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POLYNOMIALS ASSOCIATED WITH NOWHERE-ZERO FLOWS

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Abstract. In this paper we study relations between nowhere-zero $\mathbb{Z}_k$- and integer-valued flows in graphs and the functions $F_G(k)$ and $I_G(k)$ evaluating the numbers of nowhere-zero $\mathbb{Z}_k$- and k-flows in a graph G, respectively. It is known that $F_G(k)$ is a polynomial for $k > 0$. We show that $I_G(k)$ is also a polynomial and that $2^{m(G)} F_G(k) \geq I_G(k) \geq (m(G)+1) F_G(k)$, where $m(G)$ denotes the rank of the cocycle matroid of G. Finally we prove that $F_G(k+1) \geq F_G(k) \cdot k/(k-1)$ and $I_G(k+1) \geq I_G(k) \cdot k/(k-1)$ for every $k > 1$.

1. Introduction

A graph admits a nowhere-zero k-flow if its edges can be oriented and assigned values $\pm 1, \dots, \pm(k-1)$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of the graph. Using nonzero elements of an additive Abelian group A instead of integers, we get a nowhere-zero A-flow in G. By Tutte [T2], a graph admits a nowhere-zero k-flow if and only if it admits a nowhere-zero $\mathbb{Z}_k$-flow.

Nowhere-zero flows in graphs present a dual concept to the chromatic problems from graph theory, because, by Tutte [T2], a planar graph is k colorable if and only if its dual admits a nowhere-zero $\mathbb{Z}_k$-flow. Classical chromatic polynomial has also a dual concept, and it is the flow polynomial introduced by Tutte [T2]. The flow polynomial of a graph G, denoted by $F_G(k)$, determines the number of nowhere-zero A-flows in a graph G for any Abelian group A of order k. Thus a graph admits a nowhere-zero $\mathbb{Z}_k$-flow if and only if it admits a nowhere-zero A-flow for any Abelian group A of order k.

We show that the number of nowhere-zero k-flows in a graph G can be also evaluated by a polynomial $I_G(k)$ for $k > 0$. The proof does not use the contraction-deletion principle, but ideas from convex geometry. Furthermore we prove that $2^{m(G)} F_G(k) \geq I_G(k) \geq (m(G)+1) F_G(k)$, where $m(G)$ denotes the rank of the cocycle matroid of G (the number of the edges obtained after deleting from G the edges of a spanning forest).

In two introductory sections we study relations between nowhere-zero $\mathbb{Z}_k$- and k-flows in graphs. If we have a nowhere-zero k-flow in a graph G, then taking the values mod$k$ we get a nowhere-zero $\mathbb{Z}_k$-flow. If two nowhere-zero k-flows give the same $\mathbb{Z}_k$-flow in this process (i. e., their values are congruent mod$k$), we say that

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they are $\theta_{G,k}$-equivalent. There is a one-to-one correspondence between the $\theta_{G,k}$-classes and the nowhere-zero $\mathbb{Z}_k$-flows in G. Thus $F_G(k)$ evaluates also the number of the $\theta_{G,k}$-classes of G.

For every nowhere-zero k-flow φ in G, there exists just one orientation of G so that the flow has only positive values by this orientation. This is called the positive orientation of φ. The number of the directed cycles in this orientation is equal to the cardinality of the $\theta_{G,k}$-class containing φ. Furthermore, $\theta_{G,k}$ induces an equivalence relation Θ_G on the set of all positive orientations in G. Similarly as nowhere-zero k-flows are associated with positive orientations, nowhere-zero $\mathbb{Z}_k$-flows are associated with Θ_G-classes of such orientations.

For an orientation X of G, let $I_X(k)$ denote the number of nowhere-zero k-flows in G with the positive orientation X. This is also a polynomial and both $F_G(k)$ and $I_G(k)$ can be expressed as sums of certain $I_X(k)$. Polynomials $F_G(k)$, $I_G(k)$, and $I_X(k)$ satisfy the recursive rule $f(k+1) \geq f(k) \cdot k/(k-1)$ for every integral $k > 1$. This estimate is the best possible in general.

For more information about flow polynomial $F_G(k)$ and related topics see Brylawski and Oxley [BO], Biggs [Bi], Rota [R], Tutte [T5, T6] and Welsh [W]. More details about nowhere-zero flow problems can be found in Jaeger [J2], Tutte [T6], Seymour [S2], Zhang [Z] and [K2].

2. Nowhere-zero flows in graphs

In this paper we deal with finite unoriented graphs with multiple edges and loops. A graph $G = (V, E)$ consists of a set of vertices $V(G) = V$ and a set of edges $E(G) = E$. Every edge e of the graph G determines two opposite arcs arising from it after endowing e with two distinct orientations. All arcs obtained in this way are called *arcs* of G and the set of them is called the *arc set* of G and denoted by $D(G)$. Clearly, $|D(G)| = 2|E(G)|$. If x is an arc of G, then denote by x^{-1} the second arc arising from the same edge. Clearly $(x^{-1})^{-1} = x$ and $x \neq x^{-1}$ for every arc x of G. If $X \subseteq D(G)$, then denote by $X^{-1} = \{x \in D(G); x^{-1} \in X\}$. By an *orientation* of G we mean every $X \subseteq D(G)$ so that $X \cup X^{-1} = D(G)$ and $X \cap X^{-1} = \emptyset$.

If $W \subseteq V(G)$, then the set of the arcs of $D(G)$ directed from W to $V(G) \setminus W$ is denoted by $\omega_G^+(W)$. Furthermore, define $\omega_G^-(W) = (\omega_G^+(W))^{-1}$. We write $\omega_G^+(v)$, $\omega_G^-(v)$ instead of $\omega_G^+(\{v\})$, $\omega_G^-(\{v\})$, respectively.

If e is an edge of a graph G, then $G-e$ and G/e denote the graphs obtained from G after deleting and contracting the edge e, respectively. Note that $G/e = G-e$ if e is a loop.

In this paper, every Abelian group is additive. If G is a graph and A is an Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ so that

$$\partial\varphi(v) \;=\; \sum_{x \in \omega_G^+(v)} \varphi(x) \quad (v \in V(G))$$

is called the *boundary* of φ. *Support* of φ, denoted by $\sigma(\varphi)$, is the set of the edges associated with the arcs of G having nonzero values in φ. An A-chain φ is called *nowhere-zero* if $\sigma(\varphi) = E(G)$.

Lemma 1. *Let φ be an A-chain in a graph G and $W \subseteq V(G)$. Then*

$$\sum_{x \in \omega_G^+(W)} \varphi(x) \;=\; \sum_{v \in W} \partial\varphi(v).$$

Proof. Since $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$, we have

$$\sum_{x \in \omega_G^+(W)} \varphi(x) \;=\; \sum_{v \in W} \sum_{x \in \omega_G^+(v)} \varphi(x) \;=\; \sum_{v \in W} \partial\varphi(v). \quad \square$$

A (nowhere-zero) A-chain φ in G is called a *(nowhere-zero) A-flow* in G if $\partial\varphi(v) = 0$ for every vertex v of G. If k is a positive integer, then by a *(nowhere-zero) k-chain* φ in a graph G we mean a (nowhere-zero) $\mathbb{Z}$-chain in G such that $|\varphi(x)| < k$ for every $x \in D(G)$. Furthermore, if φ is a $\mathbb{Z}$-flow, then we speak about a *(nowhere-zero) k-flow* in G. Note that a graph G has a nowhere-zero 1-flow if and only if $E(G) = \emptyset$. The following lemmas are rather obvious (see cf. [K2]).

Lemma 2. *If a graph G has a bridge, then it does not admit a nowhere-zero A-flow for every Abelian group A.*

Lemma 3. *Let $X \subseteq D(G)$ be an orientation of G. Then a mapping $\psi : X \to A$ can be extended into an A-flow φ in G if and only if the following equality is satisfied for every vertex v of G*

$$\sum_{x \in \omega_G^+(v) \cap X} \psi(x) = \sum_{x \in \omega_G^-(v) \cap X} \psi(x).$$

The concept of nowhere-zero flows in graphs used here is identical with that one introduced in [K1, K2]. By Lemma 3, our approach coincides with the usual definition of nowhere-zero flows as presented, e. g., in Jaeger [J2] (also used in the introduction of this paper). The only difference is that instead of a fixed (but arbitrary) orientation of a graph G we use the set $D(G)$ as a domain for a flow.

For a graph G, let $m(G) = |E(G)| - |V(G)| + c(G)$ where $c(G)$ denotes the number of components of G. It is known that $m(G)$ is rank of the *cocycle matroid* (or *bond matroid*) of G, that is dual to the classical cycle matroid of G. In other words, $m(G)$ is the number of the edges obtained after deleting the edges of a spanning forest from G. $m(G)$ is also called the cyclomatic number of G (see [Be]). The following was proved by Tutte [T2].

Theorem 1. *For every graph G, there exists a polynomial function $F_G(k)$ of k, so that if A is an Abelian group of order $k > 0$, then $F_G(k)$ is the number of nowhere-zero A-flows in G. Furthermore, if G is bridgeless, then $F_G(k)$ has degree $m(G)$ and otherwise $F_G(k) = 0$.*

Proof. Following the usual convention we get $F_G(k) = 1$ if $E(G) = \emptyset$. By Lemma 2, $F_G(k) = 0$ if G has a bridge. Take $e \in E(G)$ which is not a bridge. If e is a loop, then $F_G(k) = (k-1)F_{G-e}(k)$ and $m(G) = m(G-e) + 1$. Otherwise

$F_G(k) = F_{G/e}(k) - F_{G-e}(k)$ and $m(G) = m(G/e) = m(G-e)+1$. Thus applying induction on $|E(G)|$ we get the statement. $\square$

$F_G(k)$ is called the *flow polynomial* of G. Note that $F_G(1) = 0$ for every graph with $E(G) \neq \emptyset$ because there does not exist a nowhere-zero A-flow in G if A is trivial (contains only zero).

Suppose φ and φ' are nowhere-zero k-flows in a graph G. Then we write $\varphi \equiv \varphi'(\theta_{G,k})$ if $\varphi(x) \equiv \varphi'(x) \pmod{k}$ for every $x \in D(G)$. $\theta_{G,k}$ is an equivalence relation on the set of nowhere-zero k-flows in G. Let us note that $[\varphi]\theta_{G,k}$ denotes the $\theta_{G,k}$-class containing φ, i. e., the set of the nowhere-zero k-flows in G which are $\theta_{G,k}$-equivalent with φ.

If ψ is a nowhere-zero $\mathbb{Z}_k$-flow in G and X an orientation of G, then ψ_X denotes the nowhere-zero k-chain in G so that $\psi_X(x) = \psi(x)$ for every $x \in X$ (for the sake of simplicity, the residue class $[n]_k$ is considered as n where $0 \leq n < k$). Clearly, $\partial\psi_X(v) \equiv 0 \pmod{k}$ for every vertex v of G.

Let φ be a nowhere-zero $\mathbb{R}$-chain in G. Then $\{x \in D(G); \varphi(x) > 0\}$ is an orientation of G. It is denoted by X_φ and called the *positive orientation of* φ. Furthermore, if φ is a nowhere-zero k-flow, then denote by $[\varphi]_k$ the nowhere-zero $\mathbb{Z}_k$-flow in G satisfying $[\varphi]_k(x) = [\varphi(x)]_k$ for every $x \in D(G)$. It is clear that $[\varphi]_k$ is well defined and that $\varphi(x) = [\varphi]_k(x)$ for every $x \in X_\varphi$. Thus $([\varphi]_k)_{X_\varphi} = \varphi$. In fact, we can prove even more.

Theorem 2. *Let G be a graph. Then the mapping $[\varphi]\theta_{G,k} \mapsto [\varphi]_k$ is a one-to-one correspondence between the $\theta_{G,k}$-classes and the nowhere-zero $\mathbb{Z}_k$-flows in G.*

Proof. If φ and φ' are nowhere-zero k-flows in G, then $\varphi \equiv \varphi'(\theta_{G,k})$ if and only if $[\varphi]_k = [\varphi']_k$. Thus it remains to show that if ψ is a nowhere-zero $\mathbb{Z}_k$-flow in G, then there exists a nowhere-zero k-flow φ in G so that $[\varphi]_k = \psi$. Choose an orientation X in G so that $\sum_{v\in V(G)} |\partial\psi_X(v)|$ is minimal. We claim that ψ_X is a k-flow. If not, then there exists a vertex w from G with $\partial\psi_X(w) \geq k$ (because, by Lemma 1, $\sum_{v\in V(G)} \partial\psi_X(v) = 0$). Let W be the set of the endpoints of all directed paths in X beginning in w (clearly $w \in W$). Then $\omega_G^+(W) \subseteq X^{-1}$ and, by Lemma 1, $\sum_{v\in W} \partial\psi_X(v) = \sum_{x\in\omega_G^+(W)} \psi_X(x) \leq 0$. Thus there exists $w' \in W$ satisfying $\partial\psi_X(w') \leq -k$. Let P be a path in X oriented from w to w'. Changing the orientations of the arcs from P we get X' from X. It satisfies $\partial\psi_{X'}(w) = \partial\psi_X(w) - k$, $\partial\psi_{X'}(w') = \partial\psi_X(w') + k$, and $\partial\psi_{X'}(v) = \partial\psi_X(v)$ for every $v \in V(G) \setminus \{w, w'\}$. Therefore $\sum_{v\in V(G)} |\partial\psi_{X'}(v)| < \sum_{v\in V(G)} |\partial\psi_X(v)|$, which contradicts the choice of X. Thus ψ_X is a nowhere-zero k-flow in G and, furthermore, $[\psi_X]_k = \psi$. $\square$

Theorem 2 in fact follows from results of Tutte [T1]. Our proof is based on ideas of Younger [Y].

Corollary 1. *The number of the $\theta_{G,k}$-classes is $F_G(k)$ for every graph G and $k > 0$.*

Proof. Follows from Theorems 1 and 2. $\square$

Let $I_G(k)$ denote the number of nowhere-zero k-flows in a graph G. By Theorem 2, $I_G(k) = 0$ if and only if $F_G(k) = 0$. Since every nowhere-zero k-flow is also a nowhere-zero $(k+1)$-flow, we get $I_G(k) \leq I_G(k+1)$ for every graph G and $k > 0$.

By Seymour [S1], $F_G(6), I_G(6) > 0$ for every bridgeless graph G (Jaeger [J1] proved that $F_G(8), I_G(8) > 0$ for such graphs). The famous 5-flow conjecture of Tutte [T2] in fact suggests that $F_G(5), I_G(5) > 0$ for every bridgeless graph G.

By a *subdivision* of an edge we mean inserting a new 2-valent vertex into the edge. Two graphs are called *homeomorphic* if they arise from the same graph after applying finitely many subdivisions of edges. Clearly, if G_1 and G_2 are homeomorphic, then $F_{G_1}(k) = F_{G_2}(k)$ and $I_{G_1}(k) = I_{G_2}(k)$ for every $k > 0$.

3. Totally cyclic orientations and $\theta_{G,k}$-classes

Circuit is a connected graph with all vertices of valency two. *Cycle* is a graph with all vertices of even valency. *Directed circuit* (*directed cycle*) is an orientation of a circuit (cycle) so that for every vertex v, the number of the arcs entering v equals the number of the arcs leaving v.

If C is a directed cycle in $D(G)$, then denote by φ_C the 2-flow in G so that $\varphi_C(x) = 1$ for $x \in C$, $\varphi_C(x) = -1$ for $x \in C^{-1}$, and $\varphi_C(x) = 0$ otherwise.

Lemma 4. *If φ and φ' are nowhere-zero k-flows in a graph G and $\varphi \equiv \varphi'(\theta_{G,k})$, then $X_\varphi \setminus X_{\varphi'}$ is a directed cycle. On the other hand if C is a directed cycle in X_φ, then there exists just one $\varphi'' \in [\varphi]\theta_{G,k}$ so that $C = X_\varphi \setminus X_{\varphi''}$. It satisfies $\varphi'' = \varphi - k\varphi_C$.*

Proof. Clearly $\varphi'(x) = \varphi(x) - k$ or $\varphi(x)$ for every $x \in X_\varphi$. Thus applying Lemma 3 for the pairs $X_\varphi, \varphi|X_\varphi$ (the restriction of φ to X_φ) and $X_\varphi, \varphi'|X_\varphi$ we get that $X_\varphi \setminus X_{\varphi'}$ is a directed cycle. The rest of the proof is trivial. $\square$

The following lemma summarizes some useful facts.

Lemma 5. *Suppose X is an orientation in a connected graph G. Then the following conditions are pairwise equivalent.*

1. *X is a positive orientation of a nowhere-zero $\mathbb{Z}$-flow in G.*
2. *X is a positive orientation of a nowhere-zero $\mathbb{R}$-flow in G.*
3. *For every $\emptyset \subset W \subset V(G)$ we have $\omega^+_G(W) \cap X \neq \emptyset \neq \omega^+_G(W) \setminus X$.*
4. *For every ordered pair (v_1, v_2) of different vertices of G there exists an oriented v_1-v_2-path in X.*
5. *Every arc of X is covered by a directed circuit of X.*

Proof. (4) $\Rightarrow$ (5), and (1) $\Rightarrow$ (2) are trivial.

(2) $\Rightarrow$ (3) Follows from the fact that if φ is a nowhere-zero $\mathbb{R}$-flow, then by Lemma 1, $0 = \sum_{x \in \omega^+_G(W)} \varphi(x)$ for each $W \subseteq V(G)$ and $\varphi(x) > 0$ for each $x \in X_\varphi$.

(3) $\Rightarrow$ (4) If there is no oriented v_1-v_2-path in X, then by the max-flow min-cut theorem, there exists $W' \subseteq V(G)$ so that $v_1 \in W'$, $v_2 \notin W'$ and $\omega^+_G(W') \cap X = \emptyset$.

(5) $\Rightarrow$ (1) If directed circuits $C_1, \dots, C_t$ cover all arcs of X, then $\varphi_{C_1} + \cdots + \varphi_{C_t}$ is a nowhere-zero $\mathbb{Z}$-flow. $\square$

Conditions (1), (2) and (5) are equivalent also for unconnected graphs. An orientation of a graph satisfying item (5) from Lemma 5 is called *totally cyclic*.

Let $\mathcal{C}(G)$ denote the set of all totally cyclic orientations of a graph G. If $X, X' \in \mathcal{C}(G)$, then we write $X \equiv X'(\Theta_G)$ if $X \setminus X'$ is a directed cycle. The following statements follows directly from Lemmas 4 and 5.

Corollary 2. *Θ_G is an equivalence relation on $\mathcal{C}(G)$.*

Corollary 3. *If $\varphi \equiv \varphi'(\theta_{G,k})$, then $X_\varphi \equiv X_{\varphi'}(\Theta_G)$. Moreover the quantity $|[\varphi]\theta_{G,k}| = |[X_\varphi]\Theta_G|$ is equal to the number of the directed cycles in X_φ.*

Corollary 4. *If $X, X' \in \mathcal{C}(G)$ and $X \equiv X'(\Theta_G)$, then X and X' have the same number of directed cycles and $\varphi \mapsto \varphi - k\varphi_{X\setminus X'}$ is a one-to-one correspondence between the sets of nowhere-zero k-flows with positive orientations X and X'.*

Note that $\emptyset$ ($\subseteq D(G)$) is also a directed cycle, which corresponds to φ in $[\varphi]\theta_{G,k}$. Thus $|[\varphi]\theta_{G,k} \geq 2|$ whenever $E(G) \neq \emptyset$. If φ is a nowhere-zero k-flow in a graph G and $r > s \geq k$, then it is also a nowhere-zero s- and r-flow and, by Corollary 3, $|[\varphi]\theta_{G,s}| = |[\varphi]\theta_{G,r}|$. Furthermore, by Lemma 4, $[\varphi]\theta_{G,r} \cap [\varphi]\theta_{G,s} = \{\varphi\}$.

In the following example we show that $|[\varphi]\theta_{G,k}|$ can be any integer ≥ 2, and a graph G can admit nowhere-zero k-flows φ, φ', φ'' and φ''' so that $X_\varphi \equiv X_{\varphi'}(\Theta_G)$, $\varphi \not\equiv \varphi'(\theta_{G,k})$ and $|[\varphi'']\theta_{G,k}| = |[X_{\varphi''}]\Theta_G| \neq |[\varphi''']\theta_{G,k}| = |[X_{\varphi'''}]\Theta_G|$.

Example 1. Let H_n be a graph consisting of two vertices v_1, v_2 and n parallel edges $e_1, \ldots, e_n$. Let $x_1, \ldots, x_n$ be the arcs arising from $e_1, \ldots, e_n$ and directed from v_1 into v_2, respectively. Take a nowhere-zero n-flow φ_n in H_n so that $\varphi_n(x_1) = n - 1$ and $\varphi_n(x_2) = \varphi_n(x_3) = \cdots = \varphi_n(x_n) = -1$. Then $X_{\varphi_n} = X_{1,n} = \{x_1, x_2^{-1}, x_3^{-1}, \ldots, x_n^{-1}\}$, and, by Corollary 3, $|[\varphi_n]\theta_{H_n,k}| = |[X_{1,n}]\Theta_{H_n}| = n$ for every $k \geq n$.

Take the graph H_4 and nowhere-zero 3-flows φ and φ' in H_4 so that $2\varphi(x_1) = \varphi(x_2) = -2\varphi(x_3) = -\varphi(x_4) = 2$ and $\varphi'(x_1) = 2\varphi'(x_2) = -\varphi'(x_3) = -2\varphi'(x_4) = 2$. Then $\varphi \not\equiv \varphi'(\theta_{H_4,k})$ for every $k \geq 3$. On the other hand $X_\varphi = X_{\varphi'} = X_{2,4} = \{x_1, x_2, x_3^{-1}, x_4^{-1}\}$, and, by Corollary 3, $|[\varphi]\theta_{H_4,k}| = |[\varphi']\theta_{H_4,k}| = |[X_{2,4}]\Theta_{H_4}| = 6$ for every $k \geq 3$. But we have $|[\varphi_4]\theta_{H_4,k}| = 4$ for every $k \geq 4$.

Remark 1. Let $\mathcal{C}_k(G)$ denote the set of positive orientations of nowhere-zero k-flows in a graph G. Denote by $\mathcal{C}_{k,\Theta}(G)$ the set of the Θ_G-classes which are subsets of $\mathcal{C}_k(G)$. Then $\mathcal{C}_{k,\Theta}(G)$ is a partition of $\mathcal{C}_k(G)$, because, by Lemma 4, if $X \in \mathcal{C}_k(G)$, then $[X]\Theta_G \subseteq \mathcal{C}_k(G)$. This is of some interest, because the problem to find a nowhere-zero k-flow in G means to find an orientation from $\mathcal{C}_k(G)$, what automatically gives the class $[X]\Theta_G$ from $\mathcal{C}_{k,\Theta}(G)$.

The following statement was proved by Hoffman [H] (see also [Be]).

Lemma 6. *Let X be an orientation of a graph G and b_x, c_x be reals (integers) so that $b_x \leq c_x$ for every $x \in X$. Then there exists an $\mathbb{R}$-flow ($\mathbb{Z}$-flow) φ in G so that $b_x \leq \varphi(x) \leq c_x$ for every $x \in X$ if and only if $\sum_{x\in\omega_G^+(W)\cap X} b_x \leq \sum_{x\in\omega_G^-(W)\cap X} c_x$ for every $W \subseteq V(G)$.*

Proof. Necessity is trivial. We prove sufficiency. Let φ be an $\mathbb{R}$-chain ($\mathbb{Z}$-chain) in G such that $b_x \leq \varphi(x) \leq c_x$ for every $x \in X$ and $\sum_{v\in V(G)} |\partial\varphi(v)|$ has the smallest possible value. We claim that φ is a flow. If not, there exists a vertex w of G such that $\partial\varphi(w) > 0$. Let $D' \subseteq D(G)$ so that $x \in X \cap D'$ iff $\varphi(x) > b_x$ and $x \in X^{-1} \cap D'$ iff $\varphi(x^{-1}) < c_{x^{-1}}$. Let W be the set of the endpoints of all directed paths in D' beginning in w ($w \in W$). Suppose that W has a vertex w' satisfying $\partial\varphi(w') < 0$. Then take a path P in D' directed from w to w'. Let $s_x = \varphi(x) - b_x$ for $x \in P \cap X$, $s_x = c_{x^{-1}} - \varphi(x^{-1})$ for $x \in P \cap X^{-1}$, and s be the minimum from $\partial\varphi(w)$, $-\partial\varphi(w')$, s_x for $x \in P$. Then $s > 0$. Take an $\mathbb{R}$-chain ($\mathbb{Z}$-chain) φ' in G such that $\varphi'(x) = \varphi(x) - s$ if $x \in P$, $\varphi'(x) = \varphi(x) + s$ if $x^{-1} \in P$, and $\varphi'(x) = \varphi(x)$ otherwise. Then $b_x \le \varphi'(x) \le c_x$ for every $x \in X$ and $\sum_{v\in V(G)} |\partial\varphi'(v)| < \sum_{v\in V(G)} |\partial\varphi(v)|$ – a contradiction. Thus $\partial\varphi(v) \ge 0$ for every $v \in W$, whence $\sum_{v\in W} \partial\varphi(v) > 0$. Furthermore, by the definition of W, $\varphi(x) = b_x$ if $x \in \omega^+_G(W) \cap X$ and $\varphi(x) = c_x$ if $x \in \omega^-_G(W) \cap X$. Thus, by Lemma 1 and the assumptions, $0 < \sum_{v\in W} \partial\varphi(v) = \sum_{x\in\omega^+_G(W)\cap X} b_x - \sum_{x\in\omega^-_G(W)\cap X} c_x \le 0$ – a contradiction. Thus φ is an $\mathbb{R}$-flow ($\mathbb{Z}$-flow) as claimed. □

Lemma 6 implies the following statement (see also [Z]).

Corollary 5. *An orientation X of a connected graph G is a positive orientation of a nowhere-zero k-flow if and only if $|\omega^+_G(W)|/|\omega^+_G(W) \cap X| \le k$ for every $\emptyset \subset W \subset V(G)$.*

It is well known that there exist $m(G)$ independent circuits in a (directed) graph (see Berge [Be] for more details). Now we prove a stronger statement for totally cyclic orientations.

An ordered set $(C_1, \ldots, C_m)$ of directed circuits is called *triangular* if every C_i contains an arc x_i $(i = 1, \ldots, m)$ so that $x_i \notin C_j$ for each $j < i$.

Lemma 7. *Let X be a totally cyclic orientation of a graph G. Then there exists a triangular set $\mathcal{C} = \{C_1, \ldots, C_{m(G)}\}$ of directed circuits in X covering all arcs of X.*

Proof. We use induction on $m(G)$. The statement holds if $m(G) = 0$ and 1. Let $m(G) > 1$. By Lemma 5, there exists a nowhere-zero $\mathbb{Z}$-flow φ in G so that $X = X_\varphi$ and a directed circuit $C \subset X$. Take $r = \min_{x\in C} \varphi(x)$ and $\varphi' = \varphi - r\varphi_C$. Let G' be the subgraph of G so that $E(G')$ is the support of φ' and $V(G') = V(G)$. Let $G_1, \ldots, G_s$ be the components of G'. Contracting the subgraphs $G_1, \ldots, G_s$ to vertices $v_1, \ldots, v_s$, respectively, and deleting the loops arising from the edges of G' we get a graph G''. Then $X' = X \cap D(G')$ and $X'' = X \cap D(G'')$ are positive orientations of nowhere-zero $\mathbb{Z}$-flows (arising from φ). X' and X'' form a partition of X. If we take spanning forests in G' and G'', then the union of their edges induces a spanning forest in G, whence $m(G) = m(G') + m(G'')$ and $m(G'), m(G'') < m(G)$. Thus, by the assumption of induction, X' and X'' have required triangular sets $\mathcal{C}'$ and $\mathcal{C}''$ of directed circuits, respectively. Every directed circuit in X' is a directed circuit in X and every directed circuit in X'' can be extended into a directed circuit in X by item (4) from Lemma 5. Thus $\mathcal{C}'$ and $\mathcal{C}''$ induce a triangular set of $m(G)$ directed circuits covering X. □

Note that Lemma 7 cannot be improved so that every circuit C_i contains an arc x_i where $x_i \notin C_j$ for $j \ne i$. It remains to take an orientation $\{x_1, x_2, x_3^{-1}, x_4^{-1}\}$ of H_4 (see Example 1).

Corollary 6. *Every totally cyclic orientation of a graph G contains at least $m(G)$ directed circuits and at least $m(G) + 1$ directed cycles.*

Proof. Follows directly from Lemma 7. □

4. Growth of flow polynomials

For every orientation X of G, let $I_X(k)$ denote the number of nowhere-zero k-flows φ in G so that $X_\varphi = X$. By Lemma 5, $I_X(k) = 0$ if X is not totally cyclic. Obviously, $I_X(k+1) \geq I_X(k)$.

Example 2. Take the graph H_n with the orientation $X_{1,n}$ from Example 1. Then $X_{1,n}$ is a positive orientation of a nowhere-zero k-flow φ in G if and only if $k \geq n$, because $\varphi(x_1) = \sum_{i=2}^{n} \varphi(x_i^{-1}) \geq n - 1 = m(H_n)$. This follows also from Corollary 5. Thus $I_X(k) = 0$ can occur for each $k \leq m(G)$, where $m(G)$ can be large enough. This contrasts with the Seymour's theorem [S1] which says that $F_G(k) > 0$ if $k \geq 6$ and G is bridgeless.

Denote by $\mathcal{C}_\Theta(G)$ the set of the equivalence classes of Θ_G. $\mathcal{C}_\Theta(G)$ is a partition of $\mathcal{C}(G)$. If $\mathcal{X} \in \mathcal{C}_\Theta(G)$ and $X, X' \in \mathcal{X}$, then, by Corollary 4, $I_X(k) = I_{X'}(k)$ for every $k > 0$. Denote this value by $I_{\mathcal{X}}(k)$. We have

$$I_G(k) = \sum_{X \in \mathcal{C}(G)} I_X(k) = \sum_{\mathcal{X} \in \mathcal{C}_\Theta(G)} I_{\mathcal{X}}(k)|\mathcal{X}| \qquad (k > 0). \tag{1}$$

Let ψ be a nowhere-zero $\mathbb{Z}_k$-flow in G. By Theorem 2, there exists a nowhere-zero k-flow φ in G so that $[\varphi]_k = \psi$. Furthermore, $[\varphi]\theta_{G,k}$ is the set of all nowhere-zero k-flows φ' in G satisfying $[\varphi']_k = \psi$. By Corollaries 3, 4 and Theorem 2, $[X_\varphi]\Theta_G$ is equal to the following three sets:

- the set of all positive orientations of the flows from $[\varphi]\theta_{G,k}$,
- the set of all orientations X of G so that the k-chain ψ_X is a k-flow,
- the set of all positive orientations of nowhere-zero k-flows φ' in G so that $[\varphi']_k = \psi$.

Thus $[X_\varphi]\Theta_G$ is called the *Θ_G-class associated with ψ* and denoted by $\mathcal{X}_\psi$. We have $|\mathcal{X}_\psi| = |[X_\varphi]\Theta_G| = |[\varphi]\theta_{G,k}|$. For every $\mathcal{X} \in \mathcal{C}_\Theta(G)$, $I_{\mathcal{X}}(k)$ is equal to the number of nowhere-zero $\mathbb{Z}_k$-flows ψ in G so that $\mathcal{X}_\psi = \mathcal{X}$. Hence

$$F_G(k) = \sum_{\mathcal{X} \in \mathcal{C}_\Theta(G)} I_{\mathcal{X}}(k) = \sum_{X \in \mathcal{C}(G)} \frac{I_X(k)}{|[X]\Theta_G|} \qquad (k > 0). \tag{2}$$

Theorem 3. *Let G be a bridgeless graph, $E(G) \neq \emptyset$, and X be a totally cyclic orientation of G. Then for every $k > 1$,*

$$I_X(k+1) \geq I_X(k) \cdot k/(k-1),$$
$$F_G(k+1) \geq F_G(k) \cdot k/(k-1),$$
$$I_G(k+1) \geq I_G(k) \cdot k/(k-1).$$

Proof. By (1) and (2), the statement remains to prove for $I_X(k)$. This holds if $I_X(k) = 0$. Suppose that $I_X(k) > 0$ and A (B) denotes the set of nowhere-zero k-flows ($(k+1)$-flows) φ in G with positive orientation X. For every $\varphi \in A$ and a directed cycle $C \neq \emptyset$ in X, define by *C-lift* of φ the flow from $B \setminus A$ of the form $\varphi + r\varphi_C$ where r is a positive integer. Let n be the number of nonempty directed cycles in X. Then we can get together $n|A|$ lifts (for every φ and C just one). But each such a lift can be obtained at most $n(k-1)$ times. Thus $|B \setminus A| \geq |A|/(k-1)$, whence $I_X(k+1) = |B| \geq |A| \cdot k/(k-1) = I_X(k) \cdot k/(k-1)$. $\square$

Corollary 7. *Let X be a totally cyclic orientation of a bridgeless graph G, $E(G) \neq \emptyset$, and $I_X(k+1)$, $F_G(k+1)$, $I_G(k+1) > 0$. Then*

$$I_X(k+1) \geq I_X(k) + 1,$$
$$F_G(k+1) \geq F_G(k) + 1,$$
$$I_G(k+1) \geq I_G(k) + m(G) + 1.$$

Proof. If $I_X(k+1) > 0$, then from the proof of Theorem 3 follows that $|B| > |A|$ and $I_X(k+1) \geq I_X(k) + 1$. If $F_G(k+1)$, $I_G(k+1) > 0$, then there exists $\mathcal{X} \in \mathcal{C}_\Theta(G)$ so that $I_\mathcal{X}(k) > 0$, thus also $I_\mathcal{X}(k+1) \geq I_\mathcal{X}(k) + 1$. Hence, by (1), (2), and Corollaries 3, 6, $F_G(k+1) \geq F_G(k) + 1$ and $I_G(k+1) \geq I_G(k) + m(G) + 1$. □

Let C_n be the circuit of length n. Then $m(C_n) = 1$ and $F_{C_n}(k) = k - 1$. C_n has just two totally cyclic orientations X_1, X_2 and $I_{X_1}(k) = I_{X_2}(k) = k - 1$. By (1), $I_{C_n}(k) = 2(k-1)$. Thus the bounds from Theorem 3 and Corollary 7 are the best possible in this case.

5. Polynomials $I_G(k)$ and $I_X(k)$

In this section we use some basic properties of convex polytopes as presented in [Sc], [KK] and [Bj]. Every polytope P in $\mathbb{R}^n$ is the convex hull of the (finite) set of its vertices. If these are integral, then P is called *integral*. Dimension of a polytope is the dimension of its affine hull. By a *variant* of a polytope P we mean every subset of P arising from it after deleting some of its faces. Every variant of a polytope is a convex set. The maximal variant of P is it itself. Denote by $\iota(P)$ the minimal nonempty variant of P, i. e., the set arising after deleting all faces of P which are proper subsets of P (note that $\iota(P) = P$ if P has dimension 0 and otherwise $\iota(P)$ is the relative interior of P). P is called a d-simplex (or a simplex) if it is a convex hull of $d+1$ affinely independent vectors. The dimension of a d-simplex is d. All faces of a simplex are simplexes. The following statement was proved by Ehrhart [E].

Lemma 8. *Let $\overline{P}$ be a variant of an integral convex polytope P. Then the number of the integral vectors from $k\overline{P} = \{k\mathbf{x}; \mathbf{x} \in \overline{P}\}$ is a polynomial function for every integral $k > 0$.*

Proof. It is known that there are simplexes $P_1, \ldots, P_r$ so that their vertices are vertices of P, their union is P, and $P_i \cap P_j$ is a face of P_i and P_j for all $i, j \in \{1, \ldots, r\}$ (in other words every polytope can be presented as a complex of simplexes of some of its vertices). Let P' be a d-simplex and $\{P'_1, \ldots, P'_s\}$ be the set of its faces. Then using induction on d we can show that $P' = \cup_{i=1}^s \iota(P'_i)$. On the other hand $\iota(P'_i) \cap \iota(P'_j) = \emptyset$ if $i \neq j$. Thus there are integral simplexes $P''_1, \ldots, P''_{r''}$ so that their vertices are vertices of P and $\{\iota(P''_1), \ldots, \iota(P''_{r''})\}$ is a decomposition of P into pairwise disjoint subsets. Furthermore, if F is a face of P, then $F \cap \iota(P''_i)$ is either $\emptyset$ or $\iota(P''_i)$. Therefore there are integral simplexes $\overline{P}_1, \ldots, \overline{P}_{\overline{r}}$ so that $\{\iota(\overline{P}_1), \ldots, \iota(\overline{P}_{\overline{r}})\}$ is a decomposition of $\overline{P}$ into pairwise disjoint subsets.

Thus the statement remains to prove if $\overline{P} = \iota(P)$ and P is an integral simplex. But using induction on the dimension of simplexes we can check that the statement remains to solve if $\overline{P} = P$ and P is an integral simplex.

Let P be an integral d-simplex in $\mathbb{R}^n$, i. e., the convex hull of affinely independent integral vectors $\mathbf{v}_0, \ldots, \mathbf{v}_d$. Take $\mathbf{x} \in kP \cap \mathbb{Z}^n$. Then $\mathbf{x} = r_0\mathbf{v}_0 + \cdots + r_d\mathbf{v}_d$ where $r_0, \ldots, r_d$ are nonnegative reals and $r_0 + \cdots + r_d = k$. Set $s_i = \lfloor r_i \rfloor$, $t_i = r_i - s_i$ $(i = 1, \ldots, d)$, $\mathbf{y} = s_0\mathbf{v}_0 + \cdots + s_d\mathbf{v}_d$, and $\mathbf{z} = t_0\mathbf{v}_0 + \ldots t_d\mathbf{v}_d$. Then $\mathbf{y} \in \mathbb{Z}^n$, $\mathbf{z} = \mathbf{x} - \mathbf{y} \in \mathbb{Z}^n$, and $t_0 + \cdots + t_d$ is a nonnegative integer $\leq d$. The numbers r_i, s_i, t_i $(i = 0, \ldots, d)$ and the vectors $\mathbf{y}$, $\mathbf{z}$ are unique, because $\mathbf{v}_0, \ldots, \mathbf{v}_d$ are affinely independent. Set

$$S = \{s_0\mathbf{v}_0 + \cdots + s_d\mathbf{v}_d;\ s_i \in \mathbb{Z},\ s_i \geq 0\},$$
$$T = \{t_0\mathbf{v}_0 + \cdots + t_d\mathbf{v}_d \in \mathbb{Z}^n;\ t_i \in \mathbb{R},\ 1 > t_i \geq 0,\ t_0 + \cdots + t_d \in \mathbb{Z}\}.$$

Note that $(0, \ldots, 0) \in T \neq \emptyset$. For every $\mathbf{z} \in T$, define $p_{\mathbf{z}} = t_0 + \cdots + t_d$ (this number is unique, because $\mathbf{v}_0, \ldots, \mathbf{v}_d$ are affinely independent). $T \subseteq \cup_{i=0}^{d}(iP \cap \mathbb{Z}^n)$ is finite and S is infinite (unless $d = 0$ and $\mathbf{v}_0 = (0, \ldots, 0)$). We have already shown that if $\mathbf{x} \in kP \cap \mathbb{Z}^n$, then there are unique vectors $\mathbf{y} \in S$ and $\mathbf{z} \in T$ so that $\mathbf{x} = \mathbf{y} + \mathbf{z}$. Thus $\mathbf{y} \in (k - p_{\mathbf{z}})P \cap S$ and $|kP \cap \mathbb{Z}^n| = \sum_{\mathbf{z} \in T} |(k - p_{\mathbf{z}})P \cap S|$ (considering $k'P = \emptyset$ if $k' < 0$ and $0P = \{(0, \ldots, 0)\}$).

It remains to show that $f(k) = |(k - p)P \cap S|$ is a polynomial for $k > 0$ if p is an integral constant $0 \leq p \leq d$. If $k \geq p$, then $f(k)$ is the number of the ordered $(d+1)$-tuples $(a_0, \ldots, a_d)$ of nonnegative integers satisfying $a_0 + \cdots + a_d = k - p$, what is the number of the ordered $(d+1)$-tuples $(a'_0, \ldots, a'_d)$ of positive integers satisfying $a'_0 + \cdots + a'_d = k - p + d + 1$. Thus $f(k) = \binom{k-p+d}{d} = \frac{(k-p+d)\ldots(k-p+1)}{d!}$ is a polynomial for $k \geq p$. Since value of this polynomial is 0 for every $0 < k < p$, the statement holds for every $k > 0$. $\square$

Note that we have in fact proved that the degree of the polynomial from Lemma 8 is equal to the dimension of P.

Theorem 4. *Let X be an orientation of a graph G. Then $I_X(k)$ is a polynomial function for $k > 0$. The degree of this polynomial is $m(G)$ if X is totally cyclic and $I_X(k) = 0$ otherwise.*

Proof. Let $U_X \subseteq \mathbb{R}^{D(G)}$ ($\overline{U}_X \subseteq \mathbb{R}^{D(G)}$) be the set of all $\mathbb{R}$-flows φ in G so that $0 \leq \varphi(x) \leq 1$ ($0 < \varphi(x) < 1$) for every $x \in X$. (Now flows are considered as vectors indexed by $D(G)$.) Then $\overline{U}_X$ is a variant of U_X and $I_X(k) = |k\overline{U}_X \cap \mathbb{Z}^{D(G)}|$. By Lemma 8, it remains to show that U_X is an integral polytope. This follows from a result of Tutte [T4] (see also [Sc, Chapter 19.3]) who proved that $U_X = (\mathbf{x} \in \mathbb{R}^{D(G)}; M\mathbf{x} \leq \mathbf{b})$ where M is a totally unimodular matrix and $\mathbf{b}$ is an integral vector (see Remark 2 for another proof). Thus $I_X(k)$ is a polynomial.

By (2), $F_G(k) \geq I_{[X]\Theta_G}(k) = I_X(k)$, thus by Theorem 1, $I_X(k)$ can have degree at most $m(G)$. By Lemma 7, there exists a triangular set $\{C_1, \ldots, C_{m(G)}\}$ of directed circuits covering X. Then for every positive integer s and an ordered $m(G)$-tuple of integers $\mathbf{a} = (a_1, \ldots, a_{m(G)})$ satisfying $1 \leq a_i \leq s$ we get a nowhere-zero $(m(G) \cdot s + 1)$-flow $\varphi_{\mathbf{a}} = \sum_{i=1}^{m(G)} a_i\varphi_{C_i}$ with the positive orientation X. If $\mathbf{a} = (a_1, \ldots, a_{m(G)}) \neq \mathbf{a}' = (a'_1, \ldots, a'_{m(G)})$ and i is the largest index satisfying $a_i \neq a'_i$, then $\varphi_{\mathbf{a}}(x_i) \neq \varphi_{\mathbf{a}'}(x_i)$ (where $x_i \in C_i$ and $x_i \notin C_j$ for every $j < i$). Therefore, $\mathbf{a} \mapsto \varphi_{\mathbf{a}}$ is an injective mapping. Thus $I_X(m(G) \cdot s + 1) \geq s^{m(G)}$ and $\liminf_{k\to\infty} I_X(k)/k^{m(G)} > 0$, whence $I_X(k)$ has degree at least $m(G)$. $\square$

Theorem 5. *Let G be a graph G. Then $I_G(k)$ is a polynomial function for $k > 0$. The degree of this polynomial is $m(G)$ if G is bridgeless and $I_G(k) = 0$ otherwise.*

Proof. Every bridgeless graph has a totally cyclic orientation (see also Lemma 11). Thus the statement follows from Theorem 4, Lemma 2 and (1). $\square$

Remark 2. Let φ be a nowhere-zero $\mathbb{R}$-flow ($\mathbb{Z}$-flow) in G with the positive orientation X. Take $m_\varphi = \max_{x \in X} \varphi(x)$ and $M_\varphi = \{x \in X; \varphi(x) = m_\varphi\}$. By Lemma 1, $|\omega_G^+(W) \cap M_\varphi| \le |\omega_G^-(W) \cap X|$ for every $W \subseteq V(G)$. Thus, by Lemma 6, there exists a $\mathbb{Z}$-flow $\overline{\varphi}$ in G so that $0 \le \overline{\varphi}(x) \le 1$ for $x \in X \setminus M_\varphi$ and $\overline{\varphi}(x) = 1$ for $x \in M_\varphi$. Then $\overline{\varphi} = \varphi_C$ where C is a directed cycle in X and $M_\varphi \subseteq C$. Let $r' = \min_{x \in C} \varphi(x)$, $r'' = \min_{x \in X \setminus C} m_\varphi - \varphi(x)$, $r = \min\{r', r''\} > 0$, and $\varphi' = \varphi - r\varphi_C$. Setting $m_{\varphi'}$ and $M_{\varphi'}$ analogously as for φ, we get that $m_{\varphi'} = m_\varphi - r$ and either $M_\varphi \subset M_{\varphi'}$ (if $r'' \le r'$) or $\sigma(\varphi') \subset E(G)$ (if $r' \le r''$, $\sigma(\varphi')$ denotes the support of φ'). Now using induction on $|E(G)|$ and $|E(G)| - |M_\varphi|$ we can show that there exist directed cycles $C_1, \dots, C_t$ in X and positive reals (integers) $r_1, \dots, r_t$ so that $m_\varphi = r_1 + \dots + r_t$ and $\varphi = r_1\varphi_{C_1} + \dots + r_t\varphi_{C_t}$. This implies that every flow $\varphi \in U_X$ is a convex hull of integral flows from U_X, whence U_X is integral.

$I_G(k)$ and $I_X(k)$ are called the *integral flow polynomials* of G and X, respectively. Similarly as the first half of Theorem 4 we can prove the following.

Theorem 6. *Let X be an orientation of a graph G and b_x, c_x be integers for every $x \in X$. Then the number of the $\mathbb{Z}$-flows φ in G satisfying $kb_x \le \varphi(x) \le kc_x$ ($kb_x < \varphi(x) < kc_x$) for every $x \in X$ is a polynomial function for integral $k > 0$.*

Example 3. Let $\mathcal{X}_{i,n}$ denote the set of the orientations of H_n (see Example 1) having just i arcs directed from v_1 to v_2. Then $\{\mathcal{X}_{1,n}, \dots, \mathcal{X}_{n-1,n}\} = \mathcal{C}_\Theta(H_n)$, $|\mathcal{X}_{i,n}| = \binom{n}{i}$ and $I_{\mathcal{X}_{i,n}}(k) = I_{\mathcal{X}_{n-i,n}}(k)$ for $i = 1, \dots, n-1$. Set $X_{i,n} = \{x_1, \dots, x_i, x_{i+1}^{-1}, \dots, x_n^{-1}\}$. Then $X_{i,n} \in \mathcal{X}_{i,n}$ and $I_{X_{i,n}}(k) = I_{\mathcal{X}_{i,n}}(k)$.

Define $p_{1,n}(1) = I_{X_{1,n}}(1)$ and $p_{1,n}(i) = I_{X_{1,n}}(i) - I_{X_{1,n}}(i-1)$ for $i > 1$. Then $p_{1,n}(i)$ is the number of the nowhere-zero i-flows φ in H_n where $\varphi(x_1) = i - 1$ and $\varphi(x_2^{-1}) + \dots + \varphi(x_n^{-1}) = i - 1$. Therefore $p_{1,n}(i)$ is the number of the ordered $(n-1)$-tuples of positive integers $(a_1, \dots, a_{n-1})$ so that $a_1 + \dots + a_{n-1} = i - 1$. Hence $p_{1,n}(i) = \binom{i-2}{n-2}$ and $I_{X_{1,n}}(k) = \sum_{i=1}^{k} p_{1,n}(i) = \sum_{i=n}^{k} \binom{i-2}{n-2} = \binom{k-1}{n-1}$.

$F_{H_2}(k) = k - 1$ and, by Theorem 1, $F_{H_n}(k) = (k-1)^{n-1} - F_{H_{n-1}}(k)$ ($n \ge 3$), whence $F_{H_n}(k) = \sum_{i=1}^{n-1} (-1)^{i-1}(k-1)^{n-i}$. Then using (1) and (2) we can evaluate all polynomials $I_{X_{i,n}}(k)$ and $I_{H_n}(k)$ for $2 \le n \le 5$, $1 \le i \le n-1$.

6. Bounds between $F_G(k)$ and $I_G(k)$

In order to obtain an upper bound for the number of directed cycles in an orientation of a graph we need to introduce another notation. Let X be an orientation of a graph G and $\mathbf{u} : V(G) \to \mathbb{Z}$. Then $Y \subseteq X$ is called a $\mathbf{u}$-*suborientation* of X if $|\omega_G^+(v) \cap Y| - |\omega_G^-(v) \cap Y| = \mathbf{u}(v)$ for every vertex v of G. Clearly, directed cycles are $\mathbf{0}$-suborientations where $\mathbf{0}(v) = 0$ for every $v \in V(G)$.

Lemma 9. *If X is an orientation of a graph G and $\mathbf{u} : V(G) \to \mathbb{Z}$, then the number of the $\mathbf{u}$-suborientations of X is at most $2^{m(G)}$.*

Proof. We use induction on $m(G)$. If $m(G) = 0$, then G is a forest and we can easily check that the number of the $\mathbf{u}$-suborientations of X is at most $1 = 2^{m(G)}$ for every $\mathbf{u} : V(G) \to \mathbb{Z}$. If $m(G) > 0$, then G has an edge e so that e is not a bridge. Thus $m(G - e) = m(G) - 1$. Let x be the arc from X arising from e. Suppose x is directed from v_1 to v_2. Take $\mathbf{u}' : V(G) \to \mathbb{Z}$ so that $\mathbf{u}' = \mathbf{u}$ if $v_1 = v_2$ and otherwise $\mathbf{u}'(v_1) = \mathbf{u}(v_1) - 1$, $\mathbf{u}'(v_2) = \mathbf{u}(v_2) + 1$ and $\mathbf{u}'(v) = \mathbf{u}(v)$ for every $v \in V(G) \setminus \{v_1, v_2\}$. Let Y be a $\mathbf{u}$-suborientation of X. Then $Y \setminus \{x\}$ is either a $\mathbf{u}'$- or a $\mathbf{u}$-suborientation of $X' = X \setminus \{x\}$ (depending on the case whether $x \in Y$ or not). Since X' is an orientation of $G - e$, then, by the assumption of induction, there are at most $2^{m(G-e)}$ $\mathbf{u}$-suborientations ($\mathbf{u}'$-suborientations) of X'. Thus there are at most $2 \cdot 2^{m(G-e)} = 2^{m(G)}$ $\mathbf{u}$-suborientations of X. $\square$

Corollary 8. *Every orientation of a graph G has at most $2^{m(G)}$ directed cycles.*

Proof. Follows from Lemma 9 and the fact that the directed cycles in X are the $\mathbf{0}$-suborientations of X. $\square$

Theorem 7. *Let G be a graph. Then $2^{m(G)} F_G(k) \geq I_G(k) \geq (m(G) + 1) F_G(k)$ for every $k > 0$.*

Proof. By Corollaries 3, 6, and 8, $2^{m(G)} \geq |\mathcal{X}| \geq m(G) + 1$ for every $\mathcal{X} \in \mathcal{C}_\Theta(G)$. Then, by (1) and (2), $2^{m(G)} F_G(k) = \sum_{\mathcal{X} \in \mathcal{C}_\Theta(G)} I_{\mathcal{X}}(k) 2^{m(G)} \geq \sum_{\mathcal{X} \in \mathcal{C}_\Theta(G)} I_{\mathcal{X}}(k) |\mathcal{X}|$ $= I_G(k) \geq \sum_{\mathcal{X} \in \mathcal{C}_\Theta(G)} I_{\mathcal{X}}(k)(m(G) + 1) = (m(G) + 1) F_G(k)$. $\square$

Note that if $m(G) = 0$, then $2^{m(G)} F_G(k) = I_G(k) = (m(G) + 1) F_G(k)$ for every $k > 0$, because $F_G(k) = I_G(k) = 1$ or 0. We give also less trivial examples which show that Corollaries 6, 8 and Theorem 7 are the best possible in a certain sense.

Example 4. Let $X_{1,n}$ be the orientation of H_n as described in Example 1. Then $X_{1,n}$ has just $n - 1$ directed circuits and $n = m(G) + 1$ directed cycles. Moreover if $n = 2, 3$, then every totally cyclic orientation of H_n has just n directed cycles. Therefore, if G is homeomorphic with H_2 or H_3, then $I_G(k) = (m(G) + 1) F_G(k)$ for every $k \geq 2$.

Remark 3. If H is a subgraph of G and φ is a nowhere-zero k-flow (A-flow) in G, then denote by $\varphi_{|H}$ the restriction of φ to $D(H)$. Clearly, $\varphi_{|H}$ is a k-chain (A-chain) in H. Suppose G is a union of two edge-disjoint subgraphs H and H' with at most one vertex in common. Then φ is a nowhere-zero k-flow (A-flow) in G if and only if $\varphi_{|H}$ and $\varphi_{|H'}$ are nowhere-zero k-flows (A-flows) in H and H', respectively. Therefore, $F_G(k) = F_H(k) \cdot F_{H'}(k)$ (see also [T6]) and $I_G(k) = I_H(k) \cdot I_{H'}(k)$. Furthermore, from Corollary 3 follows that $|[\varphi]\theta_{G,k}| = |[\varphi_{|H}]\theta_{H,k}| \cdot |[\varphi_{|H'}]\theta_{H',k}|$ for every nowhere-zero k-flow φ in G. Thus $I_G(k) \geq (m(G_1) + 1) \ldots (m(G_r) + 1) F_G(k)$ where $G_1, \ldots, G_r$ are the components or blocks of G. In this way we can improve the lower bound in Theorem 4 but not the upper bound, because $2^{m(G)} = 2^{m(G_1)} \cdot \ldots \cdot 2^{m(G_r)}$.

Example 5. Let G be a graph composed from n blocks which are circuits (or, equivalently, which are homeomorphic with H_2). Then $m(G) = n$ and, by Example 4 and Remark 3, $I_G(k) = 2^n F_G(k) = 2^{m(G)} F_G(k)$ for every $k \geq 2$.

Example 6. Suppose G is a bridgeless graph arising as follows. Take two vertex disjoint trees T_1 and T_2 and add to them n new edges joining the vertices from T_1 with vertices from T_2. Then $m(G) = n-1$. Take vertices $v_1 \in T_1$ and $v_2 \in T_2$ which are joined by an arc e. Take an orientation of T_1 (T_2) so that every arc is contained in an directed path beginning in a vertex of T_1 (v_2) and ending in v_1 (a vertex of T_2). Extend these orientations into an orientation X of G so that $\omega_G^+(V(T_1)) \cap X = \{x\}$, where x is the arc arising from e and directed from v_1 to v_2. Then $X \setminus \{x\}$ has no directed circuit, X has just $n-1$ directed circuits (each containing x) and n directed cycles. Furthermore, $|\omega_G^+(V(T_1))|/|\omega_G^+(V(T_1)) \cap X| = n$ and, by Corollary 5, $I_X(k) = 0$ for every $k \leq n-1 = m(G)$. Note that the orientation $X_{1,n}$ discussed in Examples 2 and 4 is a special case of this construction. Now G can have also other orientations having more that $m(G)+1$ directed cycles, but Corollary 6 is the best possible for the orientation X.

Example 7. Let $C_{2,n}$ be a graph arising from the circuit of length n after replacing each edge by two parallel edges. Then $m(C_{2,n}) = n+1$. Let X be a totally cyclic orientation of $C_{2,n}$. Then we can check that X contains a directed circuit C containing all vertices of $C_{2,n}$. Let Y be the set of the arcs from X which are not in C. Suppose Y contains just i (j) arcs having the same (opposite) orientations as the arcs of C ($i+j=n$, $0 \leq i,j \leq n$). Then X has just $2^i + 2^j$ directed circuits and just $2^i + 2^j + a(i,j)$ directed cycles, where $a(0,n) = a(n,0) = 2$ and $a(i,j) = 1$ otherwise. Thus Corollary 6 gives a bad estimate for every totally cyclic orientation of $C_{2,n}$. On the other hand, if i or j is 0, we get a relatively good estimate from Corollary 8.

7. Elementary cuts

If G is a graph and $W \subseteq V(G)$, then denote by $\omega_G(W)$ the set of the edges associated with $\omega_G^+(W)$. $\omega_G(W)$ is called a *cut* of G. A cut is called *elementary* if the number of components increase by one after deleting its edges from the graph. Let $\mu(G)$ denote the maximal cardinality of an elementary cut in G. Clearly $\mu(G) \leq m(G)+1$.

Lemma 10. *Let G be a connected bridgeless graph and $W \subseteq V(G)$ so that $\omega_G(W)$ is an elementary cut of G. Suppose H is the graph arising from G after contracting the edges from W and $V(G) \setminus W$ to vertices v_1 and v_2, respectively, and deleting the loops. Then every totally cyclic orientation of H can be extended into a totally cyclic orientation of G and $|\mathcal{C}_\Theta(H)| \leq |\mathcal{C}_\Theta(G)|$.*

Proof. Clearly, H is isomorphic to the graph H_n where $n = |\omega_G(W)|$. Let Y be a totally cyclic orientation of H. A vertex $v \in W$ ($\in V(G) \setminus W$) is called *Y-weak* if there exists another vertex $u \in W$ ($\in V(G) \setminus W$) so that $\omega_G^+(v) \setminus Y \subseteq \omega_G^-(u)$ ($\omega_G^-(v) \setminus Y \subseteq \omega_G^+(u)$). Note that $\omega_G^+(v) \setminus Y$ ($\omega_G^-(v) \setminus Y$) $\neq \emptyset$. Then u and v are joined by an edge e. Clearly, if Y can be extended to a cyclic orientation of G/e then it can be extended to a cyclic orientation of G. Thus the statement remains to prove for graphs with no Y-weak vertex. For such an Y take $w \in W$ and $w' \in V(G) \setminus W$ so that $\omega_G^+(w) \cap Y \neq \emptyset \neq \omega_G^-(w') \cap Y$. Using induction on $|E(G)|$ ($\geq n$) we prove the following: If all Y-weak vertices of G are from $\{w, w'\}$ (this assumption is needed for providing the induction), then Y can be extended into a totally cyclic orientation X of G so that $\omega_G^+(w)\cap X = \omega_G^+(w)\cap Y$, $\omega_G^-(w')\cap X = \omega_G^-(w')\cap Y$, and for every pair of consecutive arcs of X incident to w or w' and not arising from loops there is a directed circuit containing them. This is trivial if $|E(G)| = n$ because then $G = H_n$. The induction step is also trivial if G has a loop. Suppose G has no loop and more than n edges. Then it has an edge $e' \in E(G)\setminus \omega_G(W)$ incident to w or w' and, by the assumption of induction, G/e' has an orientation with the abovementioned properties. Adding to this orientation the arc x arising from e' and so that $x \in \omega_G^-(w)$ or $x \in \omega_G^+(w')$, we get a required orientation of G. Thus Y can be extended into a totally cyclic orientation of G.

Suppose X, X' are totally cyclic orientations of G so that $X \setminus X'$ is a directed cycle and Y, Y' are the restrictions of X, X' to $D(H)$, respectively. Then $Y \setminus Y'$ is the restriction of $X \setminus X'$ to $D(H)$, whence $Y \setminus Y'$ is a directed cycle. Thus $X \equiv X'(\Theta_G)$ implies $Y \equiv Y'(\Theta_H)$ and therefore $|\mathcal{C}_\Theta(H)| \le |\mathcal{C}_\Theta(G)|$. □

Corollary 9. *$|\mathcal{C}_\Theta(G)| \ge \mu(G) - 1$ for every connected bridgeless graph G.*

Proof. Follows from Lemma 10 and the fact that $|\mathcal{C}_\Theta(H_n)| = n - 1$ (see Example 3). □

Theorem 8. *Let G be a bridgeless graph and k a positive integer. Then $I_X(k) > 0$ for every $X \in \mathcal{C}(G)$ if and only if $k \ge \mu(G)$.*

Proof. By Corollary 5, Lemma 10 and Example 2, there exists an orientation X' of G so that $I_{X'}(k) = 0$ for every $k < \mu(G)$. Let X be a totally cyclic orientation of G and $W \subseteq V(G)$. By Berge [Be, Page 11], $\omega_G(W)$ can be partitioned into edge disjoint elementary cuts $\omega_G(W_1), \ldots, \omega_G(W_r)$. Then $|\omega_G^+(W_i)| \le \mu(G)$ and $|\omega_G^+(W_i)\cap X| \ge 1$ for $i = 1, \ldots, r$. Thus $|\omega_G^+(W)|/|\omega_G^+(W)\cap X| \le \mu(G)$ and $I_X(\mu(G)) > 0$ by Corollary 5. □

Corollary 10. *Let G be a bridgeless connected graph, $E(G) \ne \emptyset$, and $k \ge \mu(G)-1$. Then $F_G(k+1) \ge F_G(k) + \mu(G) - 1$ and $I_G(k+1) \ge I_G(k) + (\mu(G)-1)(m(G)+1)$.*

Proof. Follows from Theorem 3, Lemma 11, Corollaries 6, 9 and (1), (2). □

Lemma 11. *Every bridgeless graph has at least $2^{m(G)}$ totally cyclic orientations.*

Proof. The statement holds if $m(G) = 0$. Let $m(G) > 0$ and e be an edge of G. Deleting all bridges from $G - e$ we get a bridgeless graph G' satisfying $m(G') = m(G) - 1$. Every totally cyclic orientation of G' can be extended into two totally cyclic orientations of G. Thus the statement holds by induction on $m(G)$. □

Corollary 11. *Let G be a bridgeless connected graph, $E(G) \ne \emptyset$, and $k \ge \mu(G)-1$. Then $I_G(k+1) \ge I_G(k) + 2^{m(G)}$.*

Proof. Follows from Corollary 7, Theorem 8, Lemma 11, and (1). □

Minor of a graph G is every graph we get from G after deleting and contructing some of its edges.

Corollary 12. *If H is a minor of G and $r \ge \min\{\mu(G), 6\}$, then $F_H(r) > 0$.*

Proof. Follows from Seymour [S1], Theorem 8 and the fact that $\mu(H) \le \mu(G)$. □

If G is a planar graph and G^* the dual of G, then every elementary cut in G corresponds to a circuit in G^* and vice versa. Furthermore, G^* is hamiltonian if and only if $\mu(G) = m(G) + 1$ (as pointed out before, $\mu(G) \leq m(G) + 1$ for every graph G). For this reason we call the graphs satisfying $\mu(G) = m(G) + 1$ *cohamiltonian.* The graphs discussed in Example 6 are cohamiltonian. There are NP-complete problems to decide whether a planar graph is hamiltonian or whether a planar graph has a circuit of length at least n (see [GJ]). Thus there are NP-complete problems to decide whether a (planar) graph is cohamiltonian or whether a (planar) graph has an elementary cut of cardinality at least n.

Since it is NP-hard to evaluate $\mu(G)$, it is interesting to have estimates of $\mu(G)$ by other parameters, which are easy to compute. We have $\mu(G) \leq m(G) + 1$. If G is 2-connected and $\Delta(G) = \max_{v \in V(G)} |\omega_G^+(v)|$, then $\Delta(G) \leq \mu(G)$. For example, $\Delta(H_n) = \mu(H_n) = m(H_n) + 1 = n$. If $C_{r,n}$ is the graph arising from the circuit of length n after replacing each edge by r parallel edges, then $\Delta(C_{r,n}) = \mu(C_{r,n}) = 2r$ and $m(C_{r,n}) = n(r-1) + 1$.

8. Sets of flows

In this section we summarize some observations about flows. Let $\mathbf{P}_G$ and $\mathbf{I}_{G,k}$ denote the set of $\mathbb{Z}$-flows and $\mathbb{Z}_k$-flows in G, respectively. Considering the sum of flows φ and φ' as $(\varphi + \varphi')(x) = \varphi(x) + \varphi'(x)$, $x \in D(G)$, we get that $\mathbf{P}_G$ and $\mathbf{I}_{G,k}$ are Abelian groups. Define homomorphism $\mathfrak{f}_{G,k} : \mathbf{P}_G \to \mathbf{I}_{G,k}$ so that $(\mathfrak{f}_{G,k}(\varphi))(x) = [\varphi(x)]_k$ $(x \in D(G))$. Let $\pi_{G,k}$ denote the congruence relation on $\mathbf{P}_G$ defined by $\varphi \equiv \varphi'(\pi_{G,k})$ if $\mathfrak{f}_{G,k}(\varphi) = \mathfrak{f}_{G,k}(\varphi')$. Recall that $\sigma(\varphi)$ denotes the support of φ.

Lemma 12. *Let $\varphi, \varphi' \in \mathbf{P}_G$. Then $\sigma(\varphi) \cap \sigma(\varphi') \subseteq \sigma(\varphi + \varphi')$ and $\sigma(\varphi) \subseteq \sigma(\mathfrak{f}_{G,k}(\varphi))$.*

Proof. If $\varphi(x) = \varphi'(x) = 0$, then $(\varphi + \varphi')(x) = 0$, and similarly, if $\varphi(x) = 0$, then $(\mathfrak{f}_{G,k}(\varphi))(x) = [\varphi(x)]_k = 0$ $(x \in D(G))$. □

Let $\mathbf{R}_{G,k}$, $\mathbf{T}_{G,k}$ and $\mathbf{J}_{G,k}$ denote the sets of k-flows, nowhere-zero k-flows and nowhere-zero $\mathbb{Z}_k$-flows in G, respectively. These sets are not closed under the addition, but $\mathbf{T}_{G,k} \subseteq \mathbf{R}_{G,k} \subseteq \mathbf{P}_G$ and $\mathbf{J}_{G,k} \subseteq \mathbf{I}_{G,k}$. Let $\rho_{G,k}$ and $\theta_{G,k}$ denotes the equivalence relations arising as restrictions of $\pi_{G,k}$ into $\mathbf{R}_{G,k}$ and $\mathbf{T}_{G,k}$, respectively (note that $\theta_{G,k}$ is identical with $\theta_{G,k}$ from Section 2). In other words, $[\varphi]\rho_{G,k} = [\varphi]\pi_{G,k} \cap \mathbf{R}_{G,k}$ and $[\varphi']\theta_{G,k} = [\varphi']\pi_{G,k} \cap \mathbf{T}_{G,k}$ for every $\varphi \in \mathbf{R}_{G,k}$ and $\varphi' \in \mathbf{T}_{G,k}$, respectively.

Lemma 13. *If $\varphi \in \mathbf{R}_{G,k}$, then $\sigma(\mathfrak{f}_{G,k}(\varphi)) = \sigma(\varphi) = \sigma(\varphi')$ for every $\varphi' \in [\varphi]\rho_{G,k}$.*

Proof. Since $-k < \varphi'(x), \varphi(x) < k$ for every $x \in D(G)$, then the equalities $\varphi'(x) = 0$, $\varphi(x) = 0$ and $[\varphi(x)]_k = 0$ are pairwise equivalent. □

Lemma 14. *If $\varphi \in \mathbf{T}_{G,k}$, then $\mathfrak{f}_{G,k}(\varphi) \in \mathbf{J}_{G,k}$ and $[\varphi]\rho_{G,k} = [\varphi]\theta_{G,k}$.*

Proof. Follows directly from Lemma 13. □

Now we can generalize Theorem 2.

Theorem 9. *The mapping* $[\varphi]\rho_{G,k} \mapsto \mathfrak{f}_{G,k}(\varphi)$ *is a one-to-one correspondence between the* $\rho_{G,k}$*-classes and the elements of* $\mathbf{I}_{G,k}$*. Furthermore, this mapping preserves the supports of flows, i. e.,* $\sigma(\mathfrak{f}_{G,k}(\varphi)) = \sigma(\varphi) = \sigma(\varphi')$ *for every* $\varphi' \in [\varphi]\rho_{G,k}$.

Proof. Use Lemma 13 and apply Theorem 2 for the spanning subgraph H of G so that $E(H) = \sigma(\varphi)$. □

We can also reformulate Theorem 2 in the following form:

Theorem 10. *The mapping* $[\varphi]\theta_{G,k} \mapsto \mathfrak{f}_{G,k}(\varphi)$ *is a one-to-one correspondence between the* $\theta_{G,k}$*-classes and the elements of* $\mathbf{J}_{G,k}$.

Theorem 11. *The homomorphism* $\mathfrak{f}_{G,k}$ *is surjective.*

Proof. Follows directly from Theorem 9. □

Thus Theorem 2 is in fact a special (but the substantial) part of Theorem 9. By Theorem 11, $\mathbf{P}_G/\pi_{G,k}$ and $\mathbf{I}_{G,k}$ are isomorphic under the isomorphism $[\varphi]\pi_{G,k} \mapsto \mathfrak{f}_{G,k}(\varphi)$.

For every $X \in \mathcal{C}(G)$, let $\mathbf{T}_{X,k}$ denote the set of the nowhere-zero k-flows in G with positive orientation X and $\mathfrak{f}_{X,k}$ the restriction of $\mathfrak{f}_{G,k}$ to the set $\mathbf{T}_{X,k}$. For every $\mathcal{X} \in \mathcal{C}_\Theta(G)$, let $\mathbf{J}_{\mathcal{X},k}$ denote the set of nowhere-zero $\mathbb{Z}_k$-flows ψ in G so that $\mathcal{X}_\psi = \mathcal{X}$. Then from Lemma 4 and Corollary 4 follows that $\mathfrak{f}_{X,k}$ is a bijection $\varphi \mapsto [\varphi]_k$ from $\mathbf{T}_{X,k}$ to $\mathbf{J}_{[X]\Theta_G,k}$ and if $X \equiv X'(\Theta_G)$ then $\mathfrak{f}_{X,k} \circ \mathfrak{f}^{-1}_{X',k}$ is the mapping $\varphi \mapsto \varphi - k\varphi_{X \setminus X'}$.

Let $\mathbf{P}'_G$ be the set of the $\mathbb{R}$-flows in G. Then $\mathbf{P}'_G$ is a vector space over $\mathbb{R}$ and $\mathbf{P}_G$ is a module over $\mathbb{Z}$. By Berge [Be], their dimensions are equal to $m(G)$ and their basis can be found using Lemma 7. This also shows that the parameter $m(G)$ from Lemma 7 cannot be increased.

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