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# Algebras in the positive cone of $po$-groups

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# Algebras in the Positive Cone of *po*-Groups

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**Abstract**

We systemize a number of algebras that are especially known in the field of quantum structures and that in particular arise from the positive cones of partially ordered groups. Generalized effect algebras, generalized difference posets, cone algebras, commutative BCK-algebras with the relative cancellation property, and positive minimal clans are included in the text.

All these structures are conveniently characterizable as special cases of generalized pseudo-effect algebras, which we introduced in a previous paper. We establish the exact relations between all mentioned structures, thereby adding new structures whenever necessary to make the scheme of order complete.

Generalized pseudo-effect algebras were under certain conditions proved to be representable by means of a *po*-group. From this fact, we will easily establish representation theorems for all of the structures included in discussion.

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*Key words and phrases:* pseudo-effect algebras, generalized effect algebras, BCK-algebras, positive minimal clans, cone algebras, semiclans, *po*-groups.

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# 1 Introduction

Effect algebras [FoBe] and later generalized effect algebras [HePu] were introduced as partial additive structures modelling certain sets of positive operators of a Hilbert space, the common model of quantum physics. In the former case, this set is assumed to be bounded, as in particular the collection of quantum effects, i.e. the self-adjoint operators between zero and identity, is; in the latter case the assumption of boundedness is dropped, and an algebras of that kind may be unbounded.

In [DvVe3], an even more general structure was introduced, a so-called generalized pseudo-effect algebra. This is still a partial additive structure, but no longer necessarily commutative, and it specializes to a generalized effect algebra once commutativity is assumed. Prototypical models arise in this case within the positive cone of any – not necessarily abelian – partially ordered group.

Generalized pseudo-effect algebras (or GPE-algebras) turn out to be rather flexible; by this we mean that subclasses of them may be identified with quite a number of other well-known types of algebras.

We will, first of all, define generalized pseudo-difference posets and show them to be equivalent to GPE-algebras. These structures have two different partial difference operations; by assuming coincidence of these two differences, they specialize to the well-known generalized difference posets [HePu].

Furthermore, BCK-algebras have been a long time under study [JuMe], in particular in the frame of quantum structures [DvPu]. We will consider those among them that are commutative and possess the relative cancelation property [DvGr2]; we will refer to them here as Łukasiewicz-BCK algebras. These BCK-algebras are characterizable as generalized effect algebras fulfilling two simple conditions.

Now also Łukasiewicz-BCK algebras possess a non-commutative counterpart. The algebras of this latter kind are characterizable as a subclass of the generalized pseudo-effect algebras in an analogous manner as in the commutative case. Interestingly, it turns out that they in fact, although defined on the base of BCK-algebras, coincide with Bosbach's cone algebras [Bos1]. Moreover, there is a close connection to Bosbach's semiclans [Bos2].

Finally, we were able to include minimal clans into our discussion, introduced by Wyler [Wyl] and Schmidt [Schm]. Commutative positive minimal clans are identified with those Łukasiewicz-BCK algebras that are lattice-ordered. In an analogous way, positive minimal clans, that are not neccessarily commutative, are characterized.

| | | |
|---|---|---|
| generalized pseudo-effect (GPE) algebras $(E;+,0)$ | $\xrightarrow{\text{commutativity}}$ | generalized effect (GE) algebras $(E;+,0)$ |
| $\updownarrow$ | | $\updownarrow$ |
| generalized pseudo-difference (GPD) posets $(E;\leq,/,\backslash,0)$ | $\xrightarrow{/\,=\,\backslash}$ | generalized difference (GD) posets $(E;\leq,-,0)$ |
| infima exist, $\leq$ monotonous w.r.t. $\oslash$ and $\obslash$ $\downarrow$ | | infima exist, $\leq$ monotonous w.r.t. $*$ $\downarrow$ |
| pseudo-LBCK algebras (cone algebras) $(E;\leq,\oslash,\obslash,0)$ | $\xrightarrow{\oslash\,=\,\obslash}$ | Łukasiewicz-BCK (LBCK) algebras $(E;\leq,*,0)$ |
| suprema exist $\downarrow$ | | suprema exist $\downarrow$ |
| positive minimal clans (PMC) $(E;\leq,+,0)$ | $\xrightarrow{\text{commutativity}}$ | commutative positive minimal clans (CPMC) $(E;+,0)$ |

The relationship between all algebras included in this article are summarized in the scheme shown above. Here a doublesided arrow between two types of algebras indicates that both may be considered equivalent. Furthermore, an arrow from one type of algebra to another one means that the latter is to be considered less general than the former one and that the latter is in fact characterized among the former one by some additional axiom, which is printed next to the arrow. (The exact meaning of the axioms shown this way will be made clear only below.)

Now the most interesting point concerns representation. In [DvVe3], it was proved that a generalized pseudo-effect algebra is representable by means of a *po*-group in case it is a lower semi-lattice and fulfils a certain condition of Riesz type.

All the four structures appearing in the lower half of our scheme happen to fulfil this condition without further assumption. So to all of them, the representation theorem from [DvVe3] is applicable. In particular, several known results are reproved, and their proofs are brought onto a single line: Bosbach's representation theorem for cone algebras [Bos1] and for semiclans [Bos2], Wyler's for minimal clans [Wyl], and the one for Łukasiewicz-BCK algebras [DvGr3].

How this article is organized is easily seen by our graphical scheme; we discuss the algebras shown there line by line, i.e. a commutative one, seen on the right side, together with its non-commutative counterpart, seen on the left side, in one section each. So we start by repeating the basic facts about generalized pseudo-effect algebras in Section 2; we continue with the equivalent generalized pseudo-difference posets in Section 3; in the following Section 4, the Łukasiewicz-BCK algebras together with their non-commutative version are studied; and finally in Section 5, we turn our attention to positive minimal clans. The theorems on group representation are contained in each appropriate chapter.

# 2 Generalized Pseudo-Effect Algebras

In the frame of research in the field of quantum structures, generalized effect algebras [HePu, DvPu] were introduced, primarily to model sets of positive operators in Hilbert space.

Recently, the more general notion of a generalized pseudo-effect algebra was defined. We will repeat from [DvVe3] shortly the basic facts. The algebras of this type will in fact form the basis of this article; all structures discussed in the following section are interpretable as specializations of them. In particular, the representation theorem from [DvVe3] will be applicable a number of times.

**Definition 2.1** A structure $(E; +, 0)$, where $+$ is a partial binary operation and $0$ is a constant, is called a *generalized pseudo-effect algebra*, or a GPE-algebra for short, if, for all $a, b, c \in E$, the following holds.

(GPE1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$.

(GPE2) If $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$.

(GPE3) If $a + b$ and $a + c$ exist and are equal, then $b = c$.

If $b + a$ and $c + a$ exist and are equal, then $b = c$.

(GPE4) If $a + b$ exists and $a + b = 0$, then $a = b = 0$.

(GPE5) $a + 0$ and $0 + a$ exist and are both equal to $a$.

Furthermore, we define for any $a, b \in E$

$$a \leq b \quad \text{iff} \quad a + c = b \ \text{ for some } c \in E.$$

The order of a GPE-algebra is by (GPE2) the natural one: For any elements $a, b$ we have $a \leq b$ if and only if $a + c = b$ for some $c$ if and only if $d + a = b$ for some $d$. Moreover, $0$ is by (GPE5) the smallest element.

When talking about GPE-algebras, the following type of model should be kept in mind.

**Example 2.2** Let $(G; +, 0, \leq)$ be a partially ordered group with $G^+ \stackrel{\text{def}}{=} \{g \in G : \ g \geq 0\}$ as its positive cone; let $G_0$ a non-empty subset of $G^+$ such that for all $a, b \in G_0$, where $b \leq a$, also $a - b, -b + a \in G_0$. Then $(G_0; +, 0)$, where $+$ is the group addition restricted to those pairs of elements whose sum is again in $G_0$, and where $0$ is the group zero, is a GPE-algebra, whose order coincides with the restriction of the group order to $G_0$.

Among the GPE-algebras, the generalized effect algebras are easily identified as the commutative ones; see [DvVe3, Proposition 2.6].

For the purpose of a representation theorem the following properties were introduced in [DvVe3]. We say that a GPE-algebra $E$ fulfills

(RDP$_0$) if for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1$, $d_2 \leq b_2$ and $a = d_1 + d_2$;

(RDP$_2$) if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that

$$\begin{array}{cccc} d_1 & d_2 & \rightarrow & a_1 \\ d_3 & d_4 & \rightarrow & a_2 \\ \downarrow & \downarrow & & \\ b_1 & b_2 & & \end{array} \tag{1}$$

holds and $d_2 \wedge d_3 = 0$.

By the scheme (1) to hold we mean that $d_1+d_2 = a_1$, $d_3+d_4 = a_2$, $d_1+d_3 = b_1$, and $d_2 + d_4 = b_2$.

Analogously, the properties (RDP$_0$) and (RDP$_2$) were defined for *po*-groups [DvVe1, Definition 4.1].

In [DvVe3], two sufficient conditions were given for a GPE-algebra to originate from a *po*-group in the way shown in Example 2.2. We shall be concerned here only with one of these, given as follows. Let $E$ be a GPE-algebra.

(A) $E$ is a lower semi-lattice and fulfils (RDP$_0$).

We note that condition (A) implies (RDP$_2$); see [DvVe3, Lemma 3.3(ii)].

Now we give the representation theorem from [DvVe3].

**Theorem 2.3** Let $(E; +, 0)$ be a generalized pseudo-effect algebra fulfilling (A). Then $E$ may be isomorphically embedded into the positive cone of some $\ell$-group $G$, such that the infima and existing suprema are preserved. Assuming $E \subseteq G$, $+$ is then the restriction of the group addition of those pairs of elements of $E$ whose sum lies again in $E$, 0 is the group zero, and the order of $E$ is the restriction of the group order to $E$.

Concerning the notation used in this text, we note the following. By any sentence involving partial operations, i.e. the $+$ operation in case of GPE-algebras and any other operation in case of a structure introduced in the sequel, to hold we mean: All partial operations occuring in it are defined, and it holds. Similarly, with any sentence involving infima or suprema to hold, we include tacitly the statement that these infima and suprema exist.

Furthermore, with respect to additive structures in which associativity in the form (GPE1) holds, we will denote finite sums without brackets, whenever convenient.

## 3 Generalized Pseudo-Difference Posets

As an alternative structure to generalized effect algebras, generalized difference posets were introduced [HePu, DvPu]. The idea was to replace the sum by a difference operation.

We recall [DvPu, Definition 1.1.4] that a generalized difference poset, or a GD-poset for short, is a structure $(E; \leq, -, 0)$, where $(E; \leq)$ is a poset with the smallest element 0 and $-$ is a partial binary operation such that $b - a$ is defined for $a, b \in E$ iff $a \leq b$, in which, for any $a, b, c \in E$, (GD1) if $a \leq b$, then $b - a \leq b$, and $b - (b - a) = a$; (GD2) if $a \leq b \leq c$, then $c - b \leq c - a$, and $(c - a) - (c - b) = b - a$; (GD3) if $a \leq b, c$, then from $b - a = c - a$ it follows $b = c$.

GD-posets and GE-algebras are essentially the same. Indeed, from a GE-algebra $(E; +, 0)$, we derive a GD-poset $(E; \leq, -, 0)$ by using the same order $\leq$ and the same constant 0, and by defining for $a, b \in E$, where $a \leq b$, $b - a = c$ iff $b = a + c$. Similarly, we derive a GE-algebra from a GD-poset.

We present now a non-commutative version of a generalized difference poset. As might be expected, we will use two partial difference operations instead of one partial difference operation. The new structures will be in a one-to-one correspondence with generalized pseudo-effect algebras, in a similar sense as GD-posets are in a one-to-one correspondence with GE-algebras.

It will be possible to reformulate the Representation Theorem 2.3 for the new structure in an interesting manner, seemingly no longer referring to a Riesz property.

**Definition 3.1** A structure $(E; \leq, /, \backslash, 0)$, where $(E; \leq)$ is a poset with the smallest element 0 and $/$, $\backslash$ are partial binary operations such that, for $a, b \in E$, $b / a$ is defined iff $b \backslash a$ is defined iff $a \leq b$, is called a *generalized pseudo-difference poset*, or a GPD-poset for short, if for all $a, b, c \in E$ the following holds.

(GPD1) If $a \leq b$, then $b \,/\, a, b \setminus a \leq b$, and we have $b \setminus (b \,/\, a) = a$ and $b \,/\, (b \setminus a) = a$.

(GPD2) If $a \leq b \leq c$, then $c \,/\, b \leq c \,/\, a$ and $c \setminus b \leq c \setminus a$, and we have $(c \,/\, a) \setminus (c \,/\, b) = b \,/\, a$ and $(c \setminus a) \,/\, (c \setminus b) = b \setminus a$.

(GPD3) If $c \leq a, b$, then from $a \,/\, c = b \,/\, c$ or $a \setminus c = b \setminus c$ it follows $a = b$.

**Example 3.2** The prototypical example for GPD-posets is, in analogy to Example 2.2, the following.

Let $G$ be a partial ordered group and $G_0$ a subset of $G^+$ such that for all $a, b \in G_0$, where $b \leq a$, also $a - b, -b + a \in G_0$. Then $(G_0; \leq, \,/\,, \setminus, 0)$, where $\leq$ is the group order, for $a, b \in G_0$ such that $b \leq a$ we define

$$\begin{aligned} a \,/\, b &\stackrel{\text{def}}{=} a - b, \\ a \setminus b &\stackrel{\text{def}}{=} -b + a \end{aligned} \tag{2}$$

and 0 is the group neutral element, is a GPD-poset.

**Lemma 3.3** Let $(E; \leq, \,/\,, \setminus, 0)$ be a generalized pseudo-difference poset. Then the following holds for any $a, b, c \in E$.

(i) We have $a \,/\, a = a \setminus a = 0$ and $a \,/\, 0 = a \setminus 0 = a$.

(ii) If $a \leq b \leq c$, then $b \,/\, a \leq c \,/\, a$ and $b \setminus a \leq c \setminus a$, and we have $(c \,/\, a) \,/\, (b \,/\, a) = c \,/\, b$ and $(c \setminus a) \setminus (b \setminus a) = c \setminus b$.

(iii) If $a, b \leq c$, then from $c \,/\, a = c \,/\, b$ or $c \setminus a = c \setminus b$ it follows $a = b$.

*Proof.* 0 is the smallest element; so from (GPD1) and (GPD2) we have $0 = a \,/\, (a \setminus 0) \geq a \,/\, a$ and conclude $a \,/\, a = 0$. Similarly we see that $a \setminus a = 0$. So the first half of (i) is shown.

Let now $a, b \leq c$ and $c \,/\, a = c \,/\, b$. Then by (GPD1) we have $a = c \setminus (c \,/\, a) = c \setminus (c \,/\, b) = b$. So the first half of (iii) is shown, and similarly we prove the second one.

By the first part of (i) and (GPD1) we have $a \setminus a = 0 = a \setminus (a \,/\, 0)$, so from (iii) it follows $a = a \,/\, 0$. Similarly we see that $a = a \setminus 0$. This finishes the proof of (i).

By (GPD2) we have $b \,/\, a = (c \,/\, a) \setminus (c \,/\, b) \leq c \,/\, a$. By (GPD1) and (GPD2) we have $c \,/\, b = (c \,/\, a) \,/\, [(c \,/\, a) \setminus (c \,/\, b)] = (c \,/\, a) \,/\, (b \,/\, a)$. So the first part of (ii) is proved, and similarly, the second one is shown. □

For structures obtaining two differences, commutativity will mean the coincidence of these two operations. By this condition, GD-posets are obviously characterized among GPD-posets.

**Definition 3.4** Let $(E; \leq, \,/\,, \setminus, 0)$ be a generalized pseudo-difference poset. We say that $E$ is *commutative* if $/ \; = \; \setminus$.

**Proposition 3.5** Let $(E; \leq, \,/\,, \setminus, 0)$ be a generalized pseudo-difference poset. If $E$ is commutative, then, setting $- \; = \; / \; = \; \setminus$, $(E; \leq, -, 0)$ is a generalized difference poset.

Conversely, let $(E; \leq, -, 0)$ be a generalized difference poset. Then $(E; \leq, -, -, 0)$ is a commutative generalized pseudo-difference poset.

*Proof.* This is evident. □

We now make clear in which sense GPD-posets are in a one-to-one correspondence to GPE-algebras.

**Proposition 3.6** Let $(E; \leq, \,/\,, \setminus, 0)$ be a generalized pseudo-difference poset. We define the partial binary operation $+$ for all pairs $(b \,/\, a, a)$, where $a \leq b$, by setting $(b \,/\, a) + a \stackrel{\text{def}}{=} b$. Then we have for any $a, b, c \in E$

$$a + b \text{ is defined and equals } c \quad \begin{array}{ll} \text{iff} & b \leq c \text{ and } c \,/\, b = a \\ \text{iff} & a \leq c \text{ and } c \setminus a = b. \end{array} \tag{3}$$

Furthermore, $(E; +, 0)$ is a generalized pseudo-effect algebra whose order coincides with $\leq$.

Moreover, let $/_r$ and $\setminus_r$ be partial binary operations such that for $a, b \in E$, $b \,/_r\, a$ is defined iff $b \setminus_r a$ is defined iff $a \leq b$, in which case we require $(b \,/_r\, a) + a = a + (b \setminus_r a) = b$ to hold. Then $/_r \; = \; /$ and $\setminus_r \; = \; \setminus$.

*Proof.* $+$ is well-defined; indeed, if for $a_1, a_2, b_1, b_2 \in E$ two pairs $(b_1 \,/\, a_1, a_1)$, $(b_2 \,/\, a_2, a_2) \in E \times E$ coincide, we have $a_1 = a_2$ and, by (GPD3), $b_1 = b_2$.

To see (3) let $a, b, c \in E$. By definition, we have $a + b = c$ if and only if $b \leq c$ and $a = c / b$. Now, if $b \leq c$ and $c / b = a$, it follows by (GPD1) that $a \leq c$ and $c \backslash a = c \backslash (c / b) = b$; and if $a \leq c$ and $c \backslash a = b$, we have, again by (GPD1), $b \leq c$ and $c / b = a$.

To show (GPE1), suppose $(a+b)+c = x$ exists. By (3), this means $a+b = x / c$ and $b = (x / c) \backslash a$. By (GPD1) and (GPD2), from $a \leq x / c \leq x$ we conclude $c = x \backslash (x / c) \leq x \backslash a$; so by (GPD1) and Lemma 3.3(ii), $a = x / (x \backslash a) = (x / c) / [(x \backslash a) / c]$. By (GPD1) it follows $(x \backslash a) / c = (x / c) \backslash a = b$, which means, again by (3), $x = a + (b + c)$. Similarly, we may conclude that if the latter expression exists, also $(a + b) + c$ does, and both terms are equal.

To show (GPE2), suppose $a+b$ is defined. This means by (3) that $a, b \leq a+b$, so, again by (3), $[(a + b) / a] + a = b + [(a + b) \backslash b] = a + b$.

To show (GPE3), suppose that $a + b = a + c$. This means by definition that for some $x \in E$ we have $a = x / b = x / c$. So by Lemma 3.3(iii) $b = c$. For the other half, we procede similarly.

If $a + b = 0$, it follows by (3) that $a, b \leq 0$, so $a = b = 0$. Hence (GPE4) holds.

Since $0 \leq a$ and by Lemma 3.3(i) $a \backslash 0 = a / 0 = a$, we have by (3) that $a + 0 = a + 0 = a$. Hence (GPE5) holds.

We next show that the orders of $E$ as a GPD-poset and as a GPE-algebra coincide. Let $a, b \in E$. If $a \leq b$, we have $a + (b \backslash a) = b$ by (3). Conversely, if, for some $c \in E$, $a + c = b$, we have $a \leq b$ by (3).

Finally, for $a, b \in E$, $b / a$ is defined iff $a \leq b$, and by (3) we have $(b / a)+a = b$; so $b /_r a = b / a$ by (GPE3). So $/_r \; = \; /$, and analogously we see that $\backslash_r \; = \; \backslash$. $\square$

**Proposition 3.7** Let $(E; +, 0)$ be a generalized pseudo-effect algebra with the order $\leq$. Let $/$ and $\backslash$ be partial binary operations such that, for $a, b \in E$, $b / a$ is defined iff $b \backslash a$ is defined iff $a \leq b$, in which case we require

$$(b / a) + a = a + (b \backslash a) = b \tag{4}$$

to hold. Then $(E; \leq, /, \backslash, 0)$ is a generalized pseudo-difference poset.

Moreover, let us define the partial binary operation $+_r$ for all pairs $(b / a, a)$, where $a \leq b$, by setting $(b / a) +_r a = b$. Then $+_r = +$.

*Proof.* By [DvVe3, Lemma 2.2], $(E; \leq, 0)$ is a poset with 0 as the smallest element. Let $a, b \in E$. $b \setminus a$, $b \mathbin{/} a$ are defined iff $a \leq b$, as required for GPD-posets.

Now, to see (GPD1), assume $a \leq b$. By (4) we have $b = (b \mathbin{/} a) + a$. This implies $b \mathbin{/} a \leq b$, and again (4) gives $b = (b \mathbin{/} a) + [b \setminus (b \mathbin{/} a)]$. So we conclude by (GPE3) that $a = b \setminus (b \mathbin{/} a)$. Similarly, we see that $a = b \mathbin{/} (b \setminus a)$ holds.

Assume $a \leq b \leq c$. Then $b = a' + a$ for some $a' \in E$, and from (4) we get $(c \mathbin{/} b) + a' + a = (c \mathbin{/} b) + b = (c \mathbin{/} a) + a$, so by (GPE3), $c \mathbin{/} b \leq c \mathbin{/} a$. By (4) we have $(c \mathbin{/} b) + [(c \mathbin{/} a) \setminus (c \mathbin{/} b)] + a = (c \mathbin{/} b) + (b \mathbin{/} a) + a$. By (GPE3) the first equation of (GPD2) follows, and the second is seen analogously.

Assume $a \leq b, c$ and $b \mathbin{/} a = c \mathbin{/} a$. Then by (4) we have $b = (b \mathbin{/} a) + a = (c \mathbin{/} a) + a = c$. So (GPD3) holds.

The operation $+_r$, as seen from (GPD3), is well-defined. Now, for $a, b, c \in E$, $c + a$ is defined and equals $b$, iff $a \leq b$ and $b \mathbin{/} a = c$. It follows that $+_r$ coincides with $+$. □

As to be expected, the relationship between GPD-posets and GPE-algebras as shown in the previous two propositions specializes in the commutative case to the one between GD-posets and GE-algebras according to what was mentioned in the introduction of this chapter.

Now we make use of Theorem 2.3 to show that GPD-posets are under a certain condition isomorphically embeddable into the positive cone of some *po*-group. The condition (A) used in Theorem 2.3 possesses a reasonable form also for GPD-posets, which we now present.

Let us define the condition ($\mathrm{A}_d$) as follows for a generalized pseudo-difference poset $(E; \leq, \mathbin{/}, \setminus, 0)$.

($\mathrm{A}_d$) $E$ is a lower semi-lattice, and, for all $a, b, c \in E$, from $a \leq b$ it follows

$$a \mathbin{/} (a \wedge c) \leq b \mathbin{/} (b \wedge c) \text{ and } a \setminus (a \wedge c) \leq b \setminus (b \wedge c).$$

**Lemma 3.8** Let $(E; \leq, \mathbin{/}, \setminus, 0)$ be a GPD-poset, and let $(E; +, 0)$ be the GPE-algebra defined according to Proposition 3.6. Then $E$ fulfils ($\mathrm{A}_d$) as a GPD-poset if and only if $E$ fulfils (A) as a GPE-algebra.

*Proof.* In both cases it is assumed that $E$ is a lower semi-lattice.

Suppose first that $E$ fulfils ($\mathrm{A}_d$) as GPD-poset. Let $a, b, c \in E$ such that $c \leq a + b$. We then have $c \,/\, (b \wedge c) \leq (a + b) \,/\, [b \wedge (a + b)] = (a + b) \,/\, b = a$. Let $b_1 = b \wedge c$ and $a_1 = c \,/\, b_1$; then $a_1 \leq a$, $b_1 \leq b$, and $c = a_1 + b_1$. So ($\mathrm{RDP}_0$) holds, that is, $E$ fulfils (A) as GPE-algebra.

Suppose now that $E$ fulfils (A) as a GPE-algebra; that is, $E$ fulfils ($\mathrm{RDP}_0$). Let $a, b, c \in E$ and $a \leq b$. We then have $a \leq (b \,/\, (b \wedge c)) + (b \wedge c)$, so by ($\mathrm{RDP}_0$) there are $d_1 \leq b \,/\, (b \wedge c)$ and $d_2 \leq b \wedge c$ such that $a = d_1 + d_2$. On the other hand we have $a = d'_1 + (a \wedge c)$ for some $d'_1 \in E$; and since $d_2 \leq a, b, c$, that is, $d_2 \leq a \wedge b \wedge c = a \wedge c$, it follows $d'_1 \leq d_1$. We conclude $a \,/\, (a \wedge c) = d'_1 \leq b \,/\, (b \wedge c)$. This is the first inequation from ($\mathrm{A}_d$); the second one is seen analogously. So $E$ fulfils ($\mathrm{A}_d$) as a GPD-poset. □

**Theorem 3.9** Let $(E; \leq, \,/\,, \,\backslash\,, 0)$ be a generalized pseudo-difference poset fulfilling ($\mathrm{A}_d$). Then $E$ may be isomorphically embedded into the positive cone of an $\ell$-group $G$, such that the infima are preserved. Assuming $E \subseteq G$, the order $\leq$ of $E$ is the restriction of the group order to $E$, the operations $/$ and $\backslash$ are given by (2), and 0 is the group zero.

*Proof.* We apply Theorem 2.3 to the GPE-algebra $(E; +, 0)$, defined according to Proposition 3.6. From $+$, we get the operations $/$ and $\backslash$ back by (4), from which we conclude that, in the group $G$, $/$ and $\backslash$ are to be calculated by (2). □

# 4 Pseudo-ŁBCK algebras

We next turn our attention to a certain class of BCK-algebras. An overview of the theory of BCK-algebras may be found in [JuMe]; we are concerned here exclusively with those BCK-algebras that are commutative and that possess the relative cancellation property [DvGr2, DvPu], and we will call them ŁBCK-algebras for short.[1]

[1] We have borrowed this notation from [Pal], where a certain class of BCK-algebras is called Łukasiewicz-BCK algebra. In [DvPu, Theorem 5.2.28], it is established that this class actually coincides with what is called here ŁBCK-algebra.

We recall [DvPu, Chapter 5.1] that an ŁBCK-algebra is a structure $(E; \leq, *, 0)$, where $(E; \leq)$ is a poset with the smallest element 0 and $*$ is a binary operation, such that, for all $a, b, c \in E$, (ŁB1) $a \leq b$ iff $a * b = 0$; (ŁB2) $a * (a * b) = b * (b * a)$; (ŁB3) $(a * b) * c = (a * c) * b$; (ŁB4) from $c \leq a, b$ and $a * c = b * c$ it follows $a = b$.

As will be noted below (see the remark following Lemma 4.7), the ŁBCK-algebras may be identified with a subclass of the generalized difference posets. Indeed, the partial difference operation of any GD-poset that happens to be a lower semi-lattice may be extended canonically to a total operation; for any two elements, define a new difference by the original one applied to the first element and the infimum of the first and second one. By assuming that the order is monotone with respect to this new, total difference operation, we arrive exactly at the ŁBCK-algebras.

We shall present a non-commutative version of an ŁBCK-algebra. Again, we will use two differences instead of one. We define the new algebra in a similar manner as GPD-posets were defined on the base of GD-posets. We will then, equally as in the commutative case, identify the new algebras with GPD-posets that are lower semi-lattices and that fulfil a certain monotonicity condition.

The condition under which GPD-posets are identified with our generalized version of an ŁBCK-algebra happens to coincide with condition $(\mathrm{A}_d)$ of Chapter 3. So the new structure is, by the Theorem 3.9, representable in the positive cone of some $\ell$-group.

Interestingly, our generalization of an ŁBCK-algebra turns out to be equivalent to the so-called cone algebras introduced by Bosbach in [Bos1]. Our representation theorem will thus enable us to reprove the main result of [Bos1] that there are no more cone algebras than those that arise from positive cones of $\ell$-groups. This fact is also the key to the group representation of semiclans [Bos2].

**Definition 4.1** A structure $(E; \leq, \oslash, \obslash, 0)$, where $(E; \leq)$ is a poset with the smallest element 0 and $\oslash$, $\obslash$ are binary operations, is called a *pseudo-ŁBCK algebra* if, for all $a, b, c \in E$, the following holds.

(PŁB1) $a \leq b$ if and only if $a \oslash b = 0$ if and only if $a \obslash b = 0$.

(PŁB2) $a \oslash (a \obslash b) = b \oslash (b \obslash a)$ and $a \obslash (a \oslash b) = b \obslash (b \oslash a)$.

(PŁB3) $(a \oslash b) \obslash c = (a \obslash c) \oslash b$.

(PŁB4) If $c \leq a, b$, then from $a \oslash c = b \oslash c$ or $a \obslash c = b \obslash c$ it follows $a = b$.

**Example 4.2** The prototypical example for pseudo-ŁBCK algebras is, in analogy to Example 3.2, the following.

Let $G$ be a lattice-ordered group, and let $G_0$ be a convex subset of $G^+$ containing 0. Define for $a, b \in G^+$

$$\begin{aligned} a \oslash b &\overset{\text{def}}{=} a - (a \wedge b), \\ a \obslash b &\overset{\text{def}}{=} -(a \wedge b) + a. \end{aligned} \tag{5}$$

Then $(G_0; \leq, \oslash, \obslash, 0)$, where $\leq$ is the group order and 0 the group zero, is a pseudo-ŁBCK algebra.

**Lemma 4.3** Let $(E; \leq, \oslash, \obslash, 0)$ be a pseudo-ŁBCK algebra. Then for $a, b, c \in E$ the following holds.

(i) $a \oslash 0 = a \obslash 0 = a$ and $a \oslash a = a \obslash a = 0$.

(ii) $E$ is a lower semi-lattice; we have $a \wedge b = a \oslash (a \obslash b) = a \obslash (a \oslash b)$.
Furthermore, $a \oslash b = a \oslash (a \wedge b)$ and $a \obslash b = a \obslash (a \wedge b)$.

(iii) If $a \leq b$, then $a \oslash c \leq b \oslash c$ and $a \obslash c \leq b \obslash c$.
Suppose $c \leq a, b$. If then $a \oslash c \leq b \oslash c$ or $a \obslash c \leq b \obslash c$, it follows $a \leq b$.

(iv) If $a \leq b$, then $c \oslash b \leq c \oslash a$ and $c \obslash b \leq c \obslash a$.
In particular, $a \oslash b, a \obslash b \leq a$.

(v) $a \oslash b \leq c$ if and only if $a \obslash c \leq b$.

(vi) $(a \oslash c) \oslash (b \oslash c) = (a \oslash b) \wedge (a \oslash c)$ and $(a \obslash c) \obslash (b \obslash c) = (a \obslash b) \wedge (a \obslash c)$.

(vii) $(c \oslash a) \obslash (c \oslash b) \leq b \oslash a$ and $(c \obslash a) \oslash (c \obslash b) \leq b \obslash a$.

(viii) $(a \wedge b) \oslash c = (a \oslash c) \wedge (b \oslash c)$ and $(a \wedge b) \obslash c = (a \obslash c) \wedge (b \obslash c)$.

*Proof.* Let $a, b, c \in E$.

(i) From (PŁB1) we have $a \oslash a = a \odot a = 0$.

We next show that $a \oslash b, a \odot b \leq a$, i.e. the second part of (iv). By (PŁB3), we have $(a \oslash b) \odot a = (a \odot a) \oslash b = 0 \oslash b$; since 0 is the smallest element, we have by (PŁB1) $0 \oslash b = 0$, so again by (PŁB1) we conclude $a \oslash b \leq a$. The second inequality is shown similarly.

Now we finish the proof of (i). We have $a \oslash 0 \leq a$. On the other hand, it follows from (PŁB2) that $a \odot (a \oslash 0) = 0 \odot (0 \oslash a) = 0 \odot 0 = 0$, so $a \leq a \oslash 0$. We conclude $a = a \oslash 0$, and similarly we see that $a = a \odot 0$.

(vii) By (PŁB3) and (PŁB2) we have $(c \oslash a) \odot (c \oslash b) = [c \odot (c \oslash b)] \oslash a = [b \odot (b \oslash c)] \oslash a = (b \oslash a) \odot (b \oslash c) \leq b \oslash a$. This proves the first part, and second one is seen analogously.

(iv) Let $a \leq b$. Then $c \oslash b \leq c \oslash a$ and $c \odot b \leq c \odot a$ hold by (vii) and (PŁB1).

(v) By (PŁB1) and (PŁB3) we have $a \oslash b \leq c$ iff $(a \oslash b) \odot c = 0$ iff $(a \odot c) \oslash b = 0$ iff $a \odot c \leq b$.

We next show

(vi)$^*$ $(a \oslash c) \oslash (b \oslash c) \leq a \oslash b$ and $(a \odot c) \odot (b \odot c) \leq a \odot b$.

From (vii) we have $(a \oslash c) \odot (a \oslash b) \leq b \oslash c$. From (v), it follows $(a \oslash c) \oslash (b \oslash c) \leq a \oslash b$. This is the first part of (vi)$^*$, the second one is seen analogously.

(ii) By (iv) and (PŁB2) we have $a \oslash (a \odot b) \leq a, b$. Suppose for some $x \in E$ that $x \leq a, b$. Since then $x \odot a = 0$, we have by (i) and (PŁB2) $x = x \oslash (x \odot a) = a \oslash (a \odot x)$. Since $x \leq b$, we get by (iv) $x = a \oslash (a \odot x) \leq a \oslash (a \odot b)$. It follows $a \oslash (a \odot b) = a \wedge b$.

Analogoulsy, we show $a \odot (a \oslash b) = a \wedge b$.

By these results we have $a \oslash (a \wedge b) = a \oslash [a \odot (a \oslash b)] = a \wedge (a \oslash b) = a \oslash b$. Similarly, we see that $a \odot (a \wedge b) = a \odot b$.

(iii) Let $a \leq b$. Then $a \oslash c \leq b \oslash c$ and $a \odot c \leq b \odot c$ holds by (vi)$^*$ and (PŁB1).

Let $c \leq a, b$ and $a \oslash c \leq b \oslash c$. Then by (ii) and (PŁB3), $a \oslash c = (b \oslash c) \odot [(b \oslash c) \oslash (a \oslash c)] = (b \odot [(b \oslash c) \oslash (a \oslash c)]) \oslash c$. By (vi)$^*$, (iv) and (ii), we further conclude $b \obslash [(b \oslash c) \oslash (a \oslash c)] \geq b \obslash (b \oslash a) = a \wedge b \geq c$. Now we may apply (PLB4) to conclude $a \leq b$. So one half is proved, the other one is shown analogously.

(viii) Set $c' = a \wedge b \wedge c$. We shall show $(a \oslash c') \wedge (b \oslash c') = (a \wedge b) \oslash c'$. From (iii), (iv) and (ii), we then have $(a \wedge b) \oslash c \leq (a \oslash c) \wedge (b \oslash c) \leq (a \oslash c') \wedge (b \oslash c') = (a \wedge b) \oslash c' = (a \wedge b) \oslash c$, which proves the first part; the second one is shown in an analogous way.

By (iii) we have $(a \wedge b) \oslash c' \leq a \oslash c', b \oslash c'$. Suppose for some $x \in E$ that $x \leq a \oslash c', b \oslash c'$. Then, by (ii), $x = (a \oslash c') \wedge x = (a \oslash c') \obslash [(a \oslash c') \oslash x] = (a \obslash [(a \oslash c') \oslash x]) \oslash c' = y \oslash c'$, where we set $y = a \obslash [(a \oslash c') \oslash x]$. Since, by (iv) and (ii), $y \geq a \obslash (a \oslash c') = a \wedge c' = c'$, we conclude from $y \oslash c' \leq a \oslash c', b \oslash c'$ by (iii) that $y \leq a, b$, so $y \leq a \wedge b$ and $x = y \oslash c' \leq (a \wedge b) \oslash c'$. It follows $(a \oslash c') \wedge (b \oslash c') = (a \wedge b) \oslash c'$.

(vi) By (ii), (PLB3) and (viii), we calculate $(a \oslash b) \wedge (a \oslash c) = (a \oslash c) \oslash [(a \oslash c) \obslash (a \oslash b)] = (a \oslash c) \oslash [(a \wedge b) \oslash c] = (a \oslash c) \oslash [(a \oslash c) \wedge (b \oslash c)] = (a \oslash c) \oslash (b \oslash c)$, which is the first part. The second one is seen analogously. $\square$

Bosbach introduced in [Bos1] cone algebras in the following manner.[2] A cone algebra is a structure $(E; \oslash, \obslash)$, where $\oslash$ and $\obslash$ are binary operations, such that, for all $a, b, c \in E$, (CA1) $(c \oslash a) \oslash (b \oslash a) = (c \oslash b) \oslash (a \oslash b)$ and $(c \obslash a) \obslash (b \obslash a) = (c \obslash b) \obslash (a \obslash b)$; (CA2) $b \oslash (a \oslash a) = b \obslash (a \obslash a) = b$; (CA3) $(a \oslash b) \obslash c = (a \obslash c) \oslash b$; (CA4) $a \oslash (a \obslash b) = b \obslash (b \oslash a)$. Moreover, for $a, b \in E$, we let $a \leq b$ iff $a \oslash b = 0$.

**Proposition 4.4** Let $(E; \leq, \oslash, \obslash, 0)$ be a pseudo-LBCK algebra. Then $(E; \oslash, \obslash)$ is a cone algebra, whose order is $\leq$.

Conversely, let $(E; \oslash, \obslash)$ be a cone algebra with the order $\leq$. Then $x \obslash x = x \oslash x$ is for all $x \in E$ one and the same element; let 0 be this element. Then $(E; \leq, \oslash, \obslash, 0)$ is a pseudo-LBCK algebra.

*Proof.* Let $(E; \leq, \oslash, \obslash, 0)$ be a pseudo-LBCK algebra. (CA1) follows easily from Lemma 4.3(vi). (CA2) follows from Lemma 4.3(i). (CA3) coincides with (PLB3). (CA4) follows from Lemma 4.3(ii). So $E$ is a cone algebra. The order of $E$ as a such is by (PLB1) equal to $\leq$.

[2] We use here a slightly different notation from the one in [Bos1]; our operations translate to Bosbach's by: $a * b = b \obslash a$ and $a : b = a \oslash b$ for any elements $a, b$ of a cone algebra.

Let now $(E;\ \oslash\ ,\ \obslash\ )$ be a cone algebra with the order $\leq$. That $x \obslash x = x \oslash x$ is for any $x \in E$ the same element, to be denoted by 0, follows from [Bos1, (1.1)]. The partial order $\leq$ of $E$ is defined just that (PLB1) holds. (PLB2) holds by [Bos1, (1.10)]. (PLB3) coincides with (CA3). To show (PLB4), let $a, b, c \in E$ such that $c \leq a, b$ and $a \oslash c = b \oslash c$. Then by (CA1) and (CA2) we have $0 = (a \oslash c) \oslash (b \oslash c) = (a \oslash b) \oslash (c \oslash b) = (a \oslash b) \oslash 0 = a \oslash b$, so $a \leq b$. Similarly, we conclude $b \leq a$; it follows $a = b$. So one half of (PLB4) follows, and the other one is seen analogously. $\square$

So we see that we may consider pseudo-LBCK algebras and cone algebras as one and the same. We note that on the one hand, our axioms are simpler than Bosbach's, but on the other one, they are no longer equations only.

We will now see how LBCK-algebras are characterized among pseudo-LBCK algebras.

**Proposition 4.5** Let $(E; \leq, \oslash\ , \obslash\ , 0)$ be a pseudo-LBCK algebra. If $\oslash\ =\ \obslash$ , then, setting $* =\ \oslash\ =\ \obslash$ , $(E; \leq, *, 0)$ is an LBCK-algebra.

Conversely, let $(E; \leq, *, 0)$ be an LBCK-algebra. Then $(E; \leq, *, *, 0)$ is a pseudo-LBCK algebra.

*Proof.* This is evident. $\square$

Now we shall show that pseudo-LBCK algebras are to be considered as GPD-posets fulfilling condition $(\mathrm{A}_d)$ from Section 3.

**Proposition 4.6** Let $(E; \leq, \oslash\ , \obslash\ , 0)$ be a pseudo-LBCK algebra. Let $/$ , $\backslash$ be partial operations on $E$ such that, for $a, b \in E$, $b / a$ is defined iff $b \backslash a$ is defined iff $a \leq b$, and in which case we set

$$\begin{aligned} b / a &\stackrel{\text{def}}{=} b \oslash a \\ b \backslash a &\stackrel{\text{def}}{=} b \obslash a. \end{aligned} \tag{6}$$

Then $(E; \leq, /\ , \backslash\ , 0)$ is a generalized pseudo-difference poset fulfilling $(\mathrm{A}_d)$.

Moreover, we have for any $a, b \in E$

$$\begin{aligned} b \oslash a &= b / (a \wedge b), \\ b \obslash a &= b \backslash (a \wedge b). \end{aligned} \tag{7}$$

*Proof.* By assumption, $(E; \leq)$ is a poset with the smallest element 0.

If $a \leq b$, we have $b / a = b \oslash a \leq b$ by Lemma 4.3(iv); so $b \setminus (b / a) = b \obslash (b \oslash a) = a \wedge b = a$ by Lemma 4.3(ii). This shows one part of (GPD1), and the other one follows analogously.

Assume $a \leq b \leq c$. Then by Lemma 4.3(iv), $c \oslash b \leq c \oslash a$, so $c / b \leq c / a$, and $(c / a) \setminus (c / b) = (c \oslash a) \obslash (c \oslash b) = [c \obslash (c \oslash b)] \oslash a = (b \wedge c) \oslash a = b \oslash a = b / a$ by (PŁB3) and Lemma 4.3(ii). The second part of (GPD2) follows analogously.

Assume $a \leq b, c$ and $b / a = c / a$. Then $b \oslash a = c \oslash a$, and by (PŁB4) we have $b = c$. The second part of (GPD3) follows analogously.

Finally, for any $a, b \in E$, we have by Lemma 4.3(ii) $a \oslash b = a \oslash (a \wedge b) = a / (a \wedge b)$, which proves the first part of (7); the second one is seen analogously.

$E$ is a lower semi-lattice by Lemma 4.3(ii), which is the first part of condition $(\mathrm{A}_d)$. The second one means by (7) the following: For $a, b, c \in E$ it follows from $a \leq b$ that $a \oslash c \leq b \oslash c$ and $a \obslash c \leq b \obslash c$. But this is the content of Lemma 4.3(iii). $\square$

**Proposition 4.7** Let $(E; \leq, /, \setminus, 0)$ be a generalized pseudo-difference poset fulfilling $(\mathrm{A}_d)$. Define for any $a, b \in E$

$$\begin{aligned} b \oslash a &= b / (a \wedge b), \\ b \obslash a &= b \setminus (a \wedge b). \end{aligned} \tag{8}$$

Then $(E; \leq, \oslash, \obslash, 0)$ is a pseudo-ŁBCK algebra.

Moreover, the operations $/$, $\setminus$, whenever defined, coincide with $\oslash$ and $\obslash$, respectively.

*Proof.* By assumption, $(E; \leq)$ is a poset with the smallest element 0.

Let $a, b, c \in E$. From $a \leq b$ it follows $a \oslash b = a / (a \wedge b) = a / a = 0$ by Lemma 3.3(i). From $a \oslash b = 0$ it follows $a / (a \wedge b) = 0$, and since $a / a = 0$, we have by Lemma 3.3(iii) $a = a \wedge b$, so $a \leq b$. So $a \leq b$ iff $a \oslash b = 0$, and similarly we see that $a \leq b$ iff $a \obslash b = 0$; this proves (PŁB1).

By (GPD1), we have $a \oslash (a \obslash b) = a / (a \setminus (a \wedge b)) = a \wedge b$; similarly we see $a \obslash (a \oslash b) = a \wedge b$; from this it follows (PŁB2).

From definition (8) we see that $a \oslash b = a \oslash (b \wedge d)$ and $a \obslash b = a \obslash (b \wedge d)$ if $d \geq a$. Furthermore, we know by (GPD1) that $a \oslash b, a \obslash b \leq a$. It follows that $(a \oslash b) \obslash c = (a \oslash b') \obslash c'$, where we put $b' = a \wedge b$ and $c' = a \wedge c$.

Now from Proposition 3.6 and Lemma 3.8 we know that $(E; +, 0)$, where $+$ is given by (3), is a GPE-algebra fulfilling (A); by [DvVe3, Lemma 3.3(ii)], $(E; +, 0)$ then fulfils (RDP$_2$). So from $(a \mathbin{/} b') + b' = c' + (a \setminus c')$ we conclude, using Lemma [DvVe3, Lemma 2.4(iii)], that there are $e, f \in E$ such that $e \wedge f = 0$ and

$$\begin{array}{ccccc} (a \mathbin{/} b') \wedge c' & & e & \rightarrow & c' \\ f & & (a \setminus c') \wedge b' & \rightarrow & a \setminus c' \\ \downarrow & & \downarrow & & \\ a \mathbin{/} b' & & b' & & \end{array}$$

hold. Then $(a \oslash b') \obslash c' = (a \mathbin{/} b') \obslash c' = (a \mathbin{/} b') \setminus [(a \mathbin{/} b') \wedge c'] = f$, and also $(a \obslash c') \oslash b' = (a \setminus c') \oslash b' = (a \setminus c') \mathbin{/} [(a \setminus c') \wedge b'] = f$. So we proved (PLB3).

Now suppose $c \leq a, b$ and $a \oslash c = b \oslash c$. This means $a \mathbin{/} c = b \mathbin{/} c$, so by (GPD3) we have $a = b$. Similarly, from $a \obslash c = b \obslash c$ we conclude $a = b$. So (PLB4) holds.

The statement about the recovering of the original structure from $(E; \leq, \oslash, \obslash, 0)$ is obvious. $\square$

The relationship between pseudo-LBCK algebras and GPD-posets as shown in the previous two propositions may easily be applied to the commutative case. Evidently, we may identify LBCK-algebras with those generalized difference posets $(E; \leq, -, 0)$ in which all infima exist, by which condition we may define $a * b \stackrel{\text{def}}{=} a - (a \wedge b)$ for $a, b \in E$, and in which $a \leq b$ implies $a * c \leq b * c$ for any $a, b, c \in E$. We note that we may replace the last condition by the following one: $(a * b) * c = (a * c) * b$ for all $a, b, c \in E$. The resulting characterization of LBCK-algebras within GD-posets was established in an even more general form in [DvKi], and for the bounded case, it appeared already in [ChKo].

Pseudo-LBCK algebras clearly also may be identified with GPE-algebras fulfilling condition (A) rather than with GPD-posets fulfilling condition (A$_d$); this is an easy consequence of Propositions 3.6, 3.7 and Lemma 3.8. We note that, for the commutative case, this correspondence is established also in [JePu].

Now we turn to the representation theorem for pseudo-LBCK algebras, which is an easy application of Theorem 3.9.

**Theorem 4.8** Let $(E; \leq, \oslash, \obslash, 0)$ be a pseudo-LBCK algebra. Then $E$ may be isomorphically embedded into the positive cone of an $\ell$-group $G$, such that the infima are preserved. Assuming $E \subseteq G$, the order $\leq$ of $E$ is the restriction of the group order to $E$, 0 is the group zero, and $\oslash$, $\obslash$ are given by (5).

*Proof.* By Proposition 4.6, $(E; \leq, /, \backslash, 0)$ is a generalized pseudo-difference poset fulfilling $(\mathrm{A}_d)$, where $/$ and $\backslash$ are defined according to (6).

So by Theorem 3.9, there is an $\ell$-group $G$ such that $E$ may be identified with a subset of the positive cone $G^+$ such that $\leq$ is the group order, 0 is the group zero, and the operations $/, \backslash$ are given by (2). Now the operations $\oslash$ and $\obslash$ are recoverable from $/$ and $\backslash$ by (7), where the infima are calculated in $E$. Since by Theorem 3.9 infima are preserved from $E$ to $G$, we conclude that (5) holds. □

It is clear by Propositions 4.4 that the previous theorem holds equally well for cone algebras. This is the result from [Bos1], but the method of proof is completely different.

We conclude this chapter by comparing pseudo-LBCK algebras with still another structure, which also was introduced by Bosbach: with the so-called semiclans [Bos2].

We recall[3] that a semiclan is a structure $(E; +, \wedge)$, where $(E; \wedge)$ is a lower semi-lattice and $+$ is a partial binary operation, such that, for all $a, b, c \in E$, (SC1) from $a \leq b$ it follows that there are $d, e \in E$ such that $a + d = e + a = b$; (SC2) from $a + b = a + c$ or $b + a = c + a$ it follows $b = c$; (SC3) if $a + b$ and $a + c$ are defined, we have $(a + b) \wedge (a + c) = a + (b \wedge c)$; if $b + a$ and $c + a$ are defined, we have $(b + a) \wedge (c + a) = (b \wedge c) + a$; (SC4) $(a + b) + c$ is defined iff $a + (b + c)$ is defined, in which case both elements are equal; (SC5) if $(a \wedge b) + c = c$ and if $a \vee b$ exists, then $a \vee b = a + b = b + a$.

Any semiclan has a neutral element 0 with respect to $+$. If 0 is the smallest element with respect to the semi-lattice order, we shall call $E$ positive.

[3] We again differ in notation from Bosbach; we write semiclans additively rather than multiplicatively.

**Proposition 4.9** Let $(E; +, 0)$ be a GPE-algebra fulfilling (A). Then $(E; +, \wedge)$ is a positive semiclan with 0 as its neutral element.

Conversely, let $(E; +, \wedge)$ be a positive semiclan. Then $(E; +, 0)$, where 0 is the neutral element, is a GPE-algebra fulfilling (A) that possesses the same order.

*Proof.* Suppose first that $(E; +, 0)$ is a GPE-algebra fulfilling (A). Then $E$ is a lower semi-lattice by (A). (SC1) holds by definition of the order of $E$, (SC2) by (GPE3), (SC3) by [DvVe3, Lemma 2.4(iii)], (SC4) by (GPE1), and (SC5) holds by (A) and [DvVe3, Lemma 3.2(ii)]. So $E$ is a semiclan. Furthermore, 0 is the unique neutral element of addition by (GPD5) and (GPD3); and $E$ is positive by (GPD5).

Suppose next that $(E; +, \wedge)$ be a positive semiclan. By [Bos2, (1.1)], the 0 element is uniquely definable as the neutral element of the $+$ operation; this shows (GPE5). Furthermore, we have that, for $a, b \in E$, $a \leq b$ iff, for some $c \in E$, $a + c = b$ iff, for some $d \in E$, $d + a = b$; this holds due to (SC1), the positivity, [Bos2, (1.3)] and a symmetric version of the latter proposition. This shows (GPE2) and (GPE4). Furthermore, (GPE1) holds by (SC4); and (GPE3) holds by (SC2).

So $(E; +, 0)$ is proved to be a GPE-algebra whose order coincides with the original one. Furthermore, $E$ is a lower semi-lattice by assumption, so it only remains to show ($\mathrm{RDP}_0$). Let $a, b, c \in E$ such that $a \leq b + c$. Then $a$ and $b$ possess the upper bound $b+c$, and from this it follows by [Bos2, (1.8), (1.19)] that $a \vee b = b + [a \setminus (a \wedge b)]$. We conclude $b + [a \setminus (a \wedge b)] \leq b + c$ and so $a \setminus (a \wedge b) \leq c$. This means that the decomposition $a = (a \wedge b) + [a \setminus (a \wedge b)]$ fulfils the requirements of ($\mathrm{RDP}_0$). □

We see that positive semiclans are like cone algebras identifiable with pseudo-ŁBCK algebras. By Theorem 4.8, they are representable in the positive cone of some $\ell$-group, where then $+$ is the group addition and $\wedge$ the group order infimum.

According to [Bos2], this representation theorem may be generalized to any kind of semiclan. Let $E$ be a such; then the substructure consisting of the elements greater than 0 is a positive semiclan; the latter may be isomorphically embedded into the positive cone of some $\ell$-group. Now this embedding may be extended to an isomorphic embedding of the whole semiclan $E$ [Bos2, Section1, Theorem].

# 5 Positive Minimal Clans

The last type of structure to be considered is a subclass of the so-called clans. Clans were originally introduced by Wyler [Wyl]; we are here concerned only with minimal clans [Schm] which, in addition to that, are assumed to be positive.

Commutative positive minimal clans are characterizable as lattice-ordered generalized effect algebras fulfilling ($\mathrm{RDP}_0$). Alternatively, they may be viewed as ŁBCK-algebras that possess all suprema, i.e. that are lattice-ordered.

We shall see that these relationships generalizes to the non-commutative case in the expected manner. That is, positive minimal clans, which are not necessarily commutative, are essentially the same as lattice-ordered GPE-algebras fulfilling ($\mathrm{RDP}_0$), or as lattice-ordered pseudo-ŁBCK algebras.

Theorem 2.3 is easily usable to show that positive minimal clans are representable in the positive cone of some $\ell$-group. This was expressed by the representation theorem of Wyler [Wyl]; his method of proof in fact served as model for our Theorem 2.3 for GPE-algebras.

**Definition 5.1** A structure $(E; \leq, +, 0)$, where $(E; \leq)$ is a lattice with the smallest element 0 and $+$ is a partial binary operation, is called a *positive minimal clan*, or a PMC for short, if for all $a, b, c \in E$ the following holds.

(PMC1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$.

(PMC2) There are $d, e \in E$ such that $d + a = a + e = a \vee b$ and $d + (a \wedge b) = (a \wedge b) + e = b$.

(PMC3) If $a + b$ and $a + c$ exist and are equal, then $b = c$.

If $b + a$ and $c + a$ exist and are equal, then $b = c$.

(PMC4) $a + 0$ and $0 + a$ exist and are both equal to $a$.

(PMC5) If $a + c$, $b + c$ exist and $a \leq b$, then $a + c \leq b + c$.

If $c + a$, $c + b$ exist and $a \leq b$, then $c + a \leq c + b$.

A PMC $(E; \leq, +, 0)$ is called *commutative* if, for all $a, b \in E$, $a+b$ exists if and only if $b+a$ exists, in which case $a+b = b+a$.

**Example 5.2** The prototypical example for PMCs is, in analogy to Example 2.2, the following.

Let $G$ be a lattice-ordered group. Let $G_0$ be a convex subset of $G^+$ containing 0 that is closed under the formation of suprema (calculated in the group). Define the partial binary operation $+$ on $G_0$ as the restriction of the group addition to those pairs of elements whose sum lies again in $G_0$; let 0 the group zero; and let $\leq$ the group order. Then $(G_0; +, 0, \leq)$ is a positive minimal clan.

We now exhibit positive minimal clans as lattice-ordered GPE-algebras fulfilling (RDP$_0$).

**Proposition 5.3** Let $(E; \leq, +, 0)$ be a positive minimal clan. Then $(E; +, 0)$ is a generalized pseudo-effect algebra whose order is $\leq$. Moreover, $E$ is lattice-ordered and fulfils (RDP$_0$); or, equivalently, $E$ fulfils condition (A), and all suprema exist in $E$.

*Proof.* (GPE1), (GPE3), (GPE5) are equal to (PMC1), (PMC3), (PMC4), respectively.

Let $a, b, c \in E$. To show (GPE2), suppose that $a+b$ exists. We have $0 \leq a, b$, and so by (PMC4) and (PMC5) we conclude $a, b \leq a+b$. From (PMC2), applied to the pair $a, a+b$, we get a $d \in E$ such that $d+a = a \vee (a+b) = a+b$, and applied to the pair $b, a+b$, we get an $e \in E$ such that $b+e = a+b$.

To show (GPE4), suppose $a+b = 0$. As before, we see $a, b \leq a+b = 0$, so $a = b = 0$.

So $(E; +, 0)$ is shown to be a GPE-algebra. The order of $E$ is $\leq$; indeed, if, for some $d \in E$, $a+d = b$, then we have $a \leq a+d = b$; and if $a \leq b$, we apply (PMC2) to get some $d$ such that $b = a+d$.

This order is by assumption a lattice. It remains to show that $E$ fulfils (RDP$_0$); so let us suppose $c \leq a+b$.

We may introduce the partial operations $/$, $\backslash$ by (4). Let $a_1 = a \wedge [c / (b \wedge c)]$; then $a_1 \leq c$, and we may set $b_1 = c \backslash a_1$. Clearly $c = a_1 + b_1$ and $a_1 \leq a$. Now (PMC2) applied to the pair $b, c$ leads to $d+b = b \vee c$ and $d + (b \wedge c) = c$ for some $d$, which means $[c / (b \wedge c)] + b = b \vee c$. Furthermore, by [DvPu, Lemma 1.10.13(xiii)], we have $a_1 + b_1 = c \leq (a+b) \wedge (b \vee c) = (a+b) \wedge ([c / (b \wedge c)] + b) = (a \wedge [c / (b \wedge c)]) + b = a_1 + b$, and by [DvGr1, Lemma 2.4(b)]) we conclude $b_1 \leq b$. This finishes the proof of ($\mathrm{RDP}_0$). □

**Proposition 5.4** Let $(E; +, 0)$ be a generalized pseudo-effect algebra with the order $\leq$. Let $E$ be lattice-ordered and fulfil ($\mathrm{RDP}_0$); or, equivalently, let $E$ fulfil condition (A) and let all suprema exist in $E$. Then $(E; \leq, +, 0)$ is a positive minimal clan.

*Proof.* By assumption, $(E; \leq)$ is a lattice; and by [DvVe3, Lemma 2.2], 0 is the smallest element. (PMC1), (PMC3), (PMC4) are equal to (GPE1), (GPE3), (GPE5), respectively. (PMC5) holds due to [DvVe3, Lemma 2.4(ii)]. It remains to show (PMC2); let $a, b \in E$ be given.

We may introduce the partial operations $/$, $\backslash$ by (4). We then have $[(a \vee b) / a] + a = [(a \vee b) / b] + b$; to this equation we apply ($\mathrm{RDP}_2$), that holds in $E$ by [DvVe3, Lemma 3.3(ii)], to get $d_1, \ldots, d_4$ such that

$$\begin{array}{ccll} d_1 & d_2 & \rightarrow & (a \vee b) / a \\ d_3 & d_4 & \rightarrow & a \\ \downarrow & \downarrow & & \\ (a \vee b) / b & b & & \end{array}$$

and $d_2 \wedge d_3 = 0$ hold. Using [DvVe3, Lemma 2.4(iii)], we conclude $d_1 = [(a \vee b) / a] \wedge [(a \vee b) / b]$ and $d_4 = a \wedge b$. We claim that $d_1 = 0$; indeed, from $x \leq (a \vee b) / a, (a \vee b) / b$ for $x \in E$ we conclude, using [DvVe3, Lemma 2.4(ii)], that $x + a, x + b \leq a \vee b$ and so $a \vee b \leq (a \vee b) \backslash x$; hence $x + (a \vee b) \leq a \vee b$, which means $x = 0$.

So we have $d_2 + a = a \vee b$ and $d_2 + (a \wedge b) = b$. This proves one half of (PMC2); the other one is seen analogously. □

By specializing the last two propositions to the commutative case, we see that commutative PMCs are identifiable with lattice-ordered GE-algebras fulfilling condition (A). This result was already established in [Pu, Theorem 3.2].

We finally apply Theorem 2.3 to positive minimal clans.

**Theorem 5.5** Let $(E; \leq, +, 0)$ be a positive minimal clan. Then $E$ may be isomorphically embedded into the positive cone of an $\ell$-group $G$, such that the infima and suprema are preserved. Assuming $E \subseteq G$, $+$ is then the restriction of the group addition of those pairs of elements of $E$ whose sum lies again in $E$, $0$ is the group zero, and $\leq$ is the restriction of the group order to $E$.

*Proof.* By Proposition 5.3, $(E; +, 0)$ is a GPE-algebra with the order $\leq$, which fulfils (A). So by Theorem 2.3, $E$ may be considered a subset of the positive cone $G^+$ of some $\ell$-group $G$ such that $+$ is the group addition, $0$ is the group zero, and $\leq$ is the group order. Moreover, infima and suprema are preserved by the same theorem. □

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