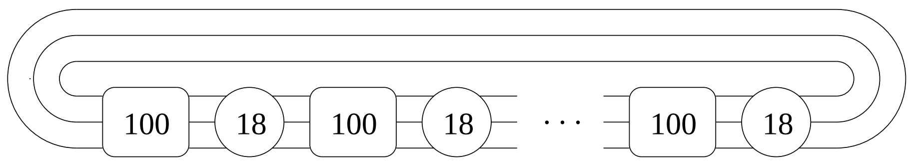

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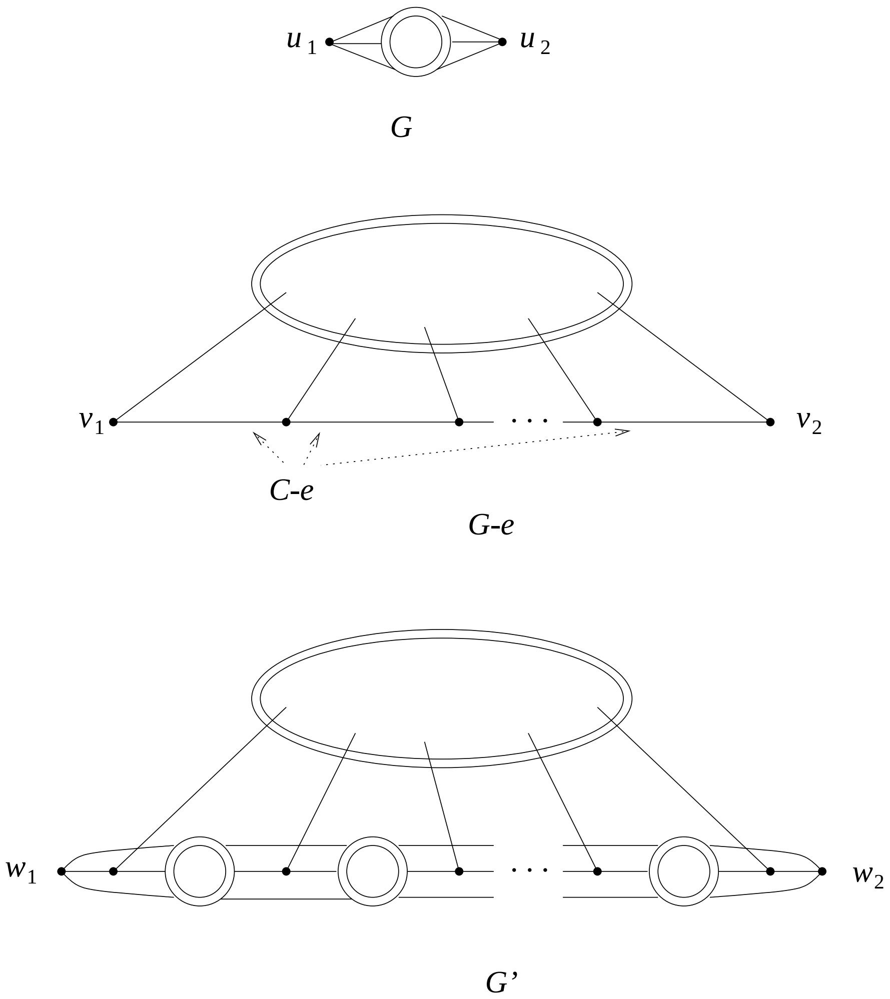

Fig. 14.1

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