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Sublinear defect principle in graph theory

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SUBLINEAR DEFECT PRINCIPLE IN GRAPH THEORY

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Abstract. We study a phenomenon consisting in the fact that many problems about graphs is enough to solve with certain errors if their number grows asymptotically slower than the order of graphs. For example, the four color theorem is equivalent with the statement that the vertices of every 5-connected planar graph of order n can be assigned four numbers so that at most b_n couples of neighboring vertices receive the same number, where $\{b_n\}$ is a sequence of nonnegative integers satisfying $\liminf_{n\to\infty} b_n/n = 0$. A similar principle holds for the Thomassen's conjecture on hamiltonian cycles in 4-connected line graphs, the conjecture of Fleischner suggesting that every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle, the nowhere-zero 5- and 3-flow problems of Tutte and the circular flow conjecture of Jaeger. Furthermore, the last two problems remain to verify for hamiltonian graphs.

1. Introduction

In this paper we study a phenomenon which can be observed by many very interesting problems about graphs. We show that they can be solved with certain approximation, or, in other words, allowing certain errors. The main principle is that the number of the errors must be, in a certain sense, asymptotically smaller than the order of graphs. For example, if we want to prove the four color theorem, we do not need to show that every planar graph is 4-colorable. It is enough to color its vertices by five colors, but the number of the vertices colored by the fifth color must be asymptotically smaller than the order of the graph. We can also allow coloring of vertices by four colors so that some couples of neighboring vertices are colored by the same color, but the number of such couples must be asymptotically smaller than the order of the graph.

A graph has a nowhere-zero k-flow if its edges can be oriented and assigned values $1, \dots, k-1$ so that the sum of the values of the incoming edges is equal to the sum of the outcoming ones for every vertex of the graph. The 5-flow (3-flow) conjecture of Tutte [Tu2] suggests that every bridgeless (4-edge-connected) graph has a nowhere-zero 5-flow (3-flow). We show that this conjecture is equivalent with the statement that every such graph has a flow with positive integral values such that the number of the edges with values ≥ 5 (≥ 3) is asymptotically smaller than the order of the graph. Similar principle holds for the circular flow conjecture of Jaeger [J2]. Furthermore we prove that the 3-flow and the circular flow conjectures remain to verify for hamiltonian graphs.

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We also deal with problems about cycles in graphs. For example, the conjecture of Thomassen [Th1], [Th2] suggests that every 4-connected line graph is hamiltonian. We show that to prove the conjecture it remains to cover the vertices of such graphs by vertex disjoint paths so that the number of the paths growths asymptotically slower than the order of the graphs. Similar property holds for a conjectures of Fleischner [F1], [F2] suggesting that every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle (which is also equivalent with the Thomassen's conjecture, by [K7]) and some other relative problems surveyed in Section 4.

Note that a similar property have been studied for hamiltonicity of 2-tough graphs in [BBHV] and for the Barnette's conjecture in [RW].

2. The general principle

Suppose $\mathcal{G}$ is a class of graphs with the following properties. There exist constants $c, d \in \mathbb{Z}^+$ (the set of positive integers) so that $\mathcal{G}$ contains only graphs of orders cn, $n \in \mathbb{Z}^+$, and $\mathcal{G}$ has at least one graph of order cn for every $n \geq d$. Number c is also called the *quantitative characteristic* of the class $\mathcal{G}$. If α is a real valued function on $\mathcal{G}$, then the pair $(\mathcal{G}, \alpha)$ is called a *graph parametric pair.*

Suppose there exists a sequence of real numbers $\{b_n\}$ so that $b_n \geq \alpha(G)$ for every graph G from $\mathcal{G}$ of order cn and $n \in \mathbb{Z}^+$. In this case we say that $\{b_n\}$ is a *bound* of α on $\mathcal{G}$. Furthermore, the sequence $\{b_n\}$ is called *zero* (resp. *sublinear*) if $b_n = 0$ for every $n \in \mathbb{Z}^+$ (reps. $b_n \geq 0$ for every $n \in \mathbb{Z}^+$ and $\liminf_{n\to\infty} b_n/n = 0$). Let $(\mathcal{G}, \alpha)_0$ (resp. $(\mathcal{G}, \alpha)_{\mathrm{s}}$) denote the statement "$\alpha$ has a zero (resp. sublinear) bound on $\mathcal{G}$".

For example, if $\tilde{\mathcal{G}}$ is the class of the cubic (3-regular) graphs (it has quantitative characteristic 2) and $\tilde{\alpha}(G) = 0$ for every $G \in \tilde{\mathcal{G}}$, then $(\tilde{\mathcal{G}}, \tilde{\alpha})$ is a graph parametric pair, and $\tilde{\alpha}$ has zero and sublinear bounds on $\tilde{\mathcal{G}}$.

Many problems about graphs can be expressed in the form $(\mathcal{G}, \alpha)_0$, where $(\mathcal{G}, \alpha)$ is a graph parametric pair. But to prove this is usually very difficult. On the other hand we can often show that $(\mathcal{G}, \alpha)_0$ and $(\mathcal{G}, \alpha)_{\mathrm{s}}$ are equivalent, thus, if we want to prove the original problem, it is enough to verify a seemingly weaker statement $(\mathcal{G}, \alpha)_{\mathrm{s}}$.

Definition 2.1. We say that α has a *sublinear defect property on* $\mathcal{G}$ if $(\mathcal{G}, \alpha)_0$ and $(\mathcal{G}, \alpha)_{\mathrm{s}}$ are equivalent.

Let $(\mathcal{G}, \alpha)_0^*$ denote the statement "there exists a graph G from $\mathcal{G}$ so that $\alpha(G) > 0$" and $(\mathcal{G}, \alpha)_{\mathrm{s}}^*$ denote the statement "there exist $r, s, n_0 \in \mathbb{Z}^+$ so that for every integer $n \geq n_0$ there exists a graph G_n from $\mathcal{G}$ of order cn satisfying $\alpha(G_n) \geq (n - s)/r$".

Proposition 2.2. *$(\mathcal{G}, \alpha)_0^*$ holds iff α has no zero bound on $\mathcal{G}$.*

Proof. Trivial. □

Proposition 2.3. *If $(\mathcal{G}, \alpha)_{\mathrm{s}}^*$ holds, then α has no sublinear bound on $\mathcal{G}$.*

Proof. If $(\mathcal{G}, \alpha)_{\mathrm{s}}^*$ holds, and $\{b_n\}$ is a bound of α on $\mathcal{G}$, then $\liminf_{n\to\infty} b_n/n \geq \liminf_{n\to\infty} \alpha(G_n)/n \geq \lim_{n\to\infty}(n - s)/(rn) = 1/r > 0$ and $\{b_n\}$ is not sublinear. □

Note that using the countable axiom of choice we can prove that $(\mathcal{G}, \alpha)_{\mathrm{s}}^*$ holds iff α has no sublinear bound on $\mathcal{G}$.

Definition 2.4. We write $(\mathcal{G},\alpha) \approx (\mathcal{G}',\alpha')$ if $(\mathcal{G},\alpha)_{\mathrm{s}}$ implies $(\mathcal{G}',\alpha')_0$ and $(\mathcal{G}',\alpha')_{\mathrm{s}}$ implies $(\mathcal{G},\alpha)_0$.

Clearly α has the sublinear defect property on $\mathcal{G}$ iff $(\mathcal{G},\alpha) \approx (\mathcal{G},\alpha)$.

Proposition 2.5. *$(\mathcal{G},\alpha) \approx (\mathcal{G}',\alpha')$ holds iff the statements $(\mathcal{G},\alpha)_0$, $(\mathcal{G},\alpha)_{\mathrm{s}}$, $(\mathcal{G}',\alpha')_0$ and $(\mathcal{G}',\alpha')_{\mathrm{s}}$ are pairwise equivalent. In particular, $(\mathcal{G},\alpha) \approx (\mathcal{G}',\alpha')$ implies that α (resp. α') has the sublinear defect property on $\mathcal{G}$ (resp. $\mathcal{G}'$).*

Proof. Follows from Definition 2.4 and the fact that every zero bound is also sublinear, i. e., $(\mathcal{G},\alpha)_0$ and $(\mathcal{G}',\alpha')_0$ imply $(\mathcal{G},\alpha)_{\mathrm{s}}$ and $(\mathcal{G}',\alpha')_{\mathrm{s}}$, respectively. □

Proposition 2.6. *If $(\mathcal{G},\alpha) \approx (\mathcal{G}',\alpha')$ and $(\mathcal{G}',\alpha') \approx (\mathcal{G}'',\alpha'')$, then $(\mathcal{G},\alpha) \approx (\mathcal{G}'',\alpha'')$.*

Proof. Follows immediately from Proposition 2.5. □

Proposition 2.7. *Let $(\mathcal{G},\alpha)$ and $(\mathcal{G}',\alpha')$ be graph parametric pairs where $\mathcal{G}'$ be a subclass of $\mathcal{G}$, both have the same quantitative characteristic, and there exist a positive constant p so that $\alpha'(G) \le p\alpha(G)$ for every $G \in \mathcal{G}$. Suppose $(\mathcal{G}',\alpha')_{\mathrm{s}}$ implies $(\mathcal{G},\alpha)_0$. Then $(\mathcal{G},\alpha) \approx (\mathcal{G},\alpha') \approx (\mathcal{G}',\alpha) \approx (\mathcal{G}',\alpha')$.*

Proof. If $(\mathcal{G}',\alpha')_{\mathrm{s}}$ holds, then, by the assumptions, so does $(\mathcal{G},\alpha)_0$ and therefore $(\mathcal{G},\alpha')_0$ and $(\mathcal{G}',\alpha)_0$ hold as well. On the other hand if one of $(\mathcal{G},\alpha)_{\mathrm{s}}$, $(\mathcal{G},\alpha')_{\mathrm{s}}$, and $(\mathcal{G}',\alpha)_{\mathrm{s}}$ holds, then from the properties of α, α', $\mathcal{G}$, and $\mathcal{G}'$ follows that $(\mathcal{G}',\alpha')_{\mathrm{s}}$ holds, what again implies $(\mathcal{G},\alpha)_0$, thus also $(\mathcal{G}',\alpha')_0$. Therefore $(\mathcal{G}',\alpha') \approx (\mathcal{G},\alpha)$, $(\mathcal{G},\alpha')$, $(\mathcal{G}',\alpha)$, and the statement holds by Proposition 2.6. □

Proposition 2.8. *Suppose $(\mathcal{G},\alpha)_0^*$ implies $(\mathcal{G}',\alpha')_{\mathrm{s}}^*$. Then $(\mathcal{G}',\alpha')_{\mathrm{s}}$ implies $(\mathcal{G},\alpha)_0$.*

Proof. Follows from Propositions 2.2 and 2.3. □

Thus, if we want to prove that α has the sublinear defect property on $\mathcal{G}$, it remains to show that $(\mathcal{G},\alpha)_0^*$ implies $(\mathcal{G}',\alpha')_{\mathrm{s}}^*$ where $\mathcal{G}'$ is a subclass of $\mathcal{G}$ and $\alpha'(G) \le p\alpha(G)$ for every $G \in \mathcal{G}$ and a positive constant p.

Proposition 2.9. *Suppose α has the sublinear defect property on $\mathcal{G}$ and there exists a graph G from $\mathcal{G}$ with $\alpha(G) > 0$. Then there exist infinitely many such graphs.*

Proof. If there is only finitely many such graphs, then α has a sublinear bound on $\mathcal{G}$, thus, by the assumptions, also a zero bound, what contradicts Proposition 2.2. □

A sequence of real numbers $\{b_n\}$ is called *strictly sublinear* if $b_n \ge 0$ for every $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} b_n/n = 0$. Practically everything what we have done until now can be reformulated so that the world "sublinear" is replaced by the words "strictly sublinear". Only Proposition 2.3 must be changed in the flowing way: "α has no strictly sublinear bound on $\mathcal{G}$ if there exist $r \in \mathbb{Z}^+$ and a strictly increasing sequence of positive integers $\{n_i\}$ so that for every $i \in \mathbb{Z}^+$ there exists a graph G_i from $\mathcal{G}$ of order n_i satisfying $\alpha(G_i) \ge n_i/r$." All graph parametric pairs $(\mathcal{G},\alpha)$ considered in this paper will have the sublinear defect property. Anyway, if somebody shows that α has a strictly sublinear bound on $\mathcal{G}$, then this bound is also sublinear.

3. Some basic notations

In this paper we allow multiple edges and loops, unless otherwise stated. An edge cut of a graph is called *cyclic* if after deleting its edges we get at least two components having cycles. A graph is called *cyclically k-edge-connected* if it does not have a cyclic edge cut of cardinality smaller than k.

We recall some notation from [K4], [K5], [K6]. By an *abstract network*, or simply a *network*, we mean a pair $\mathcal{N} = (G, U)$ where G is a graph and $U \subseteq V(G)$. The vertices from U and $V(G) - U$ are called *outer* and *inner* vertices of $\mathcal{N}$, respectively. For technical reasons, we shall deal with networks where the set U is partitioned into nonempty sets $U_1, \dots, U_n$. In this case we write $\mathcal{N} = (G, U_1, \dots, U_n)$ and call $\mathcal{N}$ the *partitioned network*. Sets $U_1, \dots, U_n$ are called *connectors* of $\mathcal{N}$.

Let $\mathcal{N} = (G, U)$ be a network. Any path $v_1 \dots v_n$ in G is called a *v_1-v_n-path*. Furthermore, if $v_1, v_n \in U$, $v_1 \neq v_2$, then the path is called *open* in $\mathcal{N}$. By a *k-polygon* in $\mathcal{N}$ (briefly a *polygon*) we mean k vertex disjoint open paths in $\mathcal{N}$.

Furthermore, if $\mathcal{N}$ is partitioned, then an open v_1-v_n-path in $\mathcal{N}$ is called *crossing* if the vertices v_1 and v_n do not belong to the same connector. A polygon in $\mathcal{N}$ is called *crossing* if it consists of crossing paths.

4. Paths and cycles

A set of pairwise vertex disjoint subgraphs $\{G_1, \dots, G_k\}$ of a graph G is called *dominating* if every edge of G is incident with at least one vertex from $G_1, \dots, G_k$. If $k = 1$, then we say that the graph G_1 is *dominating* in G.

Fleischner [F1], [F2] conjectured that every cyclically 4-edge-connected cubic graph has either a 3-edge-coloring or a dominating cycle. A seemingly stronger problem was given by Ash and Jackson [AJ] suggesting that the condition of 3-edge-colorability can be skipped. In [K7] we have shown that these two problems are equivalent.

Denote by $\mathcal{G}_{3,k}$ the class of cyclically k-edge-connected cubic graphs and by $\mathcal{G}'_{3,k}$ the class of such graphs without 3-edge-colorings. $\mathcal{G}_{3,4}$ and $\mathcal{G}'_{3,4}$ have quantitative characteristics 2, as follows, for example, from constructions presented in [K1], [K3], [K5]. Let $\alpha_1(G)$ (resp. $\alpha'_1(G)$) denote the minimal number n so that the vertices of G can be covered by $n+1$ vertex disjoint cycles (resp. paths) and $\alpha_2(G)$ (resp. $\alpha'_2(G)$) denote the minimal number n so that G has $n+1$ dominating cycles (resp. paths). An isolated vertex is considered as a cycle (resp. path) of length 0. Clearly $\alpha_2(G) \geq \alpha'_2(G)$ and the conjecture of Fleischner (resp. Ash and Jackson) suggests that $(\mathcal{G}'_{3,4}, \alpha_2)_0$ (resp. $(\mathcal{G}_{3,4}, \alpha_2)_0$).

By Fleischner and Jackson [FJ], the conjecture of Ash and Jackson is equivalent with the conjecture of Thomassen [Th1], [Th2], which suggests that every 4-connected line graph is hamiltonian. Furthermore, by Ryjáček [R], the latter problem is equivalent with the conjecture of Matthews and Sumner [MS] suggesting that every 4-connected claw-free graph is hamiltonian (a graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph). Let $\mathcal{G}_4$ (resp. $\mathcal{G}'_4$) denote the class of 4-connected claw free (resp. 4-connected line) graphs. $\mathcal{G}_4$ and $\mathcal{G}'_4$ have quantitative characteristics 1. The conjecture of Thomassen (resp. Mathews and Sumner) suggests that $(\mathcal{G}'_4, \alpha_1)_0$ (resp. $(\mathcal{G}_4, \alpha_1)_0$).

Let $L(G)$ denote the line graph of a graph G.

Lemma 4.1. *$\alpha'_1(L(G)) \geq \alpha'_2(G)$ for every cubic graph G.*

Proof. Take a family $\{P_1, \ldots, P_m\}$ of vertex disjoint paths covering $L(G)$ so that m is as small as possible. For $i = 1, \ldots, m$, let P_i be a u_i-v_i-path, and u'_i (v'_i) be a vertex from G incident with the edge corresponding to u_i (v_i). Add to G a new vertex v and $2m$ (possibly parallel) edges (v, u'_i), (v, v'_i), $i = 1, \ldots m$, obtaining a graph G'. The paths $P_1, \ldots, P_m$ can be extended into a hamiltonian cycle in $L(G')$, therefore, by Harary and Nash-Williams [HNW, Proposition 8], G' has an eulerian dominating subgraph H'. Then $H = H' - v$ is dominating in G (note that $G - V(H)$ and $G' - V(H')$ is the same graph). Let m_j be the number of the vertices of degree j in H ($j = 1, 2, 3$) and m_0 be the number of the components of H which are cycles (isolated vertices are considered to be cycles). Every cycle in H has at least one vertex joined with v by at least two edges from H' and every vertex of degree 1 or 3 in H is joined with v by at least one edge from H'. Thus $2m_0 + m_1 + m_3 \leq 2m$. Deleting from H at most m_3 edges adjacent with the vertices of degree 3 and at most m_0 edges from the cycles of H we get a dominating set of vertex disjoint paths from G of cardinality at most m, whence $\alpha'_1(L(G)) \geq \alpha'_2(G)$. $\square$

Theorem 4.2. $(\mathcal{G}_{3,4}, \alpha_2) \approx (\mathcal{G}_{3,4}, \alpha'_2) \approx (\mathcal{G}'_{3,4}, \alpha_2) \approx (\mathcal{G}'_{3,4}, \alpha'_2) \approx (\mathcal{G}_4, \alpha_1) \approx (\mathcal{G}_4, \alpha'_1) \approx (\mathcal{G}'_4, \alpha_1) \approx (\mathcal{G}'_4, \alpha'_1)$.

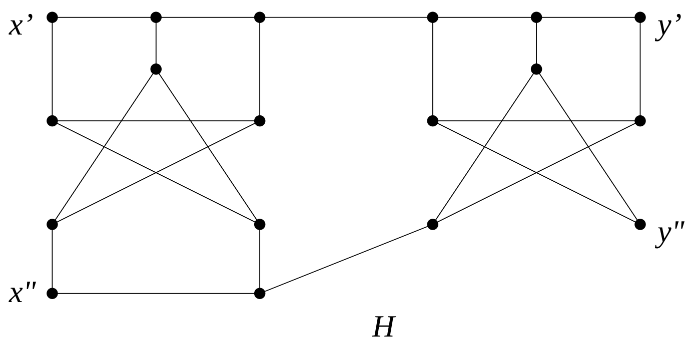

FIG. 1

Proof. Suppose there exists a graph G from $\mathcal{G}_{3,4}$ without a dominating cycle. Let $e = (u, v)$ be an edge in G and u', u'' (resp. v', v'') be the vertices neighboring u (resp. v) and not incident with e. Delete from G the vertices u and v together with the incident edges, getting a graph F with just four vertices of degree two u', u'', v', v''. Take the partitioned network $\mathcal{M} = (F, \{u', u''\}, \{v', v''\})$. Since G does not have a dominating cycle, $\mathcal{M}$ does not have a dominating crossing polygon. Similarly, delete two neighboring vertices from an arbitrary graph G' from $\mathcal{G}_{3,4}$ obtaining a graph F' with just four vertices of degree two w', w'', z', z'', and take the network $\mathcal{M}' = (F', \{w', w''\}, \{z', z''\})$. For $i = 1, \ldots, 6n$ take a copy $\mathcal{M}_i = (F_i, \{u'_i, u''_i\}, \{v'_i, v''_i\})$ of $\mathcal{M}$, one copy of $\mathcal{M}'$, and one copy of the partitioned network $(H, \{x', x''\}, \{y', y''\})$ from Fig.1 and join their outer vertices by new edges (v'_{6n}, w'), (v''_{6n}, w''), (z', x'), (z'', x''), (y', u'_1), (y'', u''_1), and (v'_i, u'_{i+1}), (v''_i, u''_{i+1}), for $i = 1, \ldots, 6n - 1$. We obtain a graph F'', indicated in Fig. 2. $F'' \in \mathcal{G}_{3,4}$ and contains H as an induced subgraph, thus, by [K7, Lemma 2], $F'' \in \mathcal{G}'_{3,4}$. Let $\{P_1, \ldots, P_m\}$ be a dominating family of vertex disjoint paths in F''. Every dominating polygon in $\mathcal{M}$ must be either a u'-u''-path or a v'-v''-path or a 2-polygon consisting of a u'-u''- and a v'-v''-paths. Thus for every $i = 1, 4, 7, \dots, 6n-2$, the three consecutive graphs F_i, F_{i+1}, F_{i+2} contain at least one end vertex from the paths $P_1, \dots, P_m$. Therefore $m \geq n$, i. e., $\alpha_1'(F'') \geq n-1$. But F' can have any even order greater than a constant. Thus there exist integers r, s and n_0 so that for every integer $n \geq n_0$ there exists a graph G_n from $\mathcal{G}_{3,4}'$ of order $2n$ satisfying $\alpha_1'(G_n) \geq (n-s)/r$. Thus $(\mathcal{G}_{3,4}, \alpha_2)_0^*$ implies $(\mathcal{G}_{3,4}', \alpha_2')_s^*$ and, by Propositions 2.7 and 2.8, $(\mathcal{G}_{3,4}, \alpha_2) \approx (\mathcal{G}_{3,4}, \alpha_2') \approx (\mathcal{G}_{3,4}', \alpha_2) \approx (\mathcal{G}_{3,4}', \alpha_2')$.

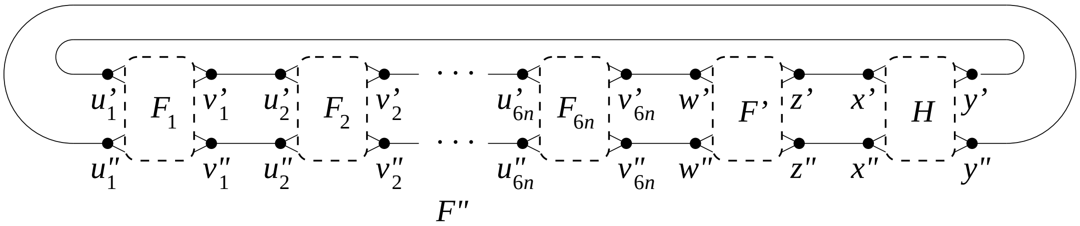

FIG. 2

By [FJ] and [R], the statements $(\mathcal{G}_{3,4}, \alpha_2)_0$, $(\mathcal{G}_4, \alpha_1)_0$, and $(\mathcal{G}_4', \alpha_1)_0$ are pairwise equivalent. Thus by Propositions 2.5–2.8, it remains to show that $(\mathcal{G}_4, \alpha_1)_0^*$ implies $(\mathcal{G}_4', \alpha_1')_s^*$. Let $(\mathcal{G}_4, \alpha_1)_0^*$ hold. Then, by [FJ], [R] and Proposition 2.2, also $(\mathcal{G}_{3,4}, \alpha_2)_0^*$ holds, whence, by the previous text, $(\mathcal{G}_{3,4}', \alpha_2')_s^*$ is satisfied, i. e., there are integers r, s, n_0 so that for every integer $n \geq n_0$ there exists a graph $G_n \in \mathcal{G}_{3,4}'$ of order $2n$ so that $\alpha_2'(G_n) \geq (n-s)/r$, and, by Lemma 4.1, $\alpha_1'(L(G_n)) \geq (n-s)/r$. Graph G_n has $3n$ edges. Let G_n' (resp. G_n'') be a graph arising from G_n after contracting one edge (resp. two edges having no vertex in common) into one vertex (resp. two vertices) of degree 4. We can check that $\alpha_1'(L(G_n'')) + 2, \alpha_1'(L(G_n')) + 1 \geq \alpha_1'(L(G_n))$ and that $L(G_n)$, $L(G_n')$, $L(G_n'')$ are 4-connected line graphs of orders $3n$, $3n-1$, $3n-2$, respectively (see also [FJ]). Thus, for every integer $k \geq 3n_0 - 2$, we have a graph $H_k \in \mathcal{G}_4'$ of order k so that $\alpha_1'(H_k) + 2 \geq (\lceil k/3 \rceil - s)/r$, i. e., $(\mathcal{G}_4', \alpha_1')_s^*$ holds. This implies the statement. $\square$

Proposition 2.5 and Theorem 4.2 imply the next theorem.

Theorem 4.3. *The following statements are pairwise equivalent.*

(1) *Every cyclically 4-edge-connected cubic graph without a 3-edge coloring has a dominating cycle.*
(2) *Every cyclically 4-edge-connected cubic graph has a dominating cycle.*
(3) *Every 4-connected line graph is hamiltonian.*
(4) *Every 4-connected claw-free graph is hamiltonian.*
(5) *There exists a sublinear sequence $\{b_n\}$ so that every cyclically 4-edge-connected cubic graph without a 3-edge coloring and of order $2n$ contains a family of at most b_n dominating paths.*
(6) *There exists a sublinear sequence $\{b_n\}$ so that every cyclically 4-edge-connected cubic graph of order $2n$ contains a family of at most b_n dominating paths.*

(7) *There exists a sublinear sequence $\{b_n\}$ so that the vertices of every 4-connected line graph of order n can be covered by at most b_n vertex disjoint paths.*
(8) *There exists a sublinear sequence $\{b_n\}$ so that the vertices of every 4-connected claw-free graph of order n can be covered by at most b_n vertex disjoint paths.*

Snark is a cyclically 4-edge-connected cubic graph with girth at least 5 and without a 3-edge-coloring. The girth of a graph is the length of its shortest cycle. It is also known that the class of snarks has quantitative characteristic 2 (see, e. g., [K5], [K1]). Note that the conjecture of Fleischner is sometimes presented in the form that every snark has a dominating cycle. The confusion is caused by the fact that some authors considered snarks as graphs from $\mathcal{G}'_{3,4}$ (see, e. g., [J4], [F1], [F2]). The "snark" variant of Fleischner's conjecture seems to be a few weaker than the original problem, because the restriction on cubic graphs with girth at most 5 is probably not trivial.

Denote by $\mathcal{G}_5$ the class of cyclically 4-edge-connected cubic graphs with girth ≥ 5 and by $\mathcal{G}'_5$ the class of snarks. Then the following statements can be proved similarly as the first part of Theorem 4.2.

Theorem 4.4. $(\mathcal{G}_5, \alpha_2) \approx (\mathcal{G}_5, \alpha'_2) \approx (\mathcal{G}'_5, \alpha_2) \approx (\mathcal{G}'_5, \alpha'_2)$.

In other words, conditions (1), (2), (5) and (6) from Theorem 4.3 remain equivalent if we consider only graphs with girth at least 5.

A set of pairwise vertex disjoint subgraphs $\{G_1, \dots, G_k\}$ of a graph G is called *cyclically dominating* if every cycle of G contains at least one vertex from $G_1, \dots, G_k$ (or, equivalently, deleting from G the vertices of the subgraphs $G_1, \dots, G_k$ we get a forest). If $k = 1$, then we say that the graph G_1 is *cyclically dominating* in G. Jaeger conjectured that every graph from $\mathcal{G}_{3,4}$ has a cyclically dominating cycle (see [FJ, Conjecture 2]). This is a weaker problem than that ones of Fleischner or Ash and Jackson. This conjecture, if true, implies the 6-flow theorem of Seymour [S], as pointed out by Jaeger (see [FJ]).

Let $\alpha_3(G)$ (resp. $\alpha'_3(G)$) denote the minimal number n so that G has $n+1$ cyclically dominating cycles (resp. paths). Clearly $\alpha_3(G) \geq \alpha'_3(G)$ and the Jaeger's conjecture suggests that $\alpha_3(G) = 0$ for every graph $G \in \mathcal{G}_{3,4}$. The following statement can be proved similarly as the first part of Theorem 4.2, replacing the words "dominating" by the words "cyclically dominating".

Theorem 4.5. $(\mathcal{G}_{3,4}, \alpha_3) \approx (\mathcal{G}_{3,4}, \alpha'_3) \approx (\mathcal{G}'_{3,4}, \alpha_3) \approx (\mathcal{G}'_{3,4}, \alpha'_3)$.

Thus conditions (1), (2), (5) and (6) from Theorem 4.3 remain equivalent if we replace the words "dominating" by the words "cyclically dominating".

An old conjecture of Barnette [B] suggests that every bipartite 3-connected cubic planar graph is hamiltonian. By Kelmans [Ke], this conjecture remains to verify for cyclically 4-edge-connected graphs. Let $\mathcal{B}$ denote the family of bipartite 3-connected cubic planar graphs and $\mathcal{B}'$ denote the family of bipartite cyclically 4-edge-connected cubic planar graphs. They have quantitative characteristics 2. The Barnette's conjecture in fact states that α_1 has a zero bound on $\mathcal{B}$. The following statement can be proved similarly as Theorem 4.1, and follows also from results of [RW].

Theorem 4.6. $(\mathcal{B}, \alpha_1) \approx (\mathcal{B}', \alpha'_1)$.

In other words, every bipartite 3-connected cubic planar graph is hamiltonian iff there exists a sublinear sequence $\{b_n\}$ so that the vertices of every bipartite cyclically 4-edge-connected cubic planar graph of order $2n$ can be covered by at most b_n vertex disjoint paths.

A well known Theorem of Tutte [Tu3] says that every 4-connected planar graph is hamiltonian. Let $\mathcal{P}_4$ denote the family of 4-connected planar graphs. $\mathcal{P}_4$ has quantitative characteristic 1.

Theorem 4.7. $(\mathcal{P}_4, \alpha_1) \approx (\mathcal{P}_4, \alpha'_1)$.

Proof. Suppose G is a nonhamiltonian graph from $\mathcal{P}_4$. Let w be a vertex of G and $w_1, \ldots, w_m$, $m \geq 4$, be the vertices neighboring w. Take the network $\mathcal{N}_1 = (G - w, \{w_1, \ldots, w_m\})$ and its copy $\mathcal{N}'_1 = (G' - w', \{w'_1, \ldots, w'_m\})$. Joining $G - w$ and $G' - w'$ by new edges $(w_3, w'_3), \ldots, (w_m, w'_m)$ we get a network $\mathcal{N}_2 = (H, \{w_1, w_2, w'_1, w'_2\})$ having no 1-polygon covering all their vertices. Deleting from G an edge $e = (x, y)$ and splitting the vertices x and y into two vertices x', x'' and y', y'' (having degrees at least 1), respectively, we get a graph H'. The partitioned network $\mathcal{N}_3 = (H', \{x', x''\}, \{y', y''\})$ does not have a crossing 2-polygon covering all vertices of H'. Adding to H and H' new vertices and edges as indicated in Fig. 3, we get a partitioned network $\mathcal{M} = (F, \{u', u''\}, \{v', v''\})$ which does not have a crossing polygon covering all vertices from F. The rest of the proof is similar as in Theorem 4.1. $\square$

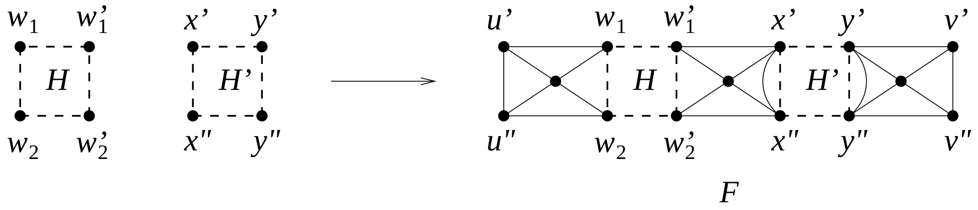

FIG. 3

5. THE 5-FLOW PROBLEM AND THE 4-COLOR THEOREM

We recall some notation from [K5]. Suppose G is a graph. Each edge e of G determines two opposite arcs arising from it after endowing e with two distinct orientations. All arcs obtained in this way are called *arcs* of G and the set of them is called the *arc set* of G and denoted by $D(G)$, i. e., $|D(G)| = 2|E(G)|$, where $E(G)$ denotes the edge set of G. If x is an arc of G, then we denote by $-x$ the opposite arc arising from the same edge. Clearly, $-(-x) = x$ for any arc x of G. Let $\omega^+(v)$ denote the set of arcs directed from v to $V(G) \setminus \{v\}$.

Let $\mathcal{N} = (G, U)$ be a network and A be an Abelian group (we deal only with additive Abelian groups). An *A-chain* in $\mathcal{N}$ is a mapping $\varphi \colon D(G) \to A$ so that $\varphi(-x) = -\varphi(x)$ for any $x \in D(G)$. Given an A-chain φ we define its *boundary* $\partial\varphi \colon V(G) \to A$ as

$$\partial\varphi(v) = \sum_{x \in \omega^+(v)} \varphi(x).$$

An A-chain in $\mathcal{N}$ is an *A-flow* if $\partial\varphi(v) = 0$ for every inner vertex v of $\mathcal{N}$. Furthermore, φ is *nowhere-zero* if $\varphi(x) \neq 0$ for every $x \in D(G)$. In [K4], [K5] was proved the following.

Lemma 5.1. *If φ is an A-flow in a network (G, U), then $\sum_{v \in U} \partial\varphi(v) = 0$.*

A (nowhere-zero) $\mathbb{Z}$-flow φ in $\mathcal{N}$ is called a *(nowhere-zero) k-flow* if $|\varphi(x)| < k$ for every $x \in D(G)$ (in [K5] we call it (nowhere-zero) unlimited k-flow).

A graph G is identified with the network $(G, \emptyset)$ in this and the next chapters. Thus, by a *(nowhere-zero) A-flow* or *k-flow* in a graph G we mean a (nowhere-zero) A-flow or k-flow, respectively, in the network $(G, \emptyset)$. This concept is also used in [K4], [K5] and it coincides with the usual definition of nowhere-zero A- and k-flows in graphs as presented in Jaeger [J4] (see [K5] for more details). In [K5, Theorem 2.5] we have generalized the classical equivalence results of Tutte [Tu4] and proved the following.

Theorem 5.2. *If (G, U) is a network, then the following statements are equivalent.*

1. *(G, U) has a nowhere-zero $\mathbb{Z}_k$-flow.*
2. *(G, U) has a nowhere-zero A-flow for any Abelian group A of order k.*
3. *(G, U) has a nowhere-zero k-flow.*

Clearly, if a network has a nowhere-zero k-flow, then it has a nowhere-zero $(k+1)$-flow. Nowhere-zero flows in graphs present a dual concept to the coloring problems, as follows from the classical results of Tutte [Tu2] (see also Zhang [Z]).

Theorem 5.3. *If G is a planar graph, then the following statements are equivalent.*

1. *G has a nowhere-zero k-flow.*
2. *G is k-face-colorable.*
3. *The dual graph of G is k-vertex-colorable.*

By a *k-reluctance* of a network (G, U), $\rho_k(G, U)$, we mean the minimal integer n so that there exists an n element subset W of $V(G)$ with the property that the network $(G, U \cup W)$ has a nowhere-zero k-flow. This parameter was introduced in [K5].

By a *k-overlap* of a network (G, U), $\sigma_k(G, U)$, we mean the minimal integer n so that there exists a $\mathbb{Z}$-flow φ in (G, U) with just $2n$ arcs satisfying $|\varphi(x)| \neq 1, \ldots, k-1$.

Clearly, a network (G, U) has a nowhere-zero k-flow iff $\rho_k(G, U) = \sigma_k(G, U) = 0$.

Lemma 5.4. *$2\sigma_k(G, U) \geq \rho_k(G, U)$ for each network (G, U) and $k \geq 2$.*

Proof. Take a nowhere-zero $\mathbb{Z}$-flow φ in (G, U) with $2n = 2\sigma_k(G, U)$ arcs satisfying $|\varphi(x)| \neq 1, \ldots, k-1$. Denote the set of these arcs by D' and the set of the vertices incident to the arcs from D' by W. Then $|D'| = 2n$ and $|W| \leq 2n$. Changing the values of the arcs from D' to 1 or -1 we get a nowhere-zero k-flow in $(G, U \cup W)$. Thus $\rho_k(G, U) \leq |W| \leq 2n = 2\sigma_k(G, U)$. $\square$

It is well known that a graph with a bridge does not have a nowhere-zero k-flow for every $k \geq 2$. On the other hand the 5-flow conjecture of Tutte [Tu2] suggests that every bridgeless graph has a nowhere-zero 5-flow. (By Seymour [S], every such graph admits a nowhere-zero 6-flow.) It is known that this conjecture remains to verify for cubic graphs (see cf. [J4]).

Let $\mathcal{G}_3$ denote the class of bridgeless cubic graphs. $\mathcal{G}_3$ and $\mathcal{G}_{3,5}$ (the class of cyclically 5-edge-connected cubic graphs – see Section 4) have quantitative characteristics 2.

Theorem 5.5. $(\mathcal{G}_3, \rho_5) \approx (\mathcal{G}_{3,5}, \rho_5) \approx (\mathcal{G}_3, \sigma_5) \approx (\mathcal{G}_{3,5}, \sigma_5)$.

Proof. By [K5, Theorem 14.2], $(\mathcal{G}_3, \rho_5)_0^*$ implies $(\mathcal{G}_{3,5}, \rho_5)_{\mathrm{s}}^*$. Thus the statement holds by Propositions 2.5–2.8 and Lemma 5.4. □

This theorem and Proposition 2.5 imply the following.

Theorem 5.6. *The following statements are pairwise equivalent.*

(1) *Every bridgeless graph has a nowhere-zero* 5*-flow.*
(2) *There exists a sublinear sequence* $\{b_n\}$ *so that every cyclically* 5*-edge-connected cubic graph of order* $2n$ *has* 5*-reluctance at most* b_n.
(3) *There exists a sublinear sequence* $\{b_n\}$ *so that every cyclically* 5*-edge-connected cubic graph of order* $2n$ *has* 5*-overlap at most* b_n.

By Theorem 5.3, the four color theorem, proved by Appel and Haken [AH], is equivalent with the statement that every bridgeless planar graph has a nowhere-zero 4-flow. It is also known (see e. g., [J4]) that it is enough to restrict our attention to cubic graphs. Let us note that the five color theorem (every planar graph is 5-vertex-colorable) has much simpler proofs (see Heawood [H] and Thomassen [Th3]).

Suppose G is a bridgeless planar graph. Let $\beta_k(G)$ denote the minimal number n so that there exists an r-face-coloring of G, $r > k$, where the number of the faces colored by colors $k+1, \dots, r$ is just n. Let $\gamma_k(G)$ denote the minimal number n so that the faces of G can be assigned numbers $1, \dots, k$ in such a way that there exist just n couples of faces assigned the same number and having an edge in common. The four color theorem says that $\beta_4(G) = \gamma_4(G) = 0$ for every planar graph G.

A cyclic k-edge cut is called *simple* if after deleting its edges we get a component equal to a cycle of length k. By a *simple* 5*-cut* graph we mean a cyclically 5-edge-connected graph where every 5-edge cut is simple. Let $\mathcal{P}_3$ denote the class of bridgeless cubic planar graphs and $\mathcal{P}_{3,5}$ denote the class of simple 5-cut cubic planar graphs. We can check that $\mathcal{P}_3$ and $\mathcal{P}_{3,5}$ have quantitative characteristics 2.

Theorem 5.7. $(\mathcal{P}_3, \rho_4) \approx (\mathcal{P}_{3,5}, \rho_4) \approx (\mathcal{P}_3, \sigma_4) \approx (\mathcal{P}_{3,5}, \sigma_4) \approx (\mathcal{P}_3, \beta_4) \approx (\mathcal{P}_{3,5}, \beta_4) \approx (\mathcal{P}_3, \gamma_4) \approx (\mathcal{P}_{3,5}, \gamma_4)$.

Proof. By [K5, Remark 15.1], $(\mathcal{P}_3, \rho_4)_0^*$ implies $(\mathcal{P}_{3,5}, \rho_4)_{\mathrm{s}}^*$. Thus, by Propositions 2.5–2.8 and Lemma 5.4, $(\mathcal{P}_3, \rho_4) \approx (\mathcal{P}_{3,5}, \rho_4) \approx (\mathcal{P}_3, \sigma_4) \approx (\mathcal{P}_{3,5}, \sigma_4)$.

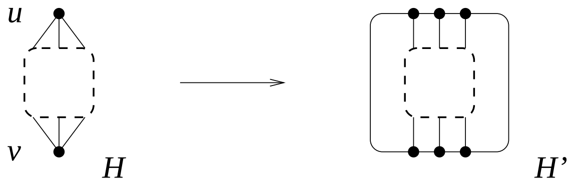

FIG. 4

Clearly, the statements $(\mathcal{P}_3, \rho_4)_0$, $(\mathcal{P}_3, \sigma_4)_0$, $(\mathcal{P}_3, \beta_4)_0$, and $(\mathcal{P}_3, \gamma_4)_0$ are pairwise equivalent. Thus, by Propositions 2.5–2.8, it remains to show that $(\mathcal{P}_3, \rho_4)_0^*$ implies $(\mathcal{P}_{3,5}, \beta_4)_{\mathrm{s}}^*$ and $(\mathcal{P}_{3,5}, \sigma_4)_{\mathrm{s}}^*$. Suppose $G \in \mathcal{P}_3$ so that $\rho_4(G) > 0$. Then, by [K5, Theorem 15.1], there exists $H \in \mathcal{P}_{3,5}$ so that $\rho_4(H) > 2$. Let H be depicted in the plane. Then its outer face has two vertices u and v having no edge in common. Split the vertices u and v into three vertices of degree 1 and join them by a new cycle of length 6 as indicated in Fig. 4, obtaining a planar graph H'. Then by every r-face-coloring of H', $r > 4$, at least one inner face of H' receives one of the colors $5, \ldots, r$ (if not, then either H or $H + (u, v)$ is 4-face-colorable and, by Theorem 5.3, either H or $H + (u, v)$ have a nowhere-zero 4-flow, i. e., $\rho_4(H) \leq 2$ – a contradiction). Take $n+1$ copies of H' and join them as indicated in Fig. 5. Furthermore, we can "replace" one copy of H' by an arbitrary graph from $\mathcal{P}_{3,5}$, obtaining a graph H_n, which is from $\mathcal{P}_{3,5}$. Then $\beta_4(H_n) \geq n$ and similarly as in Theorem 4.1 we get that $(\mathcal{P}_{3,5}, \beta_4)^*_{\mathrm{s}}$ holds. In the same way we can check that $(\mathcal{P}_{3,5}, \gamma_4)^*_{\mathrm{s}}$ holds. This proves the statement. □

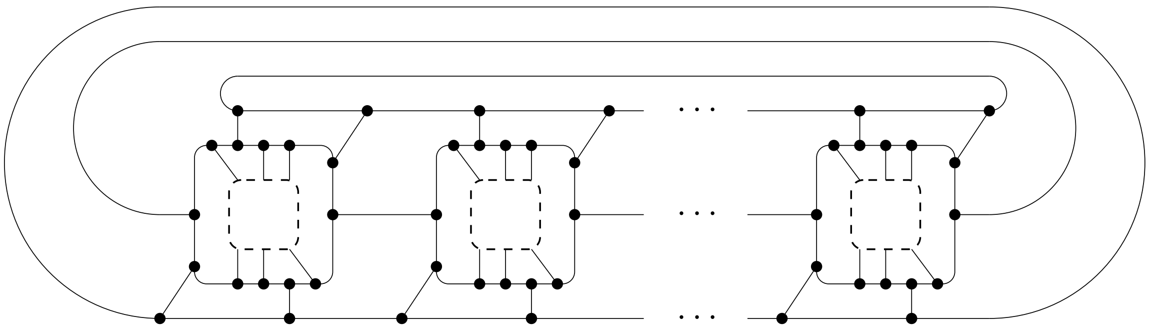

FIG. 5

Note that a similar result (considering classes of cyclically 4- and 5-edge-connected cubic graphs instead of $\mathcal{P}_{3,5}$) was obtained in [BGMR].

A graph G is called *internally* 6-*connected* if it is 5-connected and deleting the vertices of any 5-cut we get a component having just one isolated vertex.

Theorem 5.8. *The following statements are pairwise equivalent.*

(1) *Every planar graph is* 4*-vertex-colorable.*
(2) *There exists a sublinear sequence* $\{b_n\}$ *so that every simple* 5*-cut cubic planar graph of order* $2n$ *has* 4*-reluctance at most* b_n*.*
(3) *There exists a sublinear sequence* $\{b_n\}$ *so that every simple* 5*-cut cubic planar graph of order* $2n$ *has* 4*-overlap at most* b_n*.*
(4) *There exists a sublinear sequence* $\{b_n\}$ *so that the vertices of every internally* 6*-connected planar graph of order* n *can be colored by* $r \geq 5$ *colors where the colors* $5, \ldots, r$ *are used at most* b_n *times.*
(5) *There exists a sublinear sequence* $\{b_n\}$ *so that the vertices of every internally* 6*-connected planar graph of order* n *can be assigned four numbers where at most* b_n *couples of neighboring vertices receive the same number.*

Proof. Suppose G is an internally 6-connected planar graph. Adding to G some edges we get a planar triangulation G', which must be also internally 6-connected. But a planar triangulation is internally 6-connected iff its dual is from $\mathcal{P}_{3,5}$. Thus (1)–(5) are equivalent by Theorems 5.3, 5.7 and Proposition 2.5. □

6. The 3-flow and circular flow conjectures

If G is a graph and $X \subseteq D(G)$, then denote $-X = \{-x; x \in X\}$. A set $X \subseteq D(G)$ is called an *orientation* of G if $-X \cup X = D(G)$ and $-X \cap X = \emptyset$. An orientation X of G is called a *modular h-orientation* of a network (G,U) if $|\omega^+(v) \cap X| \equiv |\omega^-(v) \cap X|$ mod h for every inner vertex v of G. By [J2], [Z], a graph G (i. e., the network $(G,\emptyset)$) has a modular $2t$-orientation iff every vertex of G has even valency. Thus we consider only modular $(2t+1)$-orientations here. By a *circular k-flow* φ in (G,U) we mean a $\mathbb{Z}_k$-flow in (G,U) such that $\varphi(x) \in \{\pm 1\}$ for every $x \in D(G)$. These two notions have been introduced by Jaeger [J2] for graphs.

Theorem 6.1. *Let (G,U) be a network and t a positive integer. Then the following statements are pairwise equivalent.*

1. *(G,U) has a modular $(2t+1)$-orientation.*
2. *(G,U) has a circular $(2t+1)$-flow.*
3. *(G,U) has a $\mathbb{Z}$-flow φ such that $\varphi(x) \in \{\pm t, \pm(t+1)\}$ for every $x \in D(G)$.*

Proof. The statement was proved by Jaeger [J2] if $U = \emptyset$ (see also [Z]). Let X be a modular $(2t+1)$-orientation of (G,U). Add to G a new vertex v_0 and join every $u \in U$ with v_0 by a suitable number of parallel edges so that the new graph G' has an orientation X' satisfying $|\omega^+(v) \cap X'| \equiv |\omega^-(v) \cap X'| \mod 2t+1$ for every $v \in V(G)$. Then there exists a circular $(2t+1)$-flow φ in $(G', \{v_0\})$ so that $\varphi(x) = 1$ if $x \in X'$. By Lemma 5.1, $\partial\varphi(v_0) = 0$ whence $|\omega^+(v_0) \cap X'| \equiv |\omega^-(v_0) \cap X'| \mod 2t+1$. Thus G' satisfies (1)–(3) and therefore, (G,U) satisfies (2) and (3). Similarly we can show that (2) (resp. (3)) implies (1). $\square$

A circular flow conjecture of Jaeger [J2] suggests that every $4t$-edge-connected graph has a circular $(2t+1)$-flow. For $t = 1$ this is equivalent with the 3-flow conjecture of Tutte suggesting that every 4-edge-connected graph has a nowhere-zero 3-flow (note that the 3-flow conjecture was originally formulated in an equivalent form for graphs without 1- or 3-edge cuts, see, e. g., [J4]). For $t = 2$, the circular flow conjecture generalizes the 5-flow conjecture (see [J2]).

If (G,U) is a network, then $\overline{\rho}_{2t+1}(G,U)$ denotes the minimal integer n so that there exists an n element subset W of $V(G)$ with the property that the network $(G, U \cup W)$ has a circular $(2t+1)$-flow. (Note that, by Theorem 6.1, we get an equivalent formulation considering modular $(2t+1)$-orientations or $\mathbb{Z}$-flows φ satisfying $\varphi(x) \in \{\pm t, \pm(t+1)\}$ for every $x \in D(G)$ instead of circular $(2t+1)$-flows). By $\overline{\sigma}_{2t+1}(G,U)$ (resp. $\sigma'_{2t+1}(G,U)$) we mean the minimal integer n so that there exists a $\mathbb{Z}_{2t+1}$-flow (resp. $\mathbb{Z}$-flow) φ in (G,U) with just $2n$ arcs x satisfying $\varphi(x) \notin \{\pm 1\}$ (resp. $\varphi(x) \notin \{\pm t, \pm(t+1)\}$).

Lemma 6.2. $2\overline{\sigma}_{2t+1}(G,U) = 2\sigma'_{2t+1}(G,U) \geq \overline{\rho}_{2t+1}(G,U)$ *for every network (G,U) and $t \geq 1$.*

Proof. Let $\overline{\sigma}_{2t+1}(G,U) = n$ and φ be a $\mathbb{Z}_{2t+1}$-flow in (G,U) with just $2n$ arcs x satisfying $\varphi(x) \notin \{\pm 1\}$. Denote the set of the edges associated with these arcs by E'. Then $|E'| = n$. Each edge of e from E' can be replaced by a suitable number of parallel edges so that we get a graph G' where (G', U) has a circular $(2t+1)$-flow. Then $\sigma'_{2t+1}(G', U) = 0$ and $\sigma'_{2t+1}(G,U) \leq n$. Similarly we can check that $\overline{\sigma}_{2t+1}(G,U) \leq \sigma'_{2t+1}(G,U)$, whence $\overline{\sigma}_{2t+1}(G,U) = \sigma'_{2t+1}(G,U)$.

Let W be the set of the endpoints of the edges from E'. Then $|W| \leq 2n$ and φ can be changed to a circular $(2t+1)$-flow in $(G, U \cup W)$, whence $\overline{\rho}_{2t+1}(G, U) \leq 2n$. $\square$

The following statement was proved by Jaeger [J1] (see also [Z, Proposition 3.4.3] and [F3, Corollary VIII.48.a]).

Lemma 6.3. *Every 4-edge-connected graph has a spanning eulerian subgraph.*

Let $\mathcal{E}_k$ denote the class of k-edge-connected graphs and $\mathcal{H}_{l,k}$ denote the class of l-regular k-connected hamiltonian simple graphs, where $k \geq 4$ and $l = 2\lfloor k/2 \rfloor + 1$ (l is the smallest odd number $\geq k$). Then $\mathcal{E}_k$ and $\mathcal{H}_{l,k}$ have quantitative characteristics 1 and 2, respectively, and $\mathcal{H}_{l,k}$ is a proper subclass of $\mathcal{E}_k$.

Theorem 6.4. *Let $k \geq 4$ and $l = 2\lfloor k/2 \rfloor + 1$. Then $(\mathcal{E}_k, \overline{\rho}_{2t+1}) \approx (\mathcal{H}_{l,k}, \overline{\rho}_{2t+1}) \approx (\mathcal{E}_k, \overline{\sigma}_{2t+1}) \approx (\mathcal{H}_{l,k}, \overline{\sigma}_{2t+1}) \approx (\mathcal{E}_k, \sigma'_{2t+1}) \approx (\mathcal{H}_{l,k}, \sigma'_{2t+1})$.*

Proof. Denote $k' = \lfloor k/2 \rfloor$. Then $l = 2k'+1$. Let $G_{l,m}$, $l \leq m$, be the graph arising from a copy of $K_{1,m}$ with the vertex w of valency m and vertices $v_1, \ldots, v_m$ of valency 1 after joining each couple of vertices v_i, v_j satisfying $i - j \equiv \pm 1, \ldots, \pm k' \mod m$ by a new edge. Deleting the edges (w, v_1) and $(w, v_{k'+1})$ from $G_{l,l+1}$ and adding a new edge $(v_1, v_{k'+1})$ we get $G_{l,l-1}$. For every $m \geq k$, we have a k-connected simple graph $G_{l,m}$ with one vertex w of valency m and the rest of the vertices having valency l.

Let G be a graph from $\mathcal{E}_k$ satisfying $\overline{\rho}_{2t+1}(G) > 0$. By Lemma 6.3, G has a spanning eulerian subgraph, i. e., a trail T covering all vertices of G. Take a vertex v from G of valency $m \geq k$ and replace it by a copy of $G_{l,m}$ as follows: delete v from G and w from $G_{l,m}$ and join their neighbors by new m edges obtaining a graph G_v; furthermore, the new edges can be joined so that the trail T is extended to an eulerian trail T_v in G_v going through each vertex of the copy of $G_{l,m}$ just once. Repeating this process for every vertex v of G we get a graph G' which is l-regular, k-connected and the trail T is transformed to a hamiltonian cycle in G', i. e., $G' \in \mathcal{H}_{l,k}$. By Lemma 5.1, if G_v has a circular $(2t+1)$-flow, then it indicates such a flow in G. Thus $\overline{\rho}_{2t+1}(G') > 0$ (i. e., $(\mathcal{E}_k, \overline{\rho}_{2t+1})^*_0$ and $(\mathcal{H}_{l,k}, \overline{\rho}_{2t+1})^*_0$ are equivalent).

Let $H_{l,p}$, $l \leq p$, be the bipartite graph with partitions $\{u_1, \ldots, u_p\}$ and $\{w_1, \ldots, w_p\}$ so that u_i, w_j are joined iff $i - j \equiv 0, \ldots, l-1 \mod p$. Clearly $H_{l,p}$ is from $\mathcal{H}_{l,k}$. We can replace the vertices $w_1, \ldots, w_h$ by h copies $G'_1, \ldots, G'_h$ of G' such that the resulting graph G'' is from $\mathcal{H}_{l,k}$. Take $W \subseteq V(G'')$ satisfying $\overline{\rho}_{2t+1}(G'', W) = 0$. Then $W \cap V(G'_i) \neq \emptyset$ ($i = 1, \ldots, h$), otherwise $\overline{\rho}_{2t+1}(G'_i) = 0$ by Lemma 5.1. Hence $\overline{\rho}_{2t+1}(G'') \geq h$. Thus $(\mathcal{E}_k, \overline{\rho}_{2t+1})^*_0$ implies $(\mathcal{H}_{l,k}, \overline{\rho}_{2t+1})^*_{\mathrm{s}}$ and the statement holds by Lemma 6.2 and Propositions 2.5–2.8. $\square$

Proposition 2.5 and Theorem 6.4 imply the next theorem.

Theorem 6.5. *The following statements are pairwise equivalent.*

(1) *Every $4t$-edge-connected graph has a circular $(2t+1)$-flow.*
(2) *Every $(4t+1)$-regular $4t$-connected hamiltonian simple graph has a circular $(2t+1)$-flow.*
(3) *There exists a sublinear sequence $\{b_n\}$ so that every $(4t+1)$-regular $4t$-connected hamiltonian simple graph G of order $2n$ satisfies $\overline{\rho}_{2t+1}(G) \leq b_n$.*
(4) *There exists a sublinear sequence $\{b_n\}$ so that every $(4t+1)$-regular $4t$-connected hamiltonian simple graph G of order $2n$ satisfies $\overline{\sigma}_{2t+1}(G) \leq b_n$.*

Jaeger [J4] introduced a weaker form of the 3-flow problem suggesting that there exists $k \geq 4$ so that every k-edge-connected graph has a nowhere-zero 3-flow. This is known as the weak 3-flow conjecture. Equivalent forms of the (weak) 3-flow conjecture are presented in the following theorem.

Theorem 6.6. *Let $k \geq 4$ and $l = 2\lfloor k/2 \rfloor + 1$. Then the following statements are pairwise equivalent.*

(1) *Every k-edge-connected graph has a nowhere-zero 3-flow.*
(2) *Every l-regular k-connected hamiltonian simple graph has a nowhere-zero 3-flow.*
(3) *There exists a sublinear sequence $\{b_n\}$ so that every l-regular k-connected hamiltonian simple graph G of order $2n$ has 3-reluctance at most b_n.*
(4) *There exists a sublinear sequence $\{b_n\}$ so that every l-regular k-connected hamiltonian simple graph G of order $2n$ has 3-overlap at most b_n.*

Proof. Clearly, a network (G, U) has a nowhere-zero 3-flow iff it has a circular 3-flow. Thus $\overline{\rho}_3(G, U) = \rho_3(G, U)$, and the statement holds by Theorem 6.4, Lemma 5.4, and Propositions 2.5–2.8. □

As pointed out in [F3, Page VIII.59], the fact that the 3-flow problem remains to solve for 5-regular simple graphs was already known to Jaeger. Jaeger [J2] also proved that if every 9-edge-connected graph has a circular 5-flow, then the 5-flow conjecture is true. Thus, by Theorem 6.4 and [J2], the following holds.

Corollary 6.7. *Suppose that there exists a sublinear sequence $\{b_n\}$ so that every 9-regular 9-connected hamiltonian simple graph G of order $2n$ satisfies $\overline{\rho}_5(G) \leq b_n$. Then every bridgeless graph has a nowhere-zero 5-flow.*

Note that every graph from $\mathcal{H}_{9,9}$ has nowhere-zero 4- and 5-flows. Furthermore, if the 3-flow conjecture holds true, then it would have also a nowhere-zero 3-flow. But this still does not imply that it has a circular 5-flow. For example, let H_3 be the graph having two vertices joined by three parallel edges. Then H_3 has a circular 3-flow but no circular 5-flow. On the other hand, by Jaeger [J2] (see also [Z, Theorem 9.2.3]), if a graph has a circular $(2t+3)$-flow, then it has a circular $(2t+1)$-flow.

7. Concluding remarks

We mentioned either very famous open problems or theorems which belong to the most difficult in graph theory. But we believe that also some other problems can be expressed in language of parameters with the sublinear defect property. For example, using methods from Theorems 5.4, 6.4 and [K5] we can show that $(\mathcal{G}_3, \rho_6) \approx (\mathcal{G}_{3,4}, \rho_6) \approx (\mathcal{G}_3, \sigma_6) \approx (\mathcal{G}_{3,4}, \sigma_6)$, and, if $\mathcal{E}'_4$ denotes the class of 4-edge-connected planar graphs, then $(\mathcal{E}'_4, \rho_3) \approx (\mathcal{E}'_4, \sigma_3) \approx (\mathcal{E}'_4, \beta_3) \approx (\mathcal{E}'_4, \gamma_3)$. These statements are related to the known results of Seymour [S] and Grötzsch [G], respectively. Note that Bauer, Broersma, van den Heuvel, and Veldman [BBHV] have received a similar result in connection with the t-tough conjecture and proved a strictly sublinear defect property of this problem (using the notation from Section 2).

Now it is not possible to predict whether our techniques will lead to a direct proof of some of the problems mentioned here. But this paper certainly shows that all these problems are equivalent with seemingly much weaker statements. It would be very interesting if the sublinear defect principle allows to use probabilistic methods for the abovementioned problems.

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