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Martin Kochol

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TENSION POLYNOMIALS OF GRAPHS

Martin Kochol

Abstract. Tension polynomial $F_G(k)$ of a graph G, evaluating the number of nowhere-zero $\mathbb{Z}_k$-tensions in G, is the nontrivial divisor of the chromatic polynomial $\chi_G(k)$ of G, because $\chi_G(k) = k^{c(G)} F_G(k)$, where $c(G)$ denotes the number of components of G. In this paper we present a new characterization of these polynomials and show that $F_G(k)$ can be presented as a sum of some Ehrhart polynomials (evaluating the numbers of integral vectors in multiples of integral polytopes). In a similar way we introduce a new notion, integral tension polynomial $I_G(k)$, which evaluates the number of nowhere-zero integral tensions in G with absolute values smaller than k. Furthermore we show that $2^{r(G)} F_G(k) \geq I_G(k) \geq (r(G)+1)F_G(k)$ (where $r(G) = |V(G)| - c(G)$) and $F_G(k+1) \geq F_G(k) \cdot k/(k-1)$, $I_G(k+1) \geq I_G(k) \cdot k/(k-1)$ for every $k > 1$.

1. Introduction

The aim of this paper is to study chromatic polynomials of graphs from a new point of view. Chromatic polynomials are well known functions, that have been studied in framework of classical combinatorics and graph theory. For surveys see, e. g., Aigner [A], Biggs [Bi], Brylawski and Oxley [BO], Jaeger [J], Read and Tutte [RT], Rota [R], Tutte [T1], [T2], [T3], and Welsh [W]. Now we show that chromatic polynomials can be characterized by Ehrhart polynomials [E], that have origin in convex geometry. More precisely, Ehrhart polynomial of an integral polytope P evaluates the number of integral vectors from $kP = \{k\mathbf{x}; \mathbf{x} \in P\}$ for every positive integer k.

Chromatic problems about graphs can be expressed in framework of potentials and tensions (or potential differences). A nice introduction to this approach is in Berge [Be]. A graph G admits a nowhere-zero k-tension if its edges can be oriented and assigned values $\pm 1, \dots, \pm(k-1)$ so that for every circuit C of G the sum of the values of the edges from C oriented in one direction equals the sum of the values of the edges of C oriented oppositely. Using nonzero elements of an additive Abelian group A instead of integers, we get a nowhere-zero A-tension in G. The number of nowhere-zero A-tensions in G does not depend on the structure of A, but only on its order k, and, furthermore, this number can be evaluated by a tension polynomial $F_G(k)$ of G. This function is the nontrivial divisor of the chromatic polynomial $\chi_G(k)$ of G, because $\chi_G(k) = k^{c(G)} F_G(k)$, where $c(G)$ denotes the number of components of G (the Betti number of G of dimension zero).

In the introductory sections we study relations between nowhere-zero $\mathbb{Z}_k$- and k-tensions and acyclic orientations of graphs. If X is an acyclic orientation of a graph G, then denote by $I_X(k)$ the number of nowhere-zero k-tensions having by this

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orientation only positive values. We show that this is an Ehrhart polynomial of an integral polytope. Taking sum of $I_X(k)$ through the set of all acyclic orientations of G we get a polynomial $I_G(k)$ that evaluates the number of nowhere-zero k-tensions in G and is called the integral tension polynomial of G. On the other hand taking the sum through a special subset of acyclic orientations (representatives of classes of certain equivalence) we get $F_G(k)$. In this way we obtain a new characterization of the tension polynomial and introduce a new integral variant of this polynomial in the same time. $F_G(k)$, $I_G(k)$, and $I_X(k)$ have degrees $r(G) = |V(G)| - c(G)$ and $2^{r(G)}F_G(k) \geq I_G(k) \geq (r(G)+1)F_G(k)$. Furthermore we show that $F_G(k+1) \geq F_G(k) \cdot k/(k-1)$, $I_G(k+1) \geq I_G(k) \cdot k/(k-1)$ for every $k > 1$. These estimates are the best possible in a certain sense.

Nowhere-zero A- and k-tensions in a planar graph correspond to nowhere-zero A- and k-flows in the dual graph, respectively. Also the (integral) tension polynomials have their dual counterparts, (integral) flow polynomials, that we study in [K3].

2. Potentials, tensions, and tension polynomials

In this paper we deal with finite unoriented graphs with multiple edges and loops. A graph $G = (V, E)$ consists of a set of vertices $V(G) = V$ and a set of edges $E(G) = E$. Every edge e of the graph G determines two opposite arcs arising from it after endowing e with two distinct orientations. All arcs obtained in this way are called *arcs* of G and the set of them is called the *arc set* of G and denoted by $D(G)$. Clearly, $|D(G)| = 2|E(G)|$. If x is an arc of G, then denote by x^{-1} the second arc arising from the same edge. Clearly $(x^{-1})^{-1} = x$ and $x \neq x^{-1}$ for every arc x of G. If $X \subseteq D(G)$, then denote by $X^{-1} = \{x \in D(G); x^{-1} \in X\}$. By an *orientation* of G we mean every $X \subseteq D(G)$ so that $X \cup X^{-1} = D(G)$ and $X \cap X^{-1} = \emptyset$.

If $W \subseteq V(G)$, then the set of the arcs of $D(G)$ directed from W to $V(G) \setminus W$ is denoted by $\omega_G^+(W)$. Furthermore, define $\omega_G^-(W) = (\omega_G^+(W))^{-1}$. We write $\omega_G^+(v)$, $\omega_G^-(v)$ instead of $\omega_G^+(\{v\})$, $\omega_G^-(\{v\})$, respectively.

If e is an edge of a graph G, then $G - e$ and G/e denote the graphs obtained from G after deleting and contracting the edge e, respectively. Note that $G/e = G - e$ if e is a loop.

In this paper, every Abelian group is additive. If G is a graph and A is an Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. *Support* of φ, denoted by $\sigma(\varphi)$, is the set of the edges associated with the arcs of G having nonzero values in φ. An A-chain φ is called *nowhere-zero* if $\sigma(\varphi) = E(G)$.

By an *A-potential* in a graph G we mean a mapping $t : V(G) \to A$. The *coboundary* of t is the mapping $\delta t : D(G) \to A$ so that if x is an arc of G directed from u to v, then

$$\delta t(x) = t(v) - t(u).$$

An A-chain φ in G is a called an *A-tension* if there exists an A-potential t so that $\varphi = \delta t$. We call t a potential *attached* to φ.

The notions of coboundary and tension come from algebraic topology and physics, respectively, but express the same quantity. Our approach coincides with that one from Berge [Be]. The main difference is that instead of a fixed (but arbitrary) orientation we use the set $D(G)$ as a domain for a tension. A dual approach is used in [K1], [K2], [K3].

Lemma 1. *If t and t' are A-potentials in a graph G, then $\delta t = \delta t'$ if and only if $t-t'$ is a constant function on the components of G.*

Proof. Follows directly from definitions. $\square$

Circuit is a connected graph with all vertices of valency two. *Cycle* is a graph with all vertices of even valency. *Directed circuit* (*directed cycle*) is an orientation of a circuit (cycle) so that for every vertex v, the number of the arcs entering v equals the number of the arcs leaving v.

Lemma 2. *Let φ be an A-chain in G. Then the following statements are pairwise equivalent.*

(1) *φ is an A-tension in G.*
(2) *For every directed circuit D of G we have $\sum_{x\in D} \varphi(x) = 0$.*
(3) *For every directed cycle D of G we have $\sum_{x\in D} \varphi(x) = 0$.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. If G is connected and (3) holds, then $\sum_{x\in P} \varphi(x) = \sum_{x\in P'} \varphi(x)$ for any two directed paths $P, P' \subseteq D(G)$ having the begin and end in common. Choose $u \in V(G)$, $a \in A$, and define $t(v) = a + \sum_{x\in P} \varphi(x)$ where $P \subseteq D(G)$ is a directed path beginning in u and ending in v. Then t is an A-potential and $\delta t = \varphi$. Thus (3) $\Rightarrow$ (1). $\square$

Lemma 3. *If a graph G has a loop, then it does not admit a nowhere-zero A-tension for every Abelian group A.*

Proof. Follows directly from definitions. $\square$

If k is a positive integer, then by a (*nowhere-zero*) *k-tension* φ in a graph G we mean a (nowhere-zero) $\mathbb{Z}$-tension in G such that $|\varphi(x)| < k$ for every $x \in D(G)$.

Proposition 1. *Let G be a graph. Then the following statements are pairwise equivalent.*

(1) *G has a nowhere-zero $\mathbb{Z}_k$-tension.*
(2) *G has a nowhere-zero A-tension for any Abelian group A of order k.*
(3) *G has a nowhere-zero k-tension.*
(4) *G is k-vertex colorable.*

Proof. (4) $\Leftrightarrow$ (1), (4) $\Leftrightarrow$ (2), and (4) $\Rightarrow$ (3) are trivial. By Lemma 2, taking the values of a nowhere-zero k-tension $\mod k$ we get a nowhere-zero $\mathbb{Z}_k$-tension. Thus (3) $\Rightarrow$ (1). $\square$

For a graph G, let $r(G) = |V(G)| - c(G)$ where $c(G)$ denote the number of components of G. It is known that $r(G)$ is rank of the *cycle matroid* of G, i. e., the number of the edges of a spanning forest of G.

Theorem 1. *For every graph G, there exists a polynomial function $F_G(k)$ of k, so that if A is an Abelian group of order $k > 0$, then $F_G(k)$ is the number of nowhere-zero A-tensions in G. Furthermore, if G is loopless, then $F_G(k)$ has degree $r(G)$ and otherwise $F_G(k) = 0$.*

Proof. Following the usual convention we get $F_G(k) = 1$ if $E(G) = \emptyset$. By Lemma 3, $F_G(k) = 0$ if G has a loop. Take $e \in E(G)$ which is not a loop. If e is a bridge, then by Lemma 2, $F_G(k) = (k-1)F_{G-e}(k)$ and $r(G) = r(G-e) + 1$. Otherwise

$F_G(k) = F_{G-e}(k) - F_{G/e}(k)$ and $r(G) = r(G-e) = r(G/e)+1$. Thus applying induction on $|E(G)|$ we get the statement. $\square$

$F_G(k)$ is called the *tension polynomial* of G. By Lemma 1, the classical chromatic polynomial of G $\chi_G(k) = k^{c(G)}F_G(k)$. $F_G(1) = 0$ for every graph with $E(G) \neq \emptyset$,

Suppose that φ and φ' are nowhere-zero k-tensions in a graph G. Then we write $\varphi \equiv \varphi'(\theta_{G,k})$ if $\varphi(x) \equiv \varphi'(x) \pmod k$ for every $x \in D(G)$. $\theta_{G,k}$ is an equivalence relation on the set of nowhere-zero k-tensions in G. Let us note that $[\varphi]\theta_{G,k}$ denotes the $\theta_{G,k}$-class containing φ, i. e., the set of the nowhere-zero k-tensions in G that are $\theta_{G,k}$-equivalent with φ.

If ψ is a nowhere-zero $\mathbb{Z}_k$-tension in G and X is an orientation of G, then ψ_X denotes the nowhere-zero k-chain in G so that $\psi_X(x) = \psi(x)$ for every $x \in X$ (for the sake of simplicity, the residue class $[n]_k$ is considered as n where $0 \leq n < k$). Clearly, $\sum_{x\in D}\psi_X(x) \equiv 0 \pmod k$ for every directed cycle D of G.

Let φ be a nowhere-zero $\mathbb{R}$-chain in G. Then $\{x \in D(G); \varphi(x) > 0\}$ is an orientation of G. It is denoted by X_φ and called the *positive orientation of* φ. Furthermore, if φ is a nowhere-zero k-tension, then denote by $[\varphi]_k$ the nowhere-zero $\mathbb{Z}_k$-tension in G satisfying $[\varphi]_k(x) = [\varphi(x)]_k$ for every $x \in D(G)$. By Lemma 2, $[\varphi]_k$ is well defined and, clearly, $\varphi(x) = [\varphi]_k(x)$ for every $x \in X_\varphi$. Thus $([\varphi]_k)_{X_\varphi} = \varphi$.

Theorem 2. *Let G be a graph. Then the mapping $[\varphi]\theta_{G,k} \mapsto [\varphi]_k$ is a one-to-one correspondence between the $\theta_{G,k}$-classes and the nowhere-zero $\mathbb{Z}_k$-tensions in G.*

Proof. It remains to show that if ψ is a nowhere-zero $\mathbb{Z}_k$-tension in G, then there exists a nowhere-zero k-tension φ in G so that $[\varphi]_k = \psi$. Let t be a $\mathbb{Z}_k$-potential attached to ψ, and t' be a $\mathbb{Z}$-potential so that $t(v) = t'(v)$ for every vertex v of G (the residue class $[n]_k$ is considered as n where $0 \leq n < k$). Then $\varphi = \delta t'$ satisfies $[\varphi]_k = \psi$. $\square$

Corollary 1. *The number of the $\theta_{G,k}$-classes is $F_G(k)$ for every graph G and $k > 0$.*

Proof. Follows from Theorems 1 and 2. $\square$

3. Acyclic orientations and classes of some equivalences

By a *directed cocycle* in G we mean every $C \subseteq D(G)$ arising as a disjoint union of sets of the form $\omega_G^+(W_1), \dots, \omega_G^+(W_n)$ where $W_1, \dots, W_n \subseteq V(G)$. Furthermore, denote by φ_C the 2-tension in G so that $\varphi_C(x) = 1$ if $x \in C$, $\varphi_C(x) = -1$ if $x^{-1} \in C$, and $\varphi_C(x) = 0$ otherwise. This is a 2-tension by Lemma 2.

Lemma 4. *If φ and φ' are nowhere-zero k-tensions in a graph G and $\varphi \equiv \varphi'(\theta_{G,k})$, then $X_\varphi \setminus X_{\varphi'}$ is a directed cocycle in G. On the other hand if a directed cocycle C in G is a subset of X_φ, then there exists just one $\varphi'' \in [\varphi]\theta_{G,k}$ so that $C = X_\varphi \setminus X_{\varphi''}$. It satisfies $\varphi'' = \varphi - k\varphi_C$.*

Proof. Clearly $\varphi'(x) = \varphi(x) - k$ or $\varphi(x)$ for every $x \in X_\varphi$. Then $\overline{\varphi} = (\varphi' - \varphi)/k$ is a 2-tension in G so that $\overline{\varphi}(x) = 1$ iff $x \in X_\varphi \setminus X_{\varphi'}$. Let t be a $\mathbb{Z}$-potential attached to $\overline{\varphi}$. Without loss of generality we can assume that t has values $1, \dots, n$, where n is a positive integer. Setting $W_i = \{v \in V(G); t(v) \leq i\}$, for $i = 1, \dots, n-1$ we get that $X_\varphi \setminus X_{\varphi'}$ is a disjoint union of $\omega_G^+(W_1), \dots, \omega_G^+(W_{n-1})$ and, thus a directed cocycle. The rest of the proof is trivial. $\square$

An orientation X of G is called *acyclic* if it contains no directed circuit. It is known that every acyclic orientation X of G contains at least one source (resp. sink), i. e., a vertex v satisfying $\omega_G^+(v) \subseteq X$ (resp. $\omega_G^-(v) \subseteq X$).

Lemma 5. *Suppose X is an orientation of a graph G. Then the following conditions are pairwise equivalent.*

(1) *X is a positive orientation of a nowhere-zero $\mathbb{Z}$-tension in G.*
(2) *X is a positive orientation of a nowhere-zero $\mathbb{R}$-tension in G.*
(3) *Every $x \in X$ satisfies $x \in \omega_G^+(W) \subseteq X$ for some $W \subseteq V(G)$.*
(4) *X is an acyclic orientation of G.*

Proof. (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial. Let X be a positive orientation of a nowhere-zero $\mathbb{R}$-tension φ, t be an $\mathbb{R}$-potential attached to φ, and x be an arc directed out of u. Take $W = \{v \in V(G); t(v) \leq t(u)\}$. Then $x \in \omega_G^+(W) \subseteq X$. Thus (2) $\Rightarrow$ (3). If X is an acyclic orientation of G, then subsequently deleting the sinks and assigning them by numbers $1, \ldots, |V(G)|$ we get a $\mathbb{Z}$-potential t so that $X_{\delta t} = X$, thus (4) $\Rightarrow$ (1). $\square$

Let $\mathcal{A}(G)$ denote the set of all acyclic orientations of a graph G. If $X, X' \in \mathcal{A}(G)$, then we write $X \equiv X'(\Theta_G)$ if $X \setminus X'$ is a directed cocycle. The following statements follow directly from Lemmas 4 and 5.

Corollary 2. *Θ_G is an equivalence relation on $\mathcal{A}(G)$.*

Corollary 3. *If $\varphi \equiv \varphi'(\theta_{G,k})$, then $X_\varphi \equiv X_{\varphi'}(\Theta_G)$. Moreover the quantity $|[\varphi]\theta_{G,k}|$ $= |[X_\varphi]\Theta_G|$ is equal to the number of the directed cocycles in X_φ.*

Corollary 4. *If $X, X' \in \mathcal{A}(G)$ and $X \equiv X'(\Theta_G)$, then X and X' have the same number of directed cocycles and $\varphi \mapsto \varphi - k\varphi_{X \setminus X'}$ is a one-to-one correspondence between the sets of nowhere-zero k-tensions with positive orientations X and X'.*

Note that $\emptyset$ $(\subseteq D(G))$ is also a directed cocycle, which corresponds to φ in $[\varphi]\theta_{G,k}$. Thus $|[\varphi]\theta_{G,k} \geq 2|$ whenever $E(G) \neq \emptyset$. If φ is a nowhere-zero k-tension in a graph G and $r > s \geq k$, then it is also a nowhere-zero s- and r-tension and, by Corollary 3, $|[\varphi]\theta_{G,s}| = |[\varphi]\theta_{G,r}|$. Furthermore, by Lemma 4, $[\varphi]\theta_{G,r} \cap [\varphi]\theta_{G,s} = \{\varphi\}$.

In the following example we show that $|[\varphi]\theta_{G,k}|$ can be any integer ≥ 2, and a graph G can admit nowhere-zero k-tensions φ, φ', φ'' and φ''' so that $X_\varphi \equiv X_{\varphi'}(\Theta_G)$, $\varphi \not\equiv \varphi'(\theta_{G,k})$ and $|[\varphi'']\theta_{G,k}| = |[X_{\varphi''}]\Theta_G| \neq |[\varphi''']\theta_{G,k}| = |[X_{\varphi'''}]\Theta_G|$.

Example 1. Let G_n be the circuit of length n, $n \geq 1$, and $X = \{x_1, \ldots, x_n\}$ be a directed circuit in G_n. Take a nowhere-zero n-tension φ_n in G_n, $n \geq 2$, so that $\varphi_n(x_1) = n - 1$ and $\varphi_n(x_2) = \varphi_n(x_3) = \cdots = \varphi_n(x_n) = -1$. Then $X_{\varphi_n} = X_{1,n} = \{x_1, x_2^{-1}, x_3^{-1}, \ldots, x_n^{-1}\}$, and, by Corollary 3, $|[\varphi_n]\theta_{G_n,k}| = |[X_{1,n}]\Theta_{G_n}| = n$ for every $k \geq n$. Take nowhere-zero 3-tensions φ and φ' in G_4 so that $2\varphi(x_1) = \varphi(x_2) = -2\varphi(x_3) = -\varphi(x_4) = 2$ and $\varphi'(x_1) = 2\varphi'(x_2) = -\varphi'(x_3) = -2\varphi'(x_4) = 2$. Then $\varphi \not\equiv \varphi'(\theta_{G_4,k})$ for every $k \geq 3$. On the other hand $X_\varphi = X_{\varphi'} = X_{2,4} = \{x_1, x_2, x_3^{-1}, x_4^{-1}\}$, and, by Corollary 3, $|[\varphi]\theta_{G_4,k}| = |[\varphi']\theta_{G_4,k}| = |[X_{2,4}]\Theta_{G_4}| = 6$ for every $k \geq 3$. But we have $|[\varphi_4]\theta_{G_4,k}| = 4$ for every $k \geq 4$.

Remark 1. Let $\mathcal{A}_k(G)$ denote the set of positive orientations of nowhere-zero k-tensions in a graph G. Denote by $\mathcal{A}_{k,\Theta}(G)$ the set of the Θ_G-classes which are subsets of $\mathcal{A}_k(G)$. Then $\mathcal{A}_{k,\Theta}(G)$ is a partition of $\mathcal{A}_k(G)$, because, by Lemma 4, if $X \in \mathcal{A}_k(G)$, then $[X]\Theta_G \subseteq \mathcal{A}_k(G)$. This is of some interest, because the problem to find a nowhere-zero k-tension in G means to find an orientation from $\mathcal{A}_k(G)$, what automatically gives the class $[X]\Theta_G$ from $\mathcal{A}_{k,\Theta}(G)$.

The following statement was proved by Ghouila-Houri [GH] (see also [Be]).

Lemma 6. *Let X be an orientation of a graph G and b_x, c_x be reals (integers) so that $b_x \le c_x$ for every $x \in X$. Then there exists an $\mathbb{R}$-tension ($\mathbb{Z}$-tension) φ in G so that $b_x \le \varphi(x) \le c_x$ for every $x \in X$ if and only if $\sum_{x \in D \cap X} b_x \le \sum_{x \in D^{-1} \cap X} c_x$ for every directed circuit D in G.*

Proof. Necessity is trivial. We prove sufficiency. Without abuse of generality we can suppose that G is loopless. Set $b_x = -c_{x^{-1}}$, $c_x = -b_{x^{-1}}$ for every $x \in X^{-1}$. If P is a directed path in G (considered as a subset of $D(G)$), then denote by $b(P) = \sum_{x \in P} b_x$. Choose a vertex w of G. For every vertex v of G, denote by $t(v)$ the maximum value $b(P)$ among the paths P directed from w to v. We claim that $\varphi = \delta t$ satisfies the assumptions. If not, there exists $x \in X$ such that $\varphi(x) \notin [b_x, c_x]$. Let x be directed from u to v, and P (P') be a path directed from W to u (v) such that $b(P) = t(u)$ ($b(P') = t(v)$). Then, clearly, $t(v) \ge t(u) + b_x$, whence $\varphi(x) \ge b_x$. Thus $\varphi(x) > c_x$, i. e., $t(v) - t(u) - c_x = b(P') - b(P) - c_x > 0$. But $Q = P' \cup P^{-1} \cup \{x^{-1}\}$ is a directed cycle. By the assumptions, we get $\sum_{x \in Q \cap X} b_x \le \sum_{x \in Q^{-1} \cap X} c_x$, whence $b(P') - b(P) - c_x \le 0$ – a contradiction. Thus the statement holds. $\square$

Lemma 6 implies the following statement (see also Minty [M]).

Corollary 5. *An orientation X of a connected graph G is a positive orientation of a nowhere-zero k-tension if and only if $|D|/|D \cap X| \le k$ for every directed circuit D in G.*

It is well known that there exist $r(G)$ independent cocycles in a (directed) graph (see Berge [Be] for more details). Now we prove a stronger statement for acyclic orientations. If G is a graph and $W \subseteq V(G)$, then denote by $\omega_G(W)$ the set of the edges associated with $\omega_G^+(W)$. $\omega_G(W)$ is called a *cut* of G. A cut is called *elementary*, or a *cocircuit*, if the number of components increase by one after deleting its edges from the graph. In this case $\omega_G^+(W)$ is called a *directed cocircuit* of G. An ordered set $(C_1, \ldots, C_r)$ of directed cocircuit (cocycles) is called *triangular* if every C_i contains an arc x_i ($i = 1, \ldots, r$) so that $x_i \notin C_j$ for each $j < i$.

Lemma 7. *Let X be an acyclic orientation of a graph G. Then there exists a triangular set $(C_1, \ldots, C_{r(G)})$ of directed cocircuits in X covering all arcs of X.*

Proof. The statement remains to verify for connected G and is trivial if $r(G) = 0, 1$. We use induction on $r(G)$. Suppose $r(G) \ge 2$, v is a source of X, and $G_1, \ldots, G_s$ are the components of $G - v$. Now $r(G) = s + r(G_1) + \cdots + r(G_s)$. Take $W_i = V(G) \setminus V(G_i)$, $C_i = \omega_G^+(W_i)$, and x_i an arc from C_i for $i = 1, \ldots, s$. $C_1, \ldots, C_s$ are edge disjoint directed cocircuits. By the assumption of induction, there exists a triangular set of directed cocircuits $C_{i,1}, \ldots, C_{i,r(G_i)}$ in $X_i = X \cap D(G_i)$ covering X_i for $i = 1, \ldots, s$. Suppose $C_{i,j} = \omega_{G_i}^+(W_{i,j})$, $W_{i,j} \subseteq V(G_i)$. Then take $C'_{i,j} = C_{i,j}$ if $\omega_{G_i}^+(W_{i,j}) = \omega_G^+(W_{i,j})$ and $C'_{i,j} = \omega_G^+(W_{i,j} \cup W_i)$ otherwise ($i \in \{1, \ldots, s\}$, $j \in \{1, \ldots, r(G_i)\}$). Since $C_{i,j} \subseteq C'_{i,j}$ and $C'_{i,j} \setminus C_{i,j} \subseteq C_i$, then $(C_1, \ldots, C_s, C'_{1,1}, \ldots, C'_{1,r(G_1)}, \ldots, C'_{s,1}, \ldots, C'_{s,r(G_s)})$ satisfies the required properties. $\square$

Note that Lemma 7 cannot be improved so that every cocircuit C_i contains an arc x_i where $x_i \notin C_j$ for $j \ne i$. It remains to take an orientation $\{x_1, x_2, x_3^{-1}, x_4^{-1}\}$ of G_4 from Example 1.

Corollary 6. *Every acyclic orientation of a graph G contains at least $r(G)$ directed cocircuits and at least $r(G) + 1$ directed cocycles.*

Proof. Follows directly from Lemma 7. $\square$

4. Growth of tension polynomials

For every orientation X of G, let $I_X(k)$ denote the number of nowhere-zero k-tensions φ in G so that $X_\varphi = X$. By Lemma 5, $I_X(k) = 0$ if X is not acyclic. Obviously, $I_X(k+1) \geq I_X(k)$.

Denote by $\mathcal{A}_\Theta(G)$ the set of the equivalence classes of Θ_G. $\mathcal{A}_\Theta(G)$ is a partition of $\mathcal{A}(G)$. If $\mathcal{X} \in \mathcal{A}_\Theta(G)$ and $X, X' \in \mathcal{X}$, then, by Corollary 4, $I_X(k) = I_{X'}(k)$ for every $k > 0$. Denote this value by $I_\mathcal{X}(k)$. We have

$$I_G(k) = \sum_{X \in \mathcal{A}(G)} I_X(k) = \sum_{\mathcal{X} \in \mathcal{A}_\Theta(G)} I_\mathcal{X}(k)|\mathcal{X}| \qquad (k > 0). \tag{1}$$

Let ψ be a nowhere-zero $\mathbb{Z}_k$-tension in G. By Theorem 2, there exists a nowhere-zero k-tension φ in G so that $[\varphi]_k = \psi$. Furthermore, $[\varphi]\theta_{G,k}$ is the set of all nowhere-zero k-tensions φ' in G satisfying $[\varphi']_k = \psi$. By Corollaries 3, 4 and Theorem 2, $[X_\varphi]\Theta_G$ is equal to the following three sets:

- the set of all positive orientations of the tensions from $[\varphi]\theta_{G,k}$;
- the set of all orientations X of G so that the k-chain ψ_X is a k-tension;
- the set of all positive orientations of nowhere-zero k-tensions φ' in G so that $[\varphi']_k = \psi$.

Thus $[X_\varphi]\Theta_G$ is called the Θ_G*-class associated with* ψ and denoted by $\mathcal{X}_\psi$. We have $|\mathcal{X}_\psi| = |[X_\varphi]\Theta_G| = |[\varphi]\theta_{G,k}|$. For every $\mathcal{X} \in \mathcal{A}_\Theta(G)$, $I_\mathcal{X}(k)$ is equal to the number of nowhere-zero $\mathbb{Z}_k$-tensions ψ in G so that $\mathcal{X}_\psi = \mathcal{X}$. Hence

$$F_G(k) = \sum_{\mathcal{X} \in \mathcal{A}_\Theta(G)} I_\mathcal{X}(k) = \sum_{X \in \mathcal{A}(G)} \frac{I_X(k)}{|[X]\Theta_G|} \qquad (k > 0). \tag{2}$$

Theorem 3. *Let G be a loopless graph, $E(G) \neq \emptyset$, and X be an acyclic orientation of G. Then for every $k > 1$,*

$$I_X(k+1) \geq I_X(k) \cdot k/(k-1),$$
$$F_G(k+1) \geq F_G(k) \cdot k/(k-1),$$
$$I_G(k+1) \geq I_G(k) \cdot k/(k-1).$$

Proof. By (1) and (2), the statement remains to prove for $I_X(k)$. This holds if $I_X(k) = 0$. Suppose that $I_X(k) > 0$ and A (resp. B) denotes the set of the nowhere-zero k-tensions (resp. $(k+1)$-tensions) φ in G with positive orientation X. For every $\varphi \in A$ and a directed cocycle $C \neq \emptyset$ in X, define by C*-lift* of φ the tension from $B \setminus A$ of the form $\varphi + r\varphi_C$ where r is a positive integer. Let n be the number of nonempty directed cocycles in X. Then we can get together $n|A|$ lifts (for every φ and C just one). But each such a lift can be obtained at most $n(k-1)$ times. Thus $|B \setminus A| \geq |A|/(k-1)$, whence $I_X(k+1) = |B| \geq |A| \cdot k/(k-1) = I_X(k) \cdot k/(k-1)$. $\square$

Corollary 7. *Let X be an acyclic orientation of a loopless graph G, $E(G) \neq \emptyset$, and $I_X(k+1)$, $F_G(k+1)$, $I_G(k+1) > 0$. Then*

$$I_X(k+1) \geq I_X(k) + 1,$$
$$F_G(k+1) \geq F_G(k) + 1,$$
$$I_G(k+1) \geq I_G(k) + r(G) + 1.$$

Proof. If $I_X(k+1) > 0$, then from the proof of Theorem 3 follows that $|B| > |A|$ and $I_X(k+1) \geq I_X(k) + 1$. If $F_G(k+1)$, $I_G(k+1) > 0$, then there exists $\mathcal{X} \in \mathcal{A}_\Theta(G)$ so that $I_{\mathcal{X}}(k) > 0$, thus also $I_{\mathcal{X}}(k+1) \geq I_{\mathcal{X}}(k) + 1$. Hence, by (1), (2), and Corollaries 3, 6, $F_G(k+1) \geq F_G(k) + 1$ and $I_G(k+1) \geq I_G(k) + r(G) + 1$. $\square$

Let H_n be a graph consisting of two vertices v_1, v_2 and n parallel edges $e_1, \ldots, e_n$. Then $r(H_n) = 1$ and $F_{H_n}(k) = k - 1$. H_n has just two acyclic orientations X_1, X_2 and $I_{X_1}(k) = I_{X_2}(k) = k - 1$. By (1), $I_{H_n}(k) = 2(k-1)$. Thus the bounds from Theorem 3 and Corollary 7 are the best possible in this case.

5. Polynomials $I_G(k)$ and $I_X(k)$

In this section we use some basic properties of convex polytopes. Every polytope P in $\mathbb{R}^n$ is the convex hull of the (finite) set of its vertices. If these are integral, then P is called *integral*. Dimension of a polytope is the dimension of its affine hull. By a *variant* of a polytope P we mean every subset of P arising from it after deleting some of its faces. Every variant of a polytope is a convex set. The maximal variant of P is it itself. The following statement was proved by Ehrhart [E] (see also [K3]).

Lemma 8. *Let $\overline{P}$ be a variant of an integral convex polytope P. Then the number of the integral vectors from $k\overline{P} = \{k\mathbf{x}; \mathbf{x} \in \overline{P}\}$ is a polynomial function for every integral $k > 0$. The the degree of the polynomial is equal to the dimension of P.*

Theorem 4. *Let X be an orientation of a graph G. Then $I_X(k)$ is a polynomial function for $k > 0$. The degree of this polynomial is $r(G)$ if X is acyclic and $I_X(k) = 0$ otherwise.*

Proof. $I_X(k) = 0$ if X is not acyclic. Otherwise let $U_X \subseteq \mathbb{R}^{D(G)}$ (resp. $\overline{U}_X \subseteq \mathbb{R}^{D(G)}$) be the set of all $\mathbb{R}$-tensions φ in G so that $0 \leq \varphi(x) \leq 1$ (resp. $0 < \varphi(x) < 1$) for every $x \in X$. (Now tensions are considered as vectors indexed by $D(G)$.) Then $\overline{U}_X$ is a variant of U_X and $I_X(k) = |k\overline{U}_X \cap \mathbb{Z}^{D(G)}|$. Let P_X be the set of the pairs (φ, t) where $\varphi \in U_X$ and t is the $\mathbb{R}$-potential attached to φ and satisfying $t(v) = 0$ for a fixed vertex v of G. By [S, Chapter 19.3, Example 2], $P_X = (\mathbf{y} \in \mathbb{R}^{D(G) \cup V(G)} : M\mathbf{y} \leq \mathbf{b})$ where M is a totally unimodular matrix and $\mathbf{b}$ is an integral vector. Therefore P_X, and also U_X are integral polytopes and, by Lemma 8, $I_X(k)$ is a polynomial.

By (2), $F_G(k) \geq I_{[X]\Theta_G}(k) = I_X(k)$, thus by Theorem 1, $I_X(k)$ can have degree at most $r(G)$. By Lemma 7, there exists a triangular set $\{C_1, \ldots, C_{r(G)}\}$ of directed cocircuits covering X. Then for every positive integer s and an ordered $r(G)$-tuple of integers $\mathbf{a} = (a_1, \ldots, a_{r(G)})$ satisfying $1 \leq a_i \leq s$ we get a nowhere-zero $(r(G) \cdot s + 1)$-tension $\varphi_{\mathbf{a}} = \sum_{i=1}^{r(G)} a_i \varphi_{C_i}$ with the positive orientation X. If $\mathbf{a} = (a_1, \ldots, a_{r(G)}) \neq \mathbf{a}' = (a'_1, \ldots, a'_{r(G)})$ and i is the largest index satisfying $a_i \neq a'_i$, then $\varphi_{\mathbf{a}}(x_i) \neq \varphi_{\mathbf{a}'}(x_i)$ (where $x_i \in C_i$ and $x_i \notin C_j$ for every $j < i$). Therefore, $\mathbf{a} \mapsto \varphi_{\mathbf{a}}$ is an injective mapping, $I_X(r(G) \cdot s + 1) \geq s^{r(G)}$, and $\liminf_{k \to \infty} I_X(k)/k^{r(G)} > 0$. Hence $I_X(k)$ has degree at least $r(G)$. $\square$

Theorem 5. *Let G be a graph. Then $I_G(k)$ is a polynomial function for $k > 0$. The degree of this polynomial is $r(G)$ if G is loopless and $I_G(k) = 0$ otherwise.*

Proof. It is known that every loopless graph has an acyclic orientation (see also Lemma 11). Thus the statement follows from Theorem 4, Lemma 2 and (1). $\square$

$I_G(k)$ and $I_X(k)$ are called the *integral tension polynomials* of G and X, respectively. Similarly to the first half of Theorem 4 we can prove the following.

Theorem 6. *Let X be an orientation of a graph G and b_x, c_x be integers for every $x \in X$. Then the number of the $\mathbb{Z}$-tensions φ in G satisfying $kb_x \le \varphi(x) \le kc_x$ ($kb_x < \varphi(x) < kc_x$) for every $x \in X$ is a polynomial function for integral $k > 0$.*

Example 2. Let $\mathcal{X}_{i,n}$ denote the set of the orientations of G_n, $n \ge 2$, (see Example 1) having just i arcs directed in a chosen direction. Then $\{\mathcal{X}_{1,n}, \dots, \mathcal{X}_{n-1,n}\} = \mathcal{A}_\Theta(G_n)$, $|\mathcal{X}_{i,n}| = \binom{n}{i}$ and $I_{\mathcal{X}_{i,n}}(k) = I_{\mathcal{X}_{n-i,n}}(k)$ for $i = 1, \dots, n-1$. Set $X_{i,n} = \{x_1, \dots, x_i, x_{i+1}^{-1}, \dots, x_n^{-1}\}$. Then $X_{i,n} \in \mathcal{X}_{i,n}$ and $I_{X_{i,n}}(k) = I_{\mathcal{X}_{i,n}}(k)$.

Define $p_{1,n}(1) = I_{X_{1,n}}(1)$ and $p_{1,n}(i) = I_{X_{1,n}}(i) - I_{X_{1,n}}(i-1)$ for $i > 1$. Then $p_{1,n}(i)$ is the number of the nowhere-zero i-tensions φ in G_n where $\varphi(x_1) = i-1$ and $\varphi(x_2) + \cdots + \varphi(x_n) = 1-i$. Therefore $p_{1,n}(i)$ is the number of the ordered $(n-1)$-tuples of positive integers $(a_1, \dots, a_{n-1})$ so that $a_1 + \cdots + a_{n-1} = i-1$. Hence $p_{1,n}(i) = \binom{i-2}{n-2}$ and $I_{X_{1,n}}(k) = \sum_{i=1}^{k} p_{1,n}(i) = \sum_{i=n}^{k} \binom{i-2}{n-2} = \binom{k-1}{n-1}$.

$F_{G_2}(k) = k-1$ and, by the ideas from the proof of Theorem 1, $F_{G_n}(k) = (k-1)^{n-1} - F_{G_{n-1}}(k)$ ($n \ge 3$), whence $F_{G_n}(k) = \sum_{i=1}^{n-1} (-1)^{i-1}(k-1)^{n-i}$. Then using (1) and (2) we can evaluate all polynomials $I_{X_{i,n}}(k)$ and $I_{G_n}(k)$ for $2 \le n \le 5$, $1 \le i \le n-1$.

6. Bounds between $F_G(k)$ and $I_G(k)$

For a graph G, denote by $C(G)$ the set of all directed circuits in $D(G)$. Let X be an orientation of G and $\mathbf{u} : C(G) \to \mathbb{Z}$. Then $Y \subseteq X$ is called a $\mathbf{u}$-*suborientation* of X if $|D \cap Y| - |D^{-1} \cap Y| = \mathbf{u}(D)$ for every $D \in C(G)$.

Lemma 9. *If X is an orientation of a graph G and $\mathbf{u} : C(G) \to \mathbb{Z}$, then the number of the $\mathbf{u}$-suborientations of X is at most $2^{r(G)}$.*

Proof. We use induction on $r(G)$. The statement is trivial when $r(G) = 0$. If $r(G) > 0$, then G has an edge e not equal to a loop and $r(G/e) = r(G) - 1$. Let x be the arc from X arising from e. For every $T \subseteq D(G)$, denote by T/e the subset of $D(G/e)$ arising from T after contracting the edge e. Define from $\mathbf{u}$ two functions $\mathbf{u}', \mathbf{u}'' : C(G/e) \to \mathbb{Z}$ so that for every $D \in C(G)$, $\mathbf{u}'(D/e) = \mathbf{u}(D)$ and $\mathbf{u}''(D/e) = \mathbf{u}(D) - 1$ if $x \in D$, $\mathbf{u}''(D/e) = \mathbf{u}(D) + 1$ if $x^{-1} \in D$, and $\mathbf{u}''(D/e) = \mathbf{u}(D)$ otherwise. Let Y be a $\mathbf{u}$-suborientation of X. Then Y/e is either a $\mathbf{u}''$- or a $\mathbf{u}'$-suborientation of X/e (depending on the case whether $x \in Y$ or not). By the assumption of induction, there are at most $2^{r(G/e)}$ $\mathbf{u}'$-suborientations ($\mathbf{u}''$-suborientations) of X/e. Thus there are at most $2^{r(G)}$ $\mathbf{u}$-suborientations of X. $\square$

Corollary 8. *Every orientation of a graph G has at most $2^{r(G)}$ directed cocycles.*

Proof. Every directed cocycle in an orientation X of G is a $\mathbf{0}$-suborientation of X where $\mathbf{0}(D) = 0$ for every $D \in C(G)$. Thus the statement follows from Lemma 9. $\square$

Theorem 7. *Let G be a graph. Then $2^{r(G)} F_G(k) \ge I_G(k) \ge (r(G)+1) F_G(k)$ for every $k > 0$.*

Proof. By Corollaries 3, 6, and 8, $2^{r(G)} \ge |\mathcal{X}| \ge r(G) + 1$ for every $\mathcal{X} \in \mathcal{A}_\Theta(G)$. Then, by (1) and (2), $2^{r(G)} F_G(k) = \sum_{\mathcal{X} \in \mathcal{A}_\Theta(G)} I_\mathcal{X}(k) 2^{r(G)} \ge \sum_{\mathcal{X} \in \mathcal{A}_\Theta(G)} I_\mathcal{X}(k) |\mathcal{X}| = I_G(k) \ge \sum_{\mathcal{X} \in \mathcal{A}_\Theta(G)} I_\mathcal{X}(k)(r(G)+1) = (r(G)+1) F_G(k)$. $\square$

Note that if $r(G) = 0$, then $2^{r(G)} F_G(k) = I_G(k) = (r(G)+1) F_G(k)$ for every $k > 0$, because $F_G(k) = I_G(k) = 1$ or 0. Other examples of graphs satisfying the bounds from Theorem 7 as equalities can be obtained as dual graphs of planar graphs from [K3, Examples 4, 5, and 6].

7. Maximal circuits

Lemma 10. *Let G be a connected loopless graph and H be a circuit of G. Then every acyclic orientation of H can be extended into an acyclic orientation of G and $|\mathcal{A}_\Theta(H)| \leq |\mathcal{A}_\Theta(G)|$.*

Proof. We can easily check that every acyclic orientation of $G-v$ or $G-e$ ($v \in V(G)$, $e \in E(G)$) can be extended to an acyclic orientation of G. From this follows that every acyclic orientation of H can be extended to an acyclic orientation of G. Suppose that X, X' are acyclic orientations of G so that $X \setminus X'$ is a directed cocycle and Y, Y' are the restrictions of X, X' to $D(H)$, respectively. Then $Y \setminus Y'$ is the restriction of $X \setminus X'$ to $D(H)$, whence $Y \setminus Y'$ is a directed cocycle. Thus $X \equiv X'(\Theta_G)$ implies $Y \equiv Y'(\Theta_H)$ and therefore $|\mathcal{A}_\Theta(H)| \leq |\mathcal{A}_\Theta(G)|$. $\square$

Let $\mu(G)$ denote the length of the maximal circuit in G. By [GJ], it is an NP-complete problem to decide whether $\mu(G) \geq n$ for a graph G and an integer n.

Corollary 9. $|\mathcal{A}_\Theta(G)| \geq \mu(G) - 1$ *for every connected loopless graph G.*

Proof. Follows from Lemma 10 and the fact that $|\mathcal{A}_\Theta(G_n)| = n - 1$ (see Example 2). $\square$

Theorem 8. *Let G be a loopless graph and k be a positive integer. Then $I_X(k) > 0$ for every $X \in \mathcal{A}(G)$ if and only if $k \geq \mu(G)$.*

Proof. By Corollary 5, Lemma 10, and Example 2, there exists an orientation X' of G so that $I_{X'}(k) = 0$ for every $k < \mu(G)$. If X is an acyclic orientation of G and D is a directed circuit in G, then $|D|/|D \cap X| \leq \mu(G)$ and $I_X(\mu(G)) > 0$ by Corollary 5. $\square$

Corollary 10. *Let G be a loopless connected graph, $E(G) \neq \emptyset$, and $k \geq \mu(G) - 1$. Then $F_G(k+1) \geq F_G(k) + \mu(G) - 1$ and $I_G(k+1) \geq I_G(k) + (\mu(G) - 1)(r(G) + 1)$.*

Proof. Follows from Theorem 3, Lemma 11, Corollaries 6, 9 and (1), (2). $\square$

Lemma 11. *Every loopless graph has at least $2^{r(G)}$ acyclic orientations.*

Proof. The statement holds if $r(G) = 0$. Let $r(G) > 0$ and e be an edge of G. Deleting all loops from G/e we get a loopless graph G' and every acyclic orientation of G' can be extended to two acyclic orientations of G. Thus the statement holds by induction on $r(G)$. $\square$

Corollary 11. *Let G be a loopless connected graph, $E(G) \neq \emptyset$, and $k \geq \mu(G) - 1$. Then $I_G(k+1) \geq I_G(k) + 2^{r(G)}$.*

Proof. Follows from Corollary 7, Theorem 8, Lemma 11, and (1). $\square$

Minor of a graph G is every graph we get from G after deleting and contructing some of its edges.

Corollary 12. $F_H(\mu(G)) > 0$ *for every minor H of G.*

Proof. Follows from Theorem 8 and the fact that $\mu(H) \leq \mu(G)$. $\square$

8. Sets of tensions

In this section we summarize some observations about tensions. Let $\mathbf{P}_G$ and $\mathbf{I}_{G,k}$ denote the set of $\mathbb{Z}$-tensions and $\mathbb{Z}_k$-tensions in G, respectively. Considering the sum of tensions φ and φ' as $(\varphi+\varphi')(x) = \varphi(x)+\varphi'(x)$, $x \in D(G)$, we get that $\mathbf{P}_G$ and $\mathbf{I}_{G,k}$ are Abelian groups. Define homomorphism $\mathfrak{f}_{G,k} : \mathbf{P}_G \to \mathbf{I}_{G,k}$ so that $(\mathfrak{f}_{G,k}(\varphi))(x) = [\varphi(x)]_k$ $(x \in D(G))$. Let $\pi_{G,k}$ denote the congruence relation on $\mathbf{P}_G$ defined by $\varphi \equiv \varphi'(\pi_{G,k})$ if $\mathfrak{f}_{G,k}(\varphi) = \mathfrak{f}_{G,k}(\varphi')$. Recall that $\sigma(\varphi)$ denotes the support of φ.

Lemma 12. *Let $\varphi, \varphi' \in \mathbf{P}_G$. Then $\sigma(\varphi) \cap \sigma(\varphi') \subseteq \sigma(\varphi+\varphi')$ and $\sigma(\varphi) \subseteq \sigma(\mathfrak{f}_{G,k}(\varphi))$.*

Proof. If $\varphi(x) = \varphi'(x) = 0$, then $(\varphi+\varphi')(x) = 0$, and similarly, if $\varphi(x) = 0$, then $(\mathfrak{f}_{G,k}(\varphi))(x) = [\varphi(x)]_k = 0$ $(x \in D(G))$. □

Let $\mathbf{R}_{G,k}$, $\mathbf{T}_{G,k}$ and $\mathbf{J}_{G,k}$ denote the sets of k-tensions, nowhere-zero k-tensions and nowhere-zero $\mathbb{Z}_k$-tensions in G, respectively. These sets are not closed under the addition, but $\mathbf{T}_{G,k} \subseteq \mathbf{R}_{G,k} \subseteq \mathbf{P}_G$ and $\mathbf{J}_{G,k} \subseteq \mathbf{I}_{G,k}$. Let $\rho_{G,k}$ and $\theta_{G,k}$ denote the equivalence relations arising as restrictions of $\pi_{G,k}$ into $\mathbf{R}_{G,k}$ and $\mathbf{T}_{G,k}$, respectively (note that $\theta_{G,k}$ is identical with $\theta_{G,k}$ from Section 2). In other words, $[\varphi]\rho_{G,k} = [\varphi]\pi_{G,k} \cap \mathbf{R}_{G,k}$ and $[\varphi']\theta_{G,k} = [\varphi']\pi_{G,k} \cap \mathbf{T}_{G,k}$ for every $\varphi \in \mathbf{R}_{G,k}$ and $\varphi' \in \mathbf{T}_{G,k}$, respectively.

Lemma 13. *If $\varphi \in \mathbf{R}_{G,k}$, then $\sigma(\mathfrak{f}_{G,k}(\varphi)) = \sigma(\varphi) = \sigma(\varphi')$ for every $\varphi' \in [\varphi]\rho_{G,k}$.*

Proof. Since $-k < \varphi'(x), \varphi(x) < k$ for every $x \in D(G)$, then the equalities $\varphi'(x) = 0$, $\varphi(x) = 0$ and $[\varphi(x)]_k = 0$ are pairwise equivalent. □

Lemma 14. *If $\varphi \in \mathbf{T}_{G,k}$, then $\mathfrak{f}_{G,k}(\varphi) \in \mathbf{J}_{G,k}$ and $[\varphi]\rho_{G,k} = [\varphi]\theta_{G,k}$.*

Proof. Follows directly from Lemma 13. □

Now we can generalize Theorem 2.

Theorem 9. *The mapping $[\varphi]\rho_{G,k} \mapsto \mathfrak{f}_{G,k}(\varphi)$ is a one-to-one correspondence between the $\rho_{G,k}$-classes and the elements of $\mathbf{I}_{G,k}$. Furthermore, this mapping preserves the supports of tensions, i. e., $\sigma(\mathfrak{f}_{G,k}(\varphi)) = \sigma(\varphi) = \sigma(\varphi')$ for every $\varphi' \in [\varphi]\rho_{G,k}$.*

Proof. Use Lemma 13 and apply Theorem 2 for the spanning subgraph H of G so that $E(H) = \sigma(\varphi)$. □

We can also reformulate Theorem 2 in the following form:

Theorem 10. *The mapping $[\varphi]\theta_{G,k} \mapsto \mathfrak{f}_{G,k}(\varphi)$ is a one-to-one correspondence between the $\theta_{G,k}$-classes and the elements of $\mathbf{J}_{G,k}$.*

Theorem 11. *The homomorphism $\mathfrak{f}_{G,k}$ is surjective.*

Proof. Follows directly from Theorem 9. □

Thus Theorem 2 is in fact a special (but the substantial) part of Theorem 9. By Theorem 11, $\mathbf{P}_G/\pi_{G,k}$ and $\mathbf{I}_{G,k}$ are isomorphic under the isomorphism $[\varphi]\pi_{G,k} \mapsto \mathfrak{f}_{G,k}(\varphi)$.

For every $X \in \mathcal{A}(G)$, let $\mathbf{T}_{X,k}$ denote the set of the nowhere-zero k-tensions in G with positive orientation X and $\mathfrak{f}_{X,k}$ denote the restriction of $\mathfrak{f}_{G,k}$ to the set $\mathbf{T}_{X,k}$. For every $\mathcal{X} \in \mathcal{A}_\Theta(G)$, let $\mathbf{J}_{\mathcal{X},k}$ denote the set of nowhere-zero $\mathbb{Z}_k$-tensions ψ in G so that $\mathcal{X}_\psi = \mathcal{X}$. Then from Lemma 4 and Corollary 4 follows that $\mathfrak{f}_{X,k}$ is a bijection

$\varphi \mapsto [\varphi]_k$ from $\mathbf{T}_{X,k}$ to $\mathbf{J}_{[X]\Theta_G,k}$ and if $X \equiv X'(\Theta_G)$ then $\mathfrak{f}_{X,k} \circ \mathfrak{f}^{-1}_{X',k}$ is the mapping $\varphi \mapsto \varphi - k\varphi_{X\setminus X'}$.

Let $\mathbf{Q}_G$ be the set of the $\mathbb{R}$-tensions in G. Then $\mathbf{Q}_G$ is a vector space over $\mathbb{R}$ and $\mathbf{P}_G$ is a module over $\mathbb{Z}$. By Berge [Be], their dimensions are equal to $r(G)$ and their basis can be found using Lemma 7. This also shows that the parameter $r(G)$ from Lemma 7 cannot be increased.

Finally note that the chromatic polynomial $\chi_G(k)$ of G evaluates the number of the $\mathbb{Z}_k$-potentials in G attached to nowhere-zero $\mathbb{Z}_k$-tensions. Thus it seems to be natural to study the number of the $\mathbb{Z}$-potentials in G attached to nowhere-zero $\mathbb{Z}$-tensions. But this number is either 0 or ∞ (because if $\delta t \in \mathbf{T}_{G,k}$, then also $\delta(t+n) \in \mathbf{T}_{G,k}$ for every $n \in \mathbb{Z}$). Thus chromatic polynomials do not have a reasonable "integral" variant and from the point of view of this paper it was better to consider tensions than potentials or colorings.

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MÚ SAV, Štefánikova 49, 814 73 Bratislava 1, Slovakia

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