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Abstract

We introduce generalized pseudo-effect algebras as a non-commutative version of generalized effect algebras.

The importance of the algebras of the latter type in quantum physics is based on the fact that they reflect the inner structure of subsets of the positive cone of the group of self-adjoint operators in a Hilbert space. The new algebras are designed to model subsets of group cones as well; but now the underlying *po*-group may be chosen arbitrarily, in particular it does not need to be abelian.

We raise the question when a generalized pseudo-effect algebra is actually representable in the positive cone of a *po*-group. We are able to give an affirmative answer for two special cases. Both times a property of Riesz kind is involved, defined for our algebra in a similar manner as known for *po*-groups.

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1 Introduction

In the field of quantum structures, we are concerned with the foundational problem of quantum physics. Roughly spoken, our approach to this problem

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is the following. The basic object under study is the Hilbert space, since this is the model of all quantum theories. Now, to gain a deeper insight into the rather complicated Hilbert space formalism, we try to extract from it its physically essential parts; that is, we work on possible reductions of the somehow sophisticated analytic structure to algebraic ones. The aim thereby is to get hopefully simpler, but still physically meaningful models of quantum phenomena.

For a long time, the closed subspaces of Hilbert space have been examined, which correspond to possible measurement results in quantum experiments. These entities possess internally the structure of an orthocomplemented lattice, which gave hope to a better understanding of quantum physics insofar, as it reminds to a system of sentential-logical propositions. But the research results, although optimal, seem to be not as useful as once expected.

In more recent times, the quantum effects have been taken into consideration instead, i.e. the positive operators of a Hilbert space lying below identity. For characterizing internally the effects, it seemed to be no longer reasonable to use lattice operations, but to be natural to use their additive relations; two effects may – or may not – add up to a third one. So the set of all effects possesses the structure of a partial additive algebra, and as a such it forms a so-called effect algebra [FoBe, GiGr]. Equivalently, we may consider the differences of pairs of comparable effects; if we do so, we are lead to so-called difference posets [KoCh].

Thus the prototype among the effect algebras as well as difference posets is a unit interval of the group of self-adjoint Hilbert space operators. Now we may easily generalize these two structures by dropping the requirement of boundedness, so as to get axioms satified also by unbounded parts of the group cone [HePu]. The structures we arrive at, simply called generalized effect algebras or generalized difference posets, respectively, are the starting point of our discussion here.

What we will do is actually to generalize the mentioned structures even further. The prototypical examples of the algebras discussed here shall not neccessarily be so special as the Hilbert space operators or even as abelian group cones are. Namely, we drop the assumption of commutativity, so as to include a much broader family of models, in particular those arising from non-commutative *po*-groups. For situations in which non-commutativity actually occurs, we think in particular about any model of time-ordered sequences of whatsoever physical phenomena.

So we shall introduce what will be called generalized pseudo-effect algebras. These are still partial additive structures; they are not necessarily commutative, and by assuming commutativity they specialize to generalized effect algebras. Their prototypes arise within the positive cone of any, abelian or non-abelian, partially ordered group.

It seems then natural to ask in which cases one of our algebras is actually representable just in this way. We will prove that a group representation exists in case one among two conditions holds: The algebra is either assumed to be a lower semi-lattice and to fulfil a certain condition of Riesz type, or it is assumed to be directed and to fulfil some other, slightly different, condition again of Riesz type.

2 Generalized Pseudo-Effect Algebras

Effect algebras are of major importance in quantum physics insofar, as they reflect the internal structure of the positive operators of a Hilbert space lying below identity, i.e. of the so-called quantum effects. It was this kind of algebra that once gave motivation to introduce the generalized effect algebras; the greatest element was dropped, and a structure was obtained such that every principal order ideal was an effect algebra [HePu, DvPu]. The connection to quantum physics has been kept insofar, as the new axioms are modelled, no longer necessarily by a unit interval of the group of self-adjoint operators but, by any subset of positive operators that is closed under the formation of differences of comparable elements.

We recall that a generalized effect algebra, or GE-algebra for short, is a structure $(E; +, 0)$, where $+$ is a partial binary operation and 0 is a constant, such that, for all $a, b, c \in E$, (GE1) $a + b$ is defined iff $b + a$ is defined, in which case these elements are equal, (GE2) $(a + b) + c$ is defined iff $a + (b + c)$ is defined, in which case these elements are equal, (GE3) from $a + b = a + c$ it follows $b = c$, (GE4) from $a + b = 0$ it follows $a = b = 0$, and (GE5) $a + 0 = a$.

We shall generalize this kind of algebra by no longer requiring the axiom of commutativity. This has in fact been the idea that lead from effect algebras to pseudo-effect algebras [DvVe1], and that we now apply to the more general structure of a generalized effect algebra.

Definition 2.1 A structure $(E; +, 0)$, where $+$ is a partial binary operation and 0 is a constant, is called a *generalized pseudo-effect algebra*, or GPE-algebra for short, if, for all $a, b, c \in E$, the following holds.

(GPE1) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$.

(GPE2) If $a+b$ exists, there are elements $d, e \in E$ such that $a+b = d+a = b+e$.

(GPE3) If $a+b$ and $a+c$ exist and are equal, then $b = c$.

If $b+a$ and $c+a$ exist and are equal, then $b = c$.

(GPE4) If $a+b$ exists and $a+b = 0$, then $a = b = 0$.

(GPE5) $a+0$ and $0+a$ exist and are both equal to a.

Furthermore, we define for any $a, b \in E$

$$a \leq b \quad \text{iff} \quad a + c = b \ \text{ for some } c \in E.$$

Lemma 2.2 Let $(E; +, 0)$ be a generalized pseudo-effect algebra. Then $\leq$ is a partial order on E with 0 as smallest element.

E is naturally ordered by $\leq$, i.e., for any $a, b \in E$, $a \leq b$ if and only if $a+c = b$ for some $c \in E$ if and only if $d + a = b$ for some $d \in E$.

Proof. Let $a, b, c \in E$. We have $a \leq a$, since $a + 0 = a$ by (GPE5).

$a \leq b$ and $b \leq a$ imply $a + a_1 = b$ and $b + b_1 = a$ for some $a_1, b_1 \in E$, so $(a + a_1) + b_1 = a = a + 0$ by (GPE5); this means $a_1 + b_1 = 0$ by (GPE1) and (GPE3) and $a_1 = b_1 = 0$ by (GPE4); so $a = b$ by (GPE5).

$a \leq b$ and $b \leq c$ imply $a + a_1 = b$ and $b + b_1 = c$ for some $a_1, b_1 \in E$, so $c = (a + a_1) + b_1 = a + (a_1 + b_1)$ by (GPE1), which means $a \leq c$.

By (GPE5), 0 is the smallest element.

That $\leq$ is a natural order follows easily from (GPE2). □

Example 2.3 The prototypical GPE-algebras arise in the following way from positive cones of *po*-groups.

Let $(G;+,0,\leq)$ be a partial ordered group with $G^+ \stackrel{\text{def}}{=} \{g \in G\colon\ g \geq 0\}$ as its positive cone; let G_0 be a non-empty subset of G^+ such that for all $a, b \in G_0$, where $b \leq a$, also $a-b, -b+a \in G_0$.

Then $(G_0;+,0)$, where $+$ is the group addition restricted to those pairs of elements whose sum is again in G_0, and where 0 is the group zero, is a GPE-algebra, whose order coincides with the restriction of the group order to G_0.

Recall that a subset P_0 of a poset P is called convex, if for any two elements from P_0 all elements from P lying between these two are also contained in P_0. We note that if $(G;+,0,\leq)$ is a *po*-group and G_0 is a convex subset of G^+ containing 0, then $(G_0;+,0)$ is a GPE-algebra.

For the rest of this text, by any sentence involving partial sums to hold we mean: All these sums are defined, and it holds. Similarly, with any sentence involving infima or suprema to hold, we include tacitly the statement that these infima and suprema exist.

Furthermore, in view of (GPE1), we will denote finite sums without brackets, whenever convenient.

Lemma 2.4 Let $(E;+,0)$ be a generalized pseudo-effect algebra. For $a, b, c \in E$, the following holds.

(i) If $a+b$ exists, $a_1 \leq a$, and $b_1 \leq b$, then also $a_1 + b_1$ exists.

(ii) Suppose $b+c$ exists. Then $a \leq b$ if and only if $a+c$ exists and $a+c \leq b+c$.

Suppose $c+b$ exists. Then $a \leq b$ if and only if $c+a$ exists and $c+a \leq c+b$.

(iii) Let $(c+a) \wedge (c+b)$ exist. Then we have $c + (a \wedge b) = (c+a) \wedge (c+b)$.

Let $(a+c) \wedge (b+c)$ exist. Then we have $(a \wedge b) + c = (a+c) \wedge (b+c)$.

(iv) Let $c + (a \vee b)$ exist. Then we have $c + (a \vee b) = (c+a) \vee (c+b)$.

Let $(a \vee b) + c$ exist. Then we have $(a \vee b) + c = (a+c) \vee (b+c)$.

Proof. (i) If $a+b$ exists and $a_1'+a_1=a$ and $b_1+b_1'=b$ for some a_1' and $b_1' \in E$, it follows that $(a_1'+a_1)+(b_1+b_1')$ exists and so, by (GPE1), that a_1+b_1 exists.

(ii) Suppose $b+c$ exists. Then $a \leq b$ iff, for some d, $d+a=b$ iff, for some d, $(d+a)+c=b+c$ iff $a+c$ exists and $a+c \leq b+c$. So the first part is proved; the second is shown similarly.

(iii) As $c \leq (c+a) \wedge (c+b)$, we have for some d that $c+d=(c+a) \wedge (c+b)$. Then $c+d \leq c+a, c+b$ and, by (ii), $d \leq a, b$. Suppose $x \leq a, b$. Again by (ii), we have $c+x \leq c+a, c+b$, i.e. $c+x \leq (c+a) \wedge (c+b) = c+d$ and $x \leq d$. It follows $d = a \wedge b$.

The second part is proved similarly.

(iv) As $a, b \leq a \vee b$, we have by (ii) that $c+a$ and $c+b$ exist and $c+(a \vee b) \geq c+a, c+b$. Let $x \geq c+a, c+b$. Then for some y, z we have $x = c+a+y = c+b+z$. Since $a+y = b+z \geq a \vee b$, we get by (ii) $x \geq c+(a \vee b)$. It follows $c+(a \vee b) = (c+a) \vee (b+a)$.

The second part is proved similarly. $\square$

We will now see how generalized effect algebras are characterized among generalized pseudo-effect algebras.

Definition 2.5 Let $(E;+,0)$ be a generalized pseudo-effect algebra. We say that two elements a and b of E *commute* if $a+b$ and $b+a$ both exist and are equal.

We say that E is *commutative* if, for $a, b \in E$, $a+b$ is defined if and only if $b+a$ is defined, in which case $a+b=b+a$.

Proposition 2.6 Let $(E;+,0)$ be a generalized pseudo-effect algebra. Then E is a generalized effect algebra if and only if E is commutative.

Proof. This is evident. $\square$

We conclude this chapter by noting how generalized pseudo-effect algebras are related to pseudo-effect algebras. This relationship is, as to be expected, the same as between generalized effect algebras and effect algebras.

We recall [DvVe1] that a pseudo-effect algebra is a structure $(E;+,0,1)$, where $+$ is a partial binary operation and 0, 1 are constants, such that, for all $a,b,c\in E$, (PE1) $(a+b)+c$ is defined iff $a+(b+c)$ is defined, in which case these elements are equal, (PE2) for exactly one $d\in E$ and exactly one $e\in E$ we have $a+d=e+a=1$, (PE3) if $a+b$ exists, there are $d,e\in E$ such that $a+b=d+a=b+e$, (PE4) if $1+a$ or $a+1$ exists, then $a=0$.

Proposition 2.7 Let $(E;+,0)$ be a generalized pseudo-effect algebra, and let $u\in E$. Then $(E[u];+,0,u)$ is a pseudo-effect algebra, where $E[u]\stackrel{\text{def}}{=}\{a\in E:\ a\leq u\}$, and where $+$ is the restriction of the equally denoted operation of E to those pairs of elements of $E[u]$ whose sum is still in $E[u]$.

In particular, if a GPE-algebra $(E;+,0)$ has a greatest element 1, then $(E;+,0,1)$ is a pseudo-effect algebra.

Conversely, let $(E;+,0,1)$ be a pseudo-effect algebra. Then $(E;+,0)$ is a generalized pseudo-effect algebra.

Proof. To show the first part, let $(E;+,0)$ be a GPE-algebra and $u\in E$.

To prove that (PE1) holds in the algebra $(E[u];+,0,u)$, let $a,b,c\leq E[u]$ such that $a+b$ and $(a+b)+c$ exist in $E[u]$. By (GPE1), $b+c$ and $a+(b+c)$ exist in E and $(a+b)+c=a+(b+c)$ in E. Since $b+c\leq a+(b+c)=(a+b)+c\leq u$, also $b+c$ and $a+(b+c)$ are elements of $E[u]$, and $(a+b)+c=a+(b+c)$ holds in $E[u]$. Similarly, we see that if $b+c$ and $a+(b+c)$ are defined in $E[u]$, so are $a+b$ and $(a+b)+c$. It follows that (PE1) holds in $E[u]$.

Moreover, (PE2) holds in $E[u]$ due to (GPE2) and (GPE3). (PE3) holds due to (GPE2). If $u+a$ exists in $E[u]$, we have $u+a\leq u=u+0$ by (GPE5), and by Lemma 2.4(ii) $a\leq 0$, so $a=0$. Similarly, from the existence of $a+u$ we conclude $a=0$. So (PE4) holds.

The second part of the statement is obvious from the first one.

To show the last part, let $(E;+,0,1)$ be a pseudo-effect algebra. For the algebra $(E;+,0)$, (GPE1) follows from (PE1), (GPE2) from (PE3), and (GPE3), (GPE4), (GPE5) from [DvVe1, Lemma 1.4(v), (ii), (i)]. $\square$

3 Generalized Pseudo-Effect Algebras with Riesz Properties

For the purpose of a structure theory for generalized pseudo-effect algebras, we will introduce for these algebras four properties that may be compared to the Riesz Decomposition Property as known for *po*-groups.

Definition 3.1 Let $(E;+,0)$ be a generalized pseudo-effect algebra.

(a) For $a, b \in E$, we write $a \,\mathbf{com}\, b$ to mean that for all $a_1 \leq a$ and $b_1 \leq b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Riesz Interpolation Property*, (RIP) for short, if for any $a, b, c, d \in E$ such that $a, b, \leq c, d$ there is a $x \in E$ such that $a, b \leq x \leq c, d$.

(c) We say that E fulfils the *Weak Riesz Decomposition Property*, (RDP_0) for short, if for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1,\ d_2 \leq b_2$ and $a = d_1 + d_2$.

(d) We say that E fulfils the *Commutational Riesz Decomposition Property*, (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1,\ d_3 + d_4 = a_2,\ d_1 + d_3 = b_1,\ d_2 + d_4 = b_2$, and (ii) $d_2 \,\mathbf{com}\, d_3$.

(e) We say that E fulfils the *Strong Riesz Decomposition Property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1,\ d_3 + d_4 = a_2,\ d_1 + d_3 = b_1,\ d_2 + d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

Among these four Riesz properties, there is no pair of equivalent ones, but we may order them by strength. Furthermore, the properties (RDP_1) and (RDP_2) possess equivalent formulations, involving sums of any finite number of elements rather than only sums of pairs.

To make collections of formulas as they will appear in the sequel easier readable, we use the following abbreviation. Let $d_{ij}, a_i, b_j \in E,\ \ 1 \leq i \leq m,\ 1 \leq j \leq n$. By

$$\begin{array}{ccccc} d_{11} & \dots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \dots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \dots & b_n & & \end{array}$$

we mean $d_{i1} + \cdots + d_{in} = a_i$ for $i = 1, \dots, m$ and $d_{1j} + \cdots + d_{mj} = b_j$ for $j = 1, \dots, n$.

Lemma 3.2 Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (RDP_0).

(i) Let $a+b$ exist. From $a\,\mathbf{com}\,c$ and $b\,\mathbf{com}\,c$ it follows $(a+b)$ $\mathbf{com}$ c.

From $a \wedge c = 0$ and $b \wedge c = 0$ it follows $(a+b) \wedge c = 0$.

(ii) If $a \wedge b = 0$ and if a and b possess an upper bound, then $a+b = b+a = a \vee b$.

Proof. (i) Let $a\,\mathbf{com}\,c$ and $b\,\mathbf{com}\,c$ hold. Suppose $d \leq a+b$, $c_1 \leq c$. Due to (RDP_0) there are elements d_1 and d_2 such that $d_1 \leq a$, $d_2 \leq b$ and $d = d_1 + d_2$. By assumption each of d_1 and d_2 commutes with c_1; so also $d = d_1 + d_2$ commutes with c_1. It follows $(a+b)\,\mathbf{com}\,c$.

In a similar way, we prove the second part.

(ii) Let a, b, c be given such that $a \wedge b = 0$ and $c \geq a, b$. So for some x, $c = x + b$, and by (RDP_0) there are elements $d_1 \leq x$, $d_2 \leq b$ such that $a = d_1 + d_2$. Since $d_2 \leq a, b$ we have $d_2 = 0$, and so $a \leq x$. It follows $c = x + b \geq a + b$. In particular, $a+b$ exists, and as $a+b$ is an upper bound of a and b, it follows $a + b = a \vee b$. □

Lemma 3.3 Let $(E; +, 0)$ be a generalized pseudo-effect algebra.

(i) We have the implications
$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$
The converse of any of these implications does not hold.

(ii) Let (RDP_0) hold and E be a lower semi-lattice. Then (RDP_2) holds.

(iii) Let (RDP_1) hold. Given $a_1, \ldots, a_m, b_1, \ldots b_n \in E$, $n, m \geq 1$, such that $a_1 + \cdots + a_m = b_1 + \cdots + b_n$, there are elements $d_{11}, \ldots, d_{mn} \in E$ such that

$$\begin{array}{ccccc} d_{11} & \ldots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \ldots & b_n & & \end{array} \tag{1}$$

and such that we have

$$(d_{i+1,j} + \cdots + d_{mj}) \ \mathbf{com} \ (d_{i,j+1} + \cdots + d_{in}); \ \ 1 \leq i < m,\ 1 \leq j < n. \tag{2}$$

In case that (RDP_2) holds, we may further assume

$$(d_{i+1,j} + \cdots + d_{mj}) \wedge (d_{i,j+1} + \cdots + d_{in}) = 0; \ \ 1 \leq i < m,\ 1 \leq j < n. \tag{3}$$

Proof. (i) By Lemma 3.2(ii), (RDP_2) implies (RDP_1). It is clear that (RDP_1) implies (RDP_0).

Assume now (RDP_0), and let $a, b \leq c, d$. Then there is an $a_1 \in E$ such that $a + a_1 = c$, and from $b \leq a + a_1$ follows by (RDP_0) that there are $e \leq a$ and $\bar{b} \leq a_1$ such that $b = e + \bar{b}$. Moreover, for some $\bar{a} \in E$ we have $a = e + \bar{a}$, and for some $\bar{c}, \bar{d}$ also $c = e + \bar{c}$ and $d = e + \bar{d}$.

By Lemma 2.4(ii) we have $\bar{a}, \bar{b} \leq \bar{c}, \bar{d}$, and from $e + \bar{a} + \bar{b} \leq a + a_1 = c = e + \bar{c}$ it follows that $\bar{a} + \bar{b} \leq \bar{c}$.

Now chose $b_1 \in E$ such that $b_1 + \bar{b} = \bar{d}$, and since $\bar{a} \leq b_1 + \bar{b}$, there are $\bar{\bar{a}} \leq b_1$ and $f \leq \bar{b}$ such that $\bar{a} = \bar{\bar{a}} + f$. Chose $\bar{\bar{b}} \in E$ such that $\bar{b} = \bar{\bar{b}} + f$.

Let $\bar{x} = \bar{\bar{a}} + \bar{\bar{b}} + f$. Then $\bar{a}, \bar{b} \leq \bar{x} \leq \bar{a} + \bar{b} \leq \bar{c}$ as well as $\bar{x} \leq b_1 + \bar{b} = \bar{d}$. So $x = e + \bar{x}$ is the interpolant required to show that (RIP) holds.

By [DvVe1, Proposition 3.3(i)] and Proposition 2.7, the converse of any of the implications shown does not hold.

(ii) Let $a_1, a_2, b_1, b_2 \in E$ and suppose $a_1 + a_2 = b_1 + b_2$. Set $d_1 = a_1 \wedge b_1$ and d_2, d_3 in such a way that $a_1 = d_1 + d_2$, $b_1 = d_1 + d_3$.

We claim that $d_2 \leq b_2$. Indeed, from $a_1 \leq b_1 + b_2$ we get by (RDP$_0$) that $a_1 = e_1 + e_2$ for some $e_1 \leq b_1,\ e_2 \leq b_2$, and from $e_1 \leq a_1 \wedge b_1 = d_1$ we have $a_1 \leq d_1 + b_2$, so we conclude $d_2 \leq b_2$. Choose $d_4 \in E$ such that $d_2 + d_4 = b_2$. Similarly, we may choose $d_4' \in E$ such that $d_3 + d_4' = a_2$.

Furthermore, $d_2 \wedge d_3 = 0$ since by Lemma 2.4(iii) we have $d_1 = a_1 \wedge b_1 = (d_1 + d_2) \wedge (d_1 + d_3) = d_1 + (d_2 \wedge d_3)$. By Lemma 3.2(ii) we conclude $d_1 + d_2 + d_3 + d_4' = a_1 + a_2 = b_1 + b_2 = d_1 + d_3 + d_2 + d_4 = d_1 + d_2 + d_3 + d_4$, so $d_4' = d_4$. So d_1, d_2, d_3, d_4 fulfil the requirements of Definition 3.1(e) of (RDP$_2$).

(iii) The Lemma is trivial for $m = 1$ or $n = 1$. Let $m = n = 2$. If just (RDP$_1$) holds, there are by definition $d_{11}, d_{12}, d_{21}, d_{22} \in E$ such that (1) and (2) and holds. If even (RDP$_2$) holds, $d_{11}, \ldots, d_{22}$ may be chosen such that that (1) and (3) and holds, and since d_{12} and d_{21} possess an upper bound, also (2) is true by Lemma 3.2(ii).

Suppose now the Lemma is true for any pair of integers such that the first is less or equal to m, where $m \geq 2$, and the second is less than n, where $n \geq 3$. From this condition we shall prove that it is true also for the pair m, n. By complete induction the Lemma then follows to be true in general.

So let $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. By the induction hypothesis, there are elements $d_{11}, \ldots, d_{m,n-2}, e_1, \ldots, e_m \in E$ such that

$$\begin{array}{ccccccc} d_{11} & \ldots & d_{1,n-2} & e_1 & \rightarrow & a_1 \\ \vdots & & \vdots & \vdots & & \vdots \\ d_{m1} & \ldots & d_{m,n-2} & e_m & \rightarrow & a_m \\ \downarrow & & \downarrow & \downarrow & & \\ b_1 & \ldots & b_{n-2} & b_{n-1} + b_n & & \end{array}$$

and for $1 \leq i < m,\ 1 \leq j \leq n-2$ we have

$$(d_{i+1,j} + \cdots + d_{mj}) \ \mathbf{com}\ (d_{i,j+1} + \cdots + d_{i,n-2} + e_i)$$

and in case of (RDP$_2$) also

$$(d_{i+1,j} + \cdots + d_{mj}) \wedge (d_{i,j+1} + \cdots + d_{i,n-2} + e_i) = 0.$$

Moreover, from $e_1 + \cdots + e_m = b_{n-1} + b_n$ we have by the same hypothesis that there are elements $d_{1,n-1}, d_{1n}, \ldots, d_{m,n-1}, d_{mn} \in E$ such that

$$\begin{array}{ccccc} d_{1,n-1} & d_{1n} & \rightarrow & e_1 \\ \vdots & \vdots & & \vdots \\ d_{m,n-1} & d_{mn} & \rightarrow & e_m \\ \downarrow & \downarrow & & \\ b_{n-1} & b_n & & \end{array}$$

and for $1 \leq i \leq m-1$

$$(d_{i+1,n-1} + \cdots + d_{m,n-1}) \ \mathbf{com} \ d_{in}$$

and in case of (RDP_2) also

$$(d_{i+1,n-1} + \cdots + d_{m,n-1}) \wedge d_{in} = 0.$$

It follows that the Lemma holds for the pair m, n. □

4 Embedding of Generalized Pseudo-Effect Algebras into Positive Cones of Groups

We are concerned in this chapter with the question under which conditions generalized pseudo-effect algebra are representable as a subset of the positive cone of a *po*-group, just in the way shown in Example 2.3.

Unfortunately, the question how to characterize exactly those GPE-algebras that possess a representation by means of a *po*-group seems to be very hard to give an answer to. Here, we will introduce two different conditions, to be called (A) and (B), that at least turn out to be sufficient for GPE-algebras to be representable that way.

The conditions are defined in the following manner; E is supposed to be a generalized pseudo-effect algebra.

(A) E is a lower semi-lattice and fulfils (RDP_0).

(B) E is directed and fulfils (RDP_1).

We note that no one of these conditions implies the other one, as seen from the following examples.

Example 4.1 Set $E \stackrel{\text{def}}{=} (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N})$, and define a partial addition componentwise whenever this leads to a result in E. Then $(E; +, (0,0))$ is a GPE-algebra. Moreover, it is a lower semi-lattice; e.g. we have for $a, b \in \mathbb{N}$ $(a,0) \wedge (0,b) = (0,0)$ and $(a,0) \wedge (b,0) = (\min\{a,b\}, 0)$. As it is easily verified, E also fulfils (RDP_0). But it is not directed; for $a, b \in \mathbb{N}$ such that $a, b \geq 1$, an upper bound of $(a,0)$ and $(0,b)$ does not exist.

Example 4.2 Let E be the set of rational functions f: $[0,1] \to \mathbb{R}$ without singularities, define $+$ for any pair of functions as the pointwise addition and 0 as the zero constant function (see [DvVe1, Example 3.4]). Then $(E; +, 0)$ is a generalized effect algebra, so a fortiori a GPE-algebra. Moreover, it fulfils (RDP_1) and is directed, but it is not a lower semi-lattice.

For the proof of our representation theorem, we will use a variant of the so-called word technique. This technique was introduced by Baer [Baer] and Wyler [Wyl]; furthermore, to mention just one example, it has been applied to pseudo-effect algebras fulfilling (RDP_1) [DvVe2].

We will first embed a given algebra into a semigroup (Definition 4.3 and Lemma 4.4); then the conditions that are necessary and sufficient for extending a semigroup to a *po*-group are checked (Lemmas 4.5 and 4.6).

Definition 4.3 Let $(E; +, 0)$ be a generalized pseudo-effect algebra.

(i) A sequence $A = (a_1, \ldots, a_n)$ of finite, but non-zero, length with entries from E is called a *word* in E. We denote by $\mathcal{W}(E)$ the set of all words; that is

$$\mathcal{W}(E) \stackrel{\text{def}}{=} \{(a_1, \ldots, a_n)\colon\ a_1, \ldots, a_n \in E, n \geq 1\}.$$

We define an *addition* in $\mathcal{W}(E)$ as the concatenation; that is

$$\begin{aligned} +\colon\quad & \mathcal{W}(E) \times \mathcal{W}(E) \to \mathcal{W}(E), \\ & ((a_1, ..., a_m), (b_1, ..., b_n)) \mapsto (a_1, ..., a_m, b_1, ..., b_n). \end{aligned}$$

(ii) We call two words A and B of E *directly similar*, in symbols $A \sim B$, if one of the following conditions is fulfilled.

(α) $A = (a_1, \ldots, a_n)$ and $B = (a_1, ..., a_p + a_{p+1}, ..., a_n)$; $1 \leq p < n$.
(β) $A = (a_1, ..., a_p + a_{p+1}, ..., a_n)$ and $B = (a_1, \ldots, a_n)$; $1 \leq p < n$.
(γ) $A = (a_1, \ldots, a_n)$, $B = (a_1, ..., a_{p+1}, a_p, ..., a_n)$ and $a_p \wedge a_{p+1} = 0$; $1 \leq p < n$.

We call two words A and B *similar*, in symbols $A \simeq B$, if there are words $A_0, \ldots, A_k$, $k \geq 0$, such that $A = A_0 \sim A_1 \sim \cdots \sim A_k = B$. In such a case we say that A and B are *connected by a chain of length* k.

We set for $a_1, \ldots, a_n \in E$, $n \geq 1$,

$$[a_1, \ldots, a_n] \stackrel{\text{def}}{=} \{A \in \mathcal{W}(E)\colon\ A \simeq (a_1, \ldots, a_n)\},$$

and we put

$$\mathcal{C}(E) \stackrel{\text{def}}{=} \{[a_1, \ldots, a_n]\colon\ a_1, \ldots, a_n \in E, n \geq 1\}.$$

Lemma 4.4 Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (RDP_0).

(i) Similarity in $\mathcal{W}(E)$ is an equivalence relation compatible with $+$.

$+$ being the induced relation, $(\mathcal{C}(E); +)$ is a semigroup with the neutral element $[0]$.

(ii) For $a_1, \ldots, a_n, b \in E$, $n \geq 1$, $(a_1, \ldots, a_n) \simeq (b)$ if and only if $a_1 + \cdots + a_n$ exists and equals b.

Proof. (i) $\simeq$ is by construction an equivalence relation.

From $A_1 \simeq A$ and $B_1 \simeq B$ it follows $A_1 + B_1 \simeq A + B$, so $+$ is definable in $\mathcal{C}(E)$.

As $\mathcal{W}(E)$ is associative, so is $\mathcal{C}(E)$, i.e. $\mathcal{C}(E)$ is a semigroup. It has $[0]$ as a neutral element, because e.g. $[a_1, \ldots, a_n] + [0] = [a_1, \ldots, a_n, 0] = [a_1, \ldots, a_n]$.

(ii) If for a word $(x_1, \ldots, x_m)$ the sum of its elements $x_1 + \cdots + x_m$ exists, the same is true for any word directly similar to $(x_1, \ldots, x_m)$, and the sum remains the same. For this is evident in case the two words are related by condition (α) or (β) of Definition 4.3; and if they are related by condition (γ), the two permuted elements commute by Lemma 3.2(ii), because they possess their sum as an upper bound and we assumed (RDP_0).

So the "only if" part follows by induction on the minimal length of a chain by which (b) and $(a_1, \ldots, a_n)$ are connected.

The "if" part is obvious. $\square$

Lemma 4.5 Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (A) or (B). Let

$$(a_1, \ldots, a_m) \simeq (b_1, \ldots, b_n),$$

where $m, n \geq 1$. Then there are elements $d_{11}, \ldots, d_{mn} \in E$ such that

$$\begin{array}{ccccc} d_{11} & \ldots & d_{1n} & \rightarrow & a_1 \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mn} & \rightarrow & a_m \\ \downarrow & & \downarrow & & \\ b_1 & \ldots & b_n. & & \end{array}$$

In case that (A) holds, we may further assume that, for $1 \leq i < m,\ 1 \leq j < n$ we have

$$(d_{i+1,j} + \cdots + d_{mj}) \wedge (d_{i,j+1} + \cdots + d_{in}) = 0. \tag{4}$$

In case that (B) holds, we may further assume that, for $1 \leq i < m,\ 1 \leq j < n$ we have

$$(d_{i+1,j} + \cdots + d_{mj}) \ \mathbf{com}\ (d_{i,j+1} + \cdots + d_{in}). \tag{5}$$

Proof. The proof is by induction on the minimal length k of a chain that connects $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$.

If $k = 0$, we have $(a_1, \ldots, a_m) = (b_1, \ldots, b_n)$, and the elements in the scheme

$$\begin{array}{cccccc} a_1 & 0 & \ldots & 0 & \rightarrow & a_1 \\ 0 & a_2 & \ldots & 0 & \rightarrow & a_2 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_m & \rightarrow & a_m \\ \downarrow & \downarrow & & \downarrow & & \\ a_1 & a_2 & \ldots & a_m & & \end{array}$$

obviously fulfill the statement.

Suppose the statement is true for $k-1,\ k \geq 1$; we have to prove that it then also holds for k. So let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_n)$ be connected by a chain of length k, say $A = A_0 \sim \cdots \sim A_k = B$.

There are according to Definition 4.3(ii) three possibilities how $A_k = B = (b_1, \ldots, b_n)$ is constructed from A_{k-1}.

Ad (α). Let $A_{k-1} = (b_1, \ldots, b_p^1, b_p^2, \ldots, b_n),\ b_p = b_p^1 + b_p^2;\ 1 \leq p \leq n$. Then there are by hypothesis elements in E according to the scheme

$$\begin{array}{ccccccccc} d_{11} & \ldots & d_{1p}^1 & d_{1p}^2 & \ldots & d_{1n} & \to & a_1 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mp}^1 & d_{mp}^2 & \ldots & d_{mn} & \to & a_m \\ \downarrow & & \downarrow & \downarrow & & \downarrow & & \\ b_1 & \ldots & b_p^1 & b_p^2 & \ldots & b_n, & & \end{array} \tag{6}$$

such that the sums of the elements placed below and right from any element fulfil the zero-infimum relation or the **com**-relation, respectively. Consider now the scheme

$$\begin{array}{ccccc} d_{11} & \ldots & d_{1p}^1 + d_{1p}^2 & \ldots & d_{1n} \\ \vdots & & \vdots & & \vdots \\ d_{m1} & \ldots & d_{mp}^1 + d_{mp}^2 & \ldots & d_{mn}. \end{array}$$

Here the lines still sum up to $a_1, \ldots, a_m$, and the unchanged columns sum up as before to $b_1, \ldots, b_{p-1}, b_{p+1}, \ldots, b_n$. Furthermore, any d_{ip}^1, $1 < i \leq m$, commutes with any $d_{i'p}^2$, $1 \leq i' < i$. This follows in case of (A) from $d_{ip}^1 \wedge d_{i'p}^2 = 0$ and $d_{ip}^1, d_{i'p}^2 \leq b_p$ by Lemma 3.2(ii), and in case of (B) from $d_{ip}^1 \,\mathbf{com}\, d_{i'p}^2$.

So also the p-th column adds up to what it should:

$$\begin{aligned} b_p \ = \ b_p^1 + b_p^2 \ &= \ d_{1p}^1 + \cdots + d_{mp}^1 + d_{1p}^2 + \cdots + d_{mp}^2 \\ &= \ d_{1p}^1 + d_{1p}^2 + d_{2p}^1 + \cdots + d_{mp}^1 + d_{2p}^2 + \cdots + d_{mp}^2 \\ &= \ \ldots \\ &= \ d_{1p}^1 + d_{1p}^2 + d_{2p}^1 + d_{2p}^2 + \cdots + d_{mp}^1 + d_{mp}^2. \end{aligned}$$

In case of (A), the only new relations concerning zero infima are the following, where $1 \leq j < m$:

$$(d^1_{j+1,p} + d^2_{j+1,p} + \cdots + d^1_{mp} + d^2_{mp}) \wedge (d_{j,p+1} + \cdots + d_{jn}) = 0. \tag{7}$$

By the same reasoning as before, the first term in (7) is equal to $(d^1_{j+1,p} + \cdots + d^1_{mp}) + (d^2_{j+1,p} + \cdots + d^2_{mp})$. Here, we know by hypothesis that both the first term inside brackets and the second one possesses together with the second term in (7) a zero infimum . Now by Lemma 3.2(i), (7) follows.

In a similar manner, we argue in the case of (B).

Ad (β). Let $A_{k-1} = (b_1, \ldots, b_p + b_{p+1}, \ldots, b_n)$; $1 \leq p < n$. Then there are by hypothesis elements in E according to the scheme

$$\begin{array}{ccccccc}
d_{11} & \ldots & e_1 & \ldots & d_{1n} & \rightarrow & a_1 \\
\vdots & & \vdots & & \vdots & & \vdots \\
d_{m1} & \ldots & e_m & \ldots & d_{mn} & \rightarrow & a_m \\
\downarrow & & \downarrow & & \downarrow & & \\
b_1 & \ldots & b_p + b_{p+1} & \ldots & b_n, & &
\end{array}$$

such that the sums of the elements places below and right from any element fulfil the **com**-relation or the zero infimum relation, respectively. Now since (A) implies (RDP_2) by Lemma 3.3(ii) and (B) includes (RDP_1), we may apply Lemma 3.3(iii) to the equation

$$b_p + b_{p+1} = e_1 + \cdots + e_m$$

to get elements $d_{1p}, d_{1,p+1}, \ldots, d_{mp}, d_{m,p+1} \in E$ such that

$$\begin{array}{cccc}
d_{1p} & d_{1,p+1} & \rightarrow & e_1 \\
\vdots & \vdots & & \vdots \\
d_{mp} & d_{m,p+1} & \rightarrow & e_m \\
\downarrow & \downarrow & & \\
b_p & b_{p+1} & &
\end{array}$$

and, for $1 \leq j < m$, in case of (A)

$$(d_{j+1,p} + \cdots + d_{mp}) \wedge d_{j,p+1} = 0,$$

in case of (B)

$$(d_{j+1,p} + \cdots + d_{mp}) \ \mathbf{com} \ d_{j,p+1}.$$

So we have

$$\begin{array}{ccccccccc}
d_{11} & \dots & d_{1p} & d_{1,p+1} & \dots & d_{1n} & \to & a_1 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{m1} & \dots & d_{mp} & d_{m,p+1} & \dots & d_{mn} & \to & a_m \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \\
b_1 & \dots & b_p & b_{p+1} & \dots & b_n. & &
\end{array}$$

As a sum when deleting any summand from it gets smaller, and as the zero-infimum relation as well as the **com**-relation is additive by Lemma 3.2(i), we see that the conditions concerning these relations are again preserved herein.

Ad (γ). Let $A_{k-1} = (b_1, \dots, b_{p+1}, b_p, \dots, b_n)$, $b_p \wedge b_{p+1} = 0$; $1 \le p < n$. There are by hypothesis elements in E according to the scheme

$$\begin{array}{ccccccccc}
d_{11} & \dots & d_{1,p} & d_{1,p+1} & \dots & d_{1n} & \to & a_1 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{m1} & \dots & d_{m,p} & d_{m,p+1} & \dots & d_{mn} & \to & a_m \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \\
b_1 & \dots & b_{p+1} & b_p & \dots & b_n, & &
\end{array} \tag{8}$$

such that the sums of the elements placed below and right from any element fulfil the zero-infimum relation or the **com**-relation, respectively. Consider now the scheme that is similar to (8), but in which the p-th and the $p+1$-th column are exchanged. Then clearly the columns sum up to the elements of A_k; and by Lemma 3.2(ii) the rows sum up to the same values as before.

Now the infimum of any of the elements $d_{1,p}, \dots, d_{m,p}$ and of any of $d_{1,p+1}, \dots, d_{m,p+1}$ is zero. If (B) holds, we see by the directedness of E and by Lemma 3.2(ii) that all these pairs fulfil also the **com**-condition.

Due to the additivity of the zero-infimum relation as well as of the **com**-relation, we see that the conditions concerning these relations are again preserved. □

Lemma 4.6 Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (A) or (B). Then $\mathcal{C}(E)$ is a semigroup such that the following holds.

(i) $[0]$ is a neutral element.

(ii) Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$. From $\mathfrak{a} + \mathfrak{b} = [0]$ it follows $\mathfrak{a} = \mathfrak{b} = [0]$.

(iii) Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{C}(E)$. From $\mathfrak{a} + \mathfrak{b} = \mathfrak{a} + \mathfrak{c}$ it follows $\mathfrak{b} = \mathfrak{c}$;
from $\mathfrak{b} + \mathfrak{a} = \mathfrak{c} + \mathfrak{a}$ it follows $\mathfrak{b} = \mathfrak{c}$.

(iv) For any pair $\mathfrak{a}, \mathfrak{b} \in \mathcal{C}(E)$ there are $\mathfrak{c}, \mathfrak{d} \in \mathcal{C}(E)$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c} + \mathfrak{a} = \mathfrak{b} + \mathfrak{d}$.

Proof. By Lemma 3.3(i) and (ii), (RDP_1) holds in E in any case.

That $\mathcal{C}(E)$ is a semigroup fulfilling (i) has been proved in Lemma 4.4(i).

(ii) This follows from Lemma 4.4(ii) and (GPE4).

(iii) We may suppose $\mathfrak{a} = [a]$. Let $\mathfrak{b} = [b_1, \ldots, b_m]$, $\mathfrak{c} = [c_1, \ldots, c_n]$. We will show that $(a, b_1, \ldots, b_m) \simeq (a, c_1, \ldots, c_n)$ implies $(b_1, \ldots, b_m) \simeq (c_1, \ldots, c_n)$; then the first part follows, and the second one is proved analogously.

By our assumption there are, by Lemma 4.5, elements in E according to the following scheme:

$$\begin{array}{cccccc}
d & d_1 & \ldots & d_n & \rightarrow & a \\
e_1 & e_{11} & \ldots & e_{1n} & \rightarrow & b_1 \\
\vdots & \vdots & & \vdots & & \vdots \\
e_m & e_{m1} & \ldots & e_{mn} & \rightarrow & b_m \\
\downarrow & \downarrow & & \downarrow & & \\
a & c_1 & & c_n, & &
\end{array}$$

where any pair of elements such that one is placed further up and further right than the other one has a zero infimum or fulfils the **com**-relation, respectively.

$a = d + d_1 + \cdots + d_n = d + e_1 + \cdots + e_m$ implies $d_1 + \cdots + d_n = e_1 + \cdots + e_m$. Using this and the fact we may exchange successive elements in a word whenever they have a zero infimum or fulfil the **com**-relation, we get

$$\begin{aligned}
(b_1, \ldots, b_m) &\simeq (e_1, e_{11}, \ldots, e_{1n}, \ldots, e_m, e_{m1}, \ldots, e_{mn}) \\
&\simeq (e_1, \ldots, e_m, e_{11}, \ldots, e_{m1}, \ldots, e_{1n}, \ldots, e_{mn}) \\
&\simeq (d_1, \ldots, d_n, e_{11}, \ldots, e_{m1}, \ldots, e_{1n}, \ldots, e_{mn}) \\
&\simeq (d_1, e_{11}, \ldots, e_{m1}, \ldots, d_n, e_{1n}, \ldots, e_{mn}) \\
&\simeq (c_1, \ldots, c_n).
\end{aligned}$$

(iv) We will prove one half only; the second one is shown analogously. Moreover, we suppose that $\mathfrak{a} = [a]$ and $\mathfrak{b} = [b]$; the claim then follows by an easy induction argument. So what we will show is that there is a word $\mathfrak{c}$ such that $(a, b) = \mathfrak{c} + (a)$.

Suppose first that E fulfils (A).

Choose $\bar{a}, \bar{b}$ such that $\bar{a} + (a \wedge b) = a$ and $\bar{b} + (a \wedge b) = b$. Then $\bar{a} \wedge \bar{b} = 0$; indeed, from $x \leq \bar{a}, \bar{b}$ it follows by Lemma 2.4(ii) that $x + (a \wedge b) \leq a, b$, so $x + (a \wedge b) \leq a \wedge b$, that is $x \leq 0$, and so $x = 0$. Now choose $\bar{\bar{a}}$ such that $\bar{\bar{a}} + \bar{a} = a$. Using condition (γ) of Definition 4.3(ii), we get $(a, b) \simeq (\bar{\bar{a}}, \bar{a}, \bar{b}, a \wedge b) \sim (\bar{\bar{a}}, \bar{b}, \bar{a}, a \wedge b) \sim (\bar{\bar{a}}, \bar{b}) + (a)$.

Suppose now that E fulfils (B).

Let c be an upper bound of a and b, which means that $a + \bar{a} = \bar{b} + b = c$ for some $\bar{a}, \bar{b} \in E$.

By (RDP_1) there are $d_1, \ldots, d_4 \in E$ such that

$$\begin{array}{cccc} d_1 & d_2 & \rightarrow & a \\ d_3 & d_4 & \rightarrow & \bar{a} \\ \downarrow & \downarrow & & \\ \bar{b} & b. & & \end{array}$$

So there are elements $d_2', d_4' \in E$ such that $(a, b) = (d_1 + d_2, d_2 + d_4) \sim (d_2' + d_1, d_2, d_4) \simeq (d_2', a, d_4) \sim (d_2', a + d_4) = (d_2', d_4' + a) \sim (d_2', d_4') + (a)$. Here we used (GPE2) and the fact that due to $d_4 \leq \bar{a}$, $a + d_4$ exists. $\square$

By a theorem of Birkhoff (see [Fu, Theorem II.4]), the conditions (i) to (iv) of the preceding Lemma are necessary and sufficient that $\mathcal{C}(E)$ is the positive cone of some *po*-group. So we may give the following definition; for details we refer to [Fu].

Definition 4.7 Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (A) or (B). Let $\mathcal{G}(E)$ be the *po*-group such that $\mathcal{C}(E) = \mathcal{G}(E)^+$ and $\mathcal{G}(E) = \mathcal{C}(E) - \mathcal{C}(E) = -\mathcal{C}(E) + \mathcal{C}(E)$.

We arrive at our main theorem.

Theorem 4.8 Let $(E;+,0)$ be a generalized pseudo-effect algebra fulfilling (A) or (B). Then $\iota_E\colon\ E \to \mathcal{G}(E),\ \ a \mapsto [a]$, is an isomorphic embedding of the GPE-algebra E into the directed *po*-group $\mathcal{G}(E)$. Furthermore, we have for $a, b, c \in E$ that

$$a + b \text{ is defined and equals } c \text{ if and only if } \iota_E(a) + \iota_E(b) = \iota_E(c).$$

Proof. (i) ι_E is injective. Indeed, for $a, b \in E$, $[a] = [b]$ implies $a = b$ by Lemma 4.4(ii).

Let now $a, b \in E$. If the sum exists and $a + b = c$, then $(a, b) \sim (c)$, so $[a] + [b] = [c]$, which means $\iota_E(a) + \iota_E(b) = \iota_E(c)$. Conversely, if for some $c \in E$ we have $\iota_E(a) + \iota_E(b) = \iota_E(c)$, then $[a] + [b] = [c]$, which means $(a, b) \sim (c)$, so $a + b = c$ by Lemma 4.4(ii).

$\mathcal{G}(E)$ is directed according to [Fu, Chapter II, Proposition 1], since $\mathcal{G}(E)^+$ generates the whole of $\mathcal{G}(E)$. □

5 Properties of the Representing Group of a GPE-algebra

We shall in this section compare the properties of a GPE-algebra on the one hand and of its representing group on the other one. We are in particular concerned with the lattice operations and with Riesz properties of different kind.

First of all, we may define the Riesz properties (RIP), (RDP_0), (RDP_1), (RDP_2) as well as the **com**-relation for *po*-group in exact analogy to GPE-algebras. We refer to [DvVe1] for details. Let us note the following facts concerning these properties.

Remark 5.1 We may formulate Lemma 3.3 for *po*-groups in the following manner. Let a *po*-group G be given.

(i) We have the implications (RDP_2) $\Rightarrow$ (RDP_1) $\Rightarrow$(RDP_0) $\Leftrightarrow$ (RIP). For the proof, see [DvVe1, Proposition 4.2(i)].

(ii) G is lattice-ordered iff (RDP_2) holds. For the proof, see [DvVe1, Proposition 4.2(ii)].

(iii) Let (RDP_1) hold. Given $a_1, \ldots, a_m, b_1, \ldots b_n \in G^+$, $n, m \geq 1$, such that $a_1 + \cdots + a_m = b_1 + \cdots + b_n$, there are elements $d_{11}, \ldots, d_{mn} \in G^+$ such that the scheme (1) and the relations (2) hold. In case that (RDP_2) holds, it may be further assumed that the relations (3) hold.

This follows from Lemma 3.3(iii), since $(G^+; +, 0)$ is a GPE-algebra.

From the following lemma, we see how the lattice operations and the Riesz properties are preserved from a group to a GPE-algebra constructed from its positive cone.

Proposition 5.2 Let G be a directed *po*-group and G_0 be a convex subset of G^+ containing 0. Let $(G_0; +, 0)$ be the GPE-algebra defined according to Example 2.3.

(i) If the *po*-group G fulfils one of the properties (RIP), (RDP_0), (RDP_1), or (RDP_2), the GPE-algebra $(G_0; 0, +)$ has the equally denoted property, respectively.

(ii) Let G fulfil (RIP). Then the infimum of some $a, b \in G_0$ exists with respect to the GPE-algebra G_0 if and only if it exists with respect to the *po*-group G, and both infima are equal.

If the supremum of some $a, b \in G_0$ exists with respect to the GPE-algebra G_0, then it exists with respect to the *po*-group G, and both suprema are equal.

Proof. (i) This follows easily from the fact that G_0 is convex and that the sum operations of G_0 and of G coincide.

(ii) The order of G_0 coincides with the one of G. We will, on the other hand, distinguish between the lattice operations of G_0 and G by appropriate indices. Let $a, b \in G_0$.

Let $a \wedge_{G_0} b$ exist. So $a \wedge_{G_0} b \leq a, b$. $x \in G$ and $x \leq a, b$ implies by (RIP) that there is some $y \in G$ such that $x, 0 \leq y \leq a, b$; we then have $y \in G_0$, since G_0 is convex, whence $x \leq y \leq a \wedge_{G_0} b$. It follows $a \wedge_{G_0} b = a \wedge_G b$.

If $a \wedge_G b$ exists, it is easy to see that then $a \wedge_{G_0} b = a \wedge_G b$.

Now let $a \vee_{G_0} b$ exist. So $a, b \leq a \vee_{G_0} b$. $x \in G$ and $a, b \leq x$ implies by (RIP) that there is some $y \in G$ such that $a, b \leq y \leq x, a \vee_{G_0} b$. Since then $y \in G_0$, we have $y = a \vee_{G_0} b$, whence $a \vee_{G_0} b \leq x$. It follows $a \vee_{G_0} b = a \vee_G b$. □

It is a little bit more tedious to prove that lattice operations and the properties (RDP_1) and (RDP_2) are also preserved in the other direction, from a GPE-algebra to its representing group.

Proposition 5.3 Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (A) or (B) and ι_E: $E \to \mathcal{G}(E)$ its *po*-group representation.

(i) Let E fulfil (A). Then $\mathcal{G}(E)$ is an ℓ-group. $\iota(E)$ is a convex subset of $\mathcal{G}(E)^+$ containing 0 that generates $\mathcal{G}(E)$.

(ii) Let E fulfil (B). Then $\mathcal{G}(E)$ is a directed *po*-group fulfilling (RDP_1). $\iota(E)$ is a directed convex subset of $\mathcal{G}(E)^+$ containing 0 that generates $\mathcal{G}(E)$.

(iii) ι_E preserves infima and suprema whenever they exist.

Proof. It was already established in Theorem 4.8 that $\mathcal{G}(E)$ is a directed *po*-group.

$\iota(E)$ clearly contains 0, and it is generates $\mathcal{G}(E)$ by construction. Moreover, it is convex; indeed, from $\mathfrak{a} \in \mathcal{G}(E)$, $b \in E$ and $0 \leq \mathfrak{a} \leq [b]$ it follows $\mathfrak{a} + \mathfrak{c} = [b]$ for some $\mathfrak{c} \in \mathcal{G}(E)^+$, and so $\mathfrak{a} \in \iota(E)$ by Lemma 4.4(ii).

(i) We will show that $\mathcal{G}(E)$ fulfils (RDP_2); by [DvVe1, Proposition 4.2(ii)], it then follows that $\mathcal{G}(E)$ is lattice-ordered.

So let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{C}(E) = \mathcal{G}(E)^+$ such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{b}_1 + \mathfrak{b}_2$. Let $\mathfrak{a}_1 = [a_1^1, \ldots, a_k^1]$, $\mathfrak{a}_2 = [a_1^2, \ldots, a_l^2]$, $\mathfrak{b}_1 = [b_1^1, \ldots, b_m^1]$, $\mathfrak{b}_2 = [b_1^2, \ldots, b_n^2]$, where $a_1^1, \ldots, b_n^2 \in E$. By Lemma 4.5 there are $d_{11}^1, \ldots, d_{ln}^4 \in E$ such that

$$\begin{array}{cccccccc}
d_{11}^1 & \ldots & d_{1m}^1 & d_{11}^2 & \ldots & d_{1n}^2 & \to & a_1^1 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{k1}^1 & \ldots & d_{km}^1 & d_{k1}^2 & \ldots & d_{kn}^2 & \to & a_k^1 \\
d_{11}^3 & \ldots & d_{1m}^3 & d_{11}^4 & \ldots & d_{1n}^4 & \to & a_1^2 \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots \\
d_{l1}^3 & \ldots & d_{lm}^3 & d_{l1}^4 & \ldots & d_{ln}^4 & \to & a_l^2 \\
\downarrow & & \downarrow & \downarrow & & \downarrow & & \\
b_1^1 & \ldots & b_m^1 & b_1^2 & \ldots & b_n^2 & &
\end{array} \tag{9}$$

and such that every two elements in this diagram possess the infimum 0 if one of them is placed further up and further right than the other one.

Define now $\mathfrak{d}_1 = [d_{11}^1, \ldots, d_{1m}^1, \ldots, d_{k1}^1, \ldots, d_{km}^1]$ and in a similar way also $\mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$. Due to the commutativity condition (γ) from Definition 4.3(ii), we then have

$$\begin{array}{cccc} \mathfrak{d}_1 & \mathfrak{d}_2 & \to & \mathfrak{a}_1 \\ \mathfrak{d}_3 & \mathfrak{d}_4 & \to & \mathfrak{a}_2 \\ \downarrow & \downarrow & & \\ \mathfrak{b}_1 & \mathfrak{b}_2. & & \end{array} \tag{10}$$

This already shows that (RDP_0) holds in $\mathcal{G}(E)$. By Remark 5.1(i) and Proposition 5.2(ii), we conclude that $\mathfrak{d}_2 \wedge \mathfrak{d}_3 = 0$. So (RDP_2) is proved.

(ii) $\iota(E)$ is directed, since E was supposed to be directed. It remains to prove (RDP_1).

So let $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{b}_1, \mathfrak{b}_2 \in \mathcal{C}(E)$ such that $\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{b}_1 + \mathfrak{b}_2$. Let $\mathfrak{a}_1 = [a_1^1, \ldots, a_k^1]$, $\mathfrak{a}_2 = [a_1^2, \ldots, a_l^2]$, $\mathfrak{b}_1 = [b_1^1, \ldots, b_m^1]$, $\mathfrak{b}_2 = [b_1^2, \ldots, b_n^2]$, where $a_1^1, \ldots, b_n^2 \in E$. By Lemma 4.5 there are $d_{11}^1, \ldots, d_{ln}^4 \in E$ such that (9) holds and such that every two elements in that diagram fulfil the **com**-condition if one of them is placed further up and further right than the other one.

Define now $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3, \mathfrak{d}_4$ as in the proof of (i). Due to the **com**-conditions, (10) then holds.

Now, for some $\mathfrak{x}, \mathfrak{y} \in \mathcal{G}(E)$, let $0 \le \mathfrak{x} \le \mathfrak{d}_2$ and $0 \le \mathfrak{y} \le \mathfrak{d}_3$. So $\mathfrak{x} + \mathfrak{x}' = \mathfrak{d}_2$ for some $\mathfrak{x}' \in \mathcal{C}(E)$, and we may apply to this equation Lemma 4.5 to conclude that $\mathfrak{x}$ is representable as a word such that every element in it lies below some element in $\mathfrak{d}_2$. The same applies to $\mathfrak{y}$ and $\mathfrak{d}_3$.

Now for every $x, y \in E$ such that x lies below an element occuring in the word $\mathfrak{d}_2$ and y lies below an element occuring in $\mathfrak{d}_3$, we have $x + y = y + x$. We conclude that $\mathfrak{x} + \mathfrak{y} = \mathfrak{y} + \mathfrak{x}$. So we have proved $\mathfrak{d}_2 \,\mathbf{com}\, \mathfrak{d}_3$. So (RDP_1) holds in $\mathcal{G}(E)$.

(iii) Since (RIP) holds in $\mathcal{G}(E)$ by parts (i), (ii) and Remark 5.1, this part follows from Proposition 5.2. $\square$

6 Categorical Equivalences of GPE-Algebras and *po*-Groups

After having discussed some correlations of GPE-algebras and their representing groups in the previous chapter, we now turn our attention to the question how the homomorphisms of these two types of structures are related to each other.

Definition 6.1 A mapping $\varphi\colon\ E \to F$ between GPE-algebras is called a homomorphism if (i) for any $a, b \in E$ such that $a+b$ is defined, also $\varphi(a)+\varphi(b)$ is defined and $\varphi(a+b) = \varphi(a) + \varphi(b)$, and (ii) $\varphi(0) = 0$.

A mapping $\varphi\colon\ E \to G$ of a GPE-algebra into a *po*-group is called a homomorphism if (i) $\varphi(E) \subseteq G^+$, (ii) for any $a, b \in E$ such that $a + b$ is defined, $\varphi(a + b) = \varphi(a) + \varphi(b)$, and (iii) $\varphi(0) = 0$.

We first show that the *po*-group $\mathcal{G}(E)$ representing a GPE-algebra E has the universal property, i.e. any homomorphism from E into another *po*-group factors through $\mathcal{G}(E)$.

Theorem 6.2 Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (A) or (B), and let $\iota_E\colon\ E \to \mathcal{G}(E)$ its *po*-group representation.

(i) Let E fulfil (A). Then for every infimum preserving homomorphism $h\colon\ E \to H$ from E into some ℓ-group H there is a unique ℓ-group homomorphism $h^{\mathcal{G}(E)}\colon\ \mathcal{G}(E) \to H$ such that $h = h^{\mathcal{G}(E)} \circ \iota_E$.

(ii) Let E fulfil (B). Then for every homomorphism $h\colon\ E \to H$ from E into some *po*-group H there is a unique *po*-group homomorphism $h^{\mathcal{G}(E)}$: $\mathcal{G}(E) \to H$ such that $h = h^{\mathcal{G}(E)} \circ \iota_E$.

Proof. (i) Let H be an ℓ-group and $h\colon\ E \to H$ a homomorphism that preserves the infima.

Since $\iota_E\colon\ E \to \mathcal{G}(E),\ \ a \mapsto [a]$, is injective, we may define $h^E\colon\ \iota_E(E) \to H,\ \ [a] \mapsto h(a)$. Then h^E preserves $+$ whenever defined in $\iota_E(E)$, and $h^E(0) = 0$. By Proposition 5.2(ii), h^E preserves infima.

Now we may extend h^E to the function $h^{\mathcal{C}(E)}\colon\ \mathcal{C}(E) \to H,\ [a_1, \ldots, a_k] \mapsto h(a_1) + \cdots + h(a_k)$. Indeed, the latter sum remains the same when chosing, rather than the word $(a_1, \ldots, a_k)$, a directly similar one, or consequently even a similar one, instead. $h^{\mathcal{C}(E)}$ by construction preserves $+$.

We have to show that $h^{\mathcal{C}(E)}$ preserves infima. So let $\mathfrak{a} = [a^1_1, \ldots, a^1_k]$ and $\mathfrak{b} = [b^1_1, \ldots, b^1_m]$ from $\mathcal{C}(E)$ be given; we have to show that $h^{\mathcal{C}(E)}(\mathfrak{a} \wedge \mathfrak{b}) = h^{\mathcal{C}(E)}(\mathfrak{a}) \wedge h^{\mathcal{C}(E)}(\mathfrak{b})$. By Proposition 5.3(i), $\mathcal{G}(E)$ is lattice ordered, so in particular it is directed and fulfils (RDP$_2$). Let $\mathfrak{a}' = [a^2_1, \ldots, a^2_l]$ and $\mathfrak{b}' = [b^2_1, \ldots, b^2_n]$ be chosen from $\mathcal{C}(E)$ such that $\mathfrak{a} + \mathfrak{a}' = \mathfrak{b} + \mathfrak{b}'$. Then by Lemma 4.5 there are $d^1_{11}, \ldots, d^4_{ln} \in E$ such that the scheme (9) holds and such that every pair of elements in it one of which is further up and further right than the other one possesses the infimum 0.

From the zero infimum relations we get $\mathfrak{a} \wedge \mathfrak{b} = [a^1_1, \ldots, a^1_k] \wedge [b^1_1, \ldots, b^1_m] = [d^1_{11}, \ldots, d^1_{1m}, \ldots, d^1_{k1}, \ldots, d^1_{km}]$. It follows $h^{\mathcal{C}(E)}(\mathfrak{a} \wedge \mathfrak{b}) = h^{\mathcal{C}(E)}([d^1_{11}]) + \cdots + h^{\mathcal{C}(E)}([d^1_{km}]) = h(d^1_{11}) + \cdots + h(d^1_{km})$. On the other hand, the scheme (9) and the zero infimum relations still hold if all elements are replaced by their images under h. So we also have $h^{\mathcal{C}(E)}(\mathfrak{a}) \wedge h^{\mathcal{C}(E)}(\mathfrak{b}) = [h(a^1_1) + \cdots + h(a^1_k)] \wedge [h(b^1_1) + \cdots + h(b^1_m)] = h(d^1_{11}) + \cdots + h(d^1_{km})$. We conclude $h^{\mathcal{C}(E)}(\mathfrak{a} \wedge \mathfrak{b}) = h^{\mathcal{C}(E)}(\mathfrak{a}) \wedge h^{\mathcal{C}(E)}(\mathfrak{b})$.

We may further extend $h^{\mathcal{C}(E)}$ to $h^{\mathcal{G}(E)}\colon\ \mathcal{G}(E) \to H$ by defining $h^{\mathcal{G}(E)}(\mathfrak{a}-\mathfrak{b}) \overset{\text{def}}{=} h^{\mathcal{C}(E)}(\mathfrak{a}) - h^{\mathcal{C}(E)}(\mathfrak{b})$, where $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}(E)^+$. For suppose $\mathfrak{a} - \mathfrak{b} = \mathfrak{c} - \mathfrak{d}$ for some $\mathfrak{c}, \mathfrak{d} \in \mathcal{G}(E)^+$; then setting $\mathfrak{b}' = \mathfrak{a} + \mathfrak{b} - \mathfrak{a} \in \mathcal{G}(E)^+$, we have $\mathfrak{a} + \mathfrak{b} = \mathfrak{b}' + \mathfrak{a}$ and hence $h^{\mathcal{C}(E)}(\mathfrak{a}) + h^{\mathcal{C}(E)}(\mathfrak{b}) = h^{\mathcal{C}(E)}(\mathfrak{b}') + h^{\mathcal{C}(E)}(\mathfrak{a})$; and we have $\mathfrak{a} + \mathfrak{d} = \mathfrak{b}' + \mathfrak{c}$ and hence $h^{\mathcal{C}(E)}(\mathfrak{a}) + h^{\mathcal{C}(E)}(\mathfrak{d}) = h^{\mathcal{C}(E)}(\mathfrak{b}') + h^{\mathcal{C}(E)}(\mathfrak{c})$; so we conclude $h^{\mathcal{C}(E)}(\mathfrak{a}) - h^{\mathcal{C}(E)}(\mathfrak{b}) = -h^{\mathcal{C}(E)}(\mathfrak{b}') + h^{\mathcal{C}(E)}(\mathfrak{a}) = h^{\mathcal{C}(E)}(\mathfrak{c}) - h^{\mathcal{C}(E)}(\mathfrak{d})$.

$h^{\mathcal{G}(E)}$ preserves $+$ on $\mathcal{G}(E)^+$, and by a similar kind of argumentation as in the last paragraph we see that it preserves $+$ on the whole group $\mathcal{G}(E)$.

We still have to prove that $h^{\mathcal{G}(E)}$ preserves infima. So let $\mathfrak{a}, \mathfrak{b} \in \mathcal{G}(E)$. Put $\mathfrak{c} = \mathfrak{a} \wedge \mathfrak{b}$. Then we have $\mathfrak{a} - \mathfrak{c}, \mathfrak{b} - \mathfrak{c} \geq 0$, and so $h^{\mathcal{G}(E)}(\mathfrak{a} \wedge \mathfrak{b}) = h^{\mathcal{G}(E)}[(\mathfrak{a} - \mathfrak{c}) \wedge (\mathfrak{b} - \mathfrak{c})] + \mathfrak{c}) = [h^{\mathcal{G}(E)}(\mathfrak{a}) - h^{\mathcal{G}(E)}(\mathfrak{c})] \wedge [h^{\mathcal{G}(E)}(\mathfrak{b}) - h^{\mathcal{G}(E)}(\mathfrak{c})] + h^{\mathcal{G}(E)}(\mathfrak{c}) = h^{\mathcal{G}(E)}(\mathfrak{a}) \wedge h^{\mathcal{G}(E)}(\mathfrak{b})$.

So $h^{\mathcal{G}(E)}$ is proved to be an ℓ-group homomorphism such that $h = h^{\mathcal{G}(E)} \circ \iota_E$, and it is unique since $\iota_E(E)$ generates $\mathcal{G}(E)$.

(ii) Let H be a *po*-group and $h\colon\ E \to H$ a homomorphism. Similarly as in part (i), we define h^E and extend it to $h^{\mathcal{C}(E)}$ and further to $h^{\mathcal{G}(E)}$, to see that this is the unique *po*-group homomorphism from $\mathcal{G}(E)$ to H such that $h = h^{\mathcal{G}(E)} \circ \iota_E$. □

We now establish two categorical equivalences, where our objects are GPE-algebras on the one side and *po*-groups on the other side.

Definition 6.3 Let $\mathcal{GPE}$A be the category whose objects are the generalized pseudo-effect algebras fulfilling condition (A) and whose morphisms are the infimum preserving GPE-algebra homomorphisms.

Let $\mathcal{GPE}$B be the category whose objects are the generalized pseudo-effect algebras fulfilling condition (B) and whose morphisms are the GPE-algebra homomorphisms.

Let $\mathcal{LG}$ be the category whose objects are the pairs (G, G_0) of an ℓ-group G and of a convex subset G_0 of G^+ containing 0 that generates G, and whose morphisms $\varphi\colon\ (G, G_0) \to (H, H_0)$ are the ℓ-group homomorphisms $\varphi\colon\ G \to H$ such that $\varphi(G_0) \subseteq H_0$.

Let $\mathcal{POG}$R1 be the category whose objects are the pairs (G, G_0) of a directed *po*-group G that fulfils (RDP$_1$) and of a directed convex subset G_0 of G^+ containing 0 that generates G, and whose morphisms $\varphi\colon\ (G, G_0) \to (H, H_0)$ are the *po*-group homomorphisms $\varphi\colon\ G \to H$ such that $\varphi(G_0) \subseteq H_0$.

Finally, let $\Delta_A\colon\ \mathcal{GPE}\mathrm{A} \to \mathcal{LG}$ and $\Delta_B\colon\ \mathcal{GPE}\mathrm{B} \to \mathcal{POG}\mathrm{R1}$ be the functors that assign to every object E from their domain the pair $(\mathcal{G}(E), E)$ and to every morphism $\varphi\colon\ E \to F$ from their domain the group homomorphism $\psi\colon\ \mathcal{G}(E) \to \mathcal{G}(F)$ subject to the condition $\psi \circ \iota_E = \iota_F \circ \varphi$.

It is clear from Proposition 5.3 and Theorem 6.2, that Δ_A and Δ_B are well-defined as functors between categories.

Theorem 6.4 The functor Δ_A establishes a categorical equivalence between $\mathcal{GPE}$A and $\mathcal{LG}$.

The functor Δ_B establishes a categorical equivalence between $\mathcal{GPE}$B and $\mathcal{POG}$R1.

Proof. It follows from the uniqueness part of Theorem 6.2, that Δ_A and Δ_B are full and faithful functors.

We prove next that Δ_A is isomorphism-dense. Let a pair (G, G_0) from $\mathcal{LG}$ be given; we have to show that the image of some object from $\mathcal{GPE}$A is isomorphic to (G, G_0). Now by Proposition 5.2, $(G_0; +, 0)$ is a GPE-algebra fulfilling (A); so let us show that $(\mathcal{G}(G_0), \iota_{G_0}(G_0))$ is isomorphic to (G, G_0). By Theorem 6.2 the map κ: $\iota_{G_0}(G_0) \to G$, $[a] \mapsto a$, is extendible to an ℓ-group homomorphism $\kappa^{\mathcal{G}(G_0)}$: $\mathcal{G}(G_0) \to G$. The latter is surjective since G_0 was assumed to generate G; and we have $\kappa^{\mathcal{G}(G_0)}(\iota_{G_0}(G_0)) = G_0$. It is also injective; indeed, $\kappa^{\mathcal{G}(G_0)}([a_1, \ldots a_m] - [b_1, \ldots b_n]) = 0$ for some $a_1, \ldots a_m, b_1, \ldots b_n \in G_0$ means $a_1 + \cdots + a_m = b_1 + \cdots + b_n$; and this implies by Remark 5.1(iii) that $(a_1, \ldots, a_m) \simeq (b_1, \ldots, b_n)$, hence $[a_1, \ldots, a_m] - [b_1, \ldots, b_n] = 0$.

We see similarly that also Δ_B is isomorphism-dense. $\square$

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