Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 4/2001

Anatolij Dvurečenskij
Pekka Lahti
Kari Ylinen

The uniqueness question in the multidimensional moment problem with applications to phase space observables

July 11, 2001

PREPRINT SERIES

THE UNIQUENESS QUESTION IN THE MULTIDIMENSIONAL MOMENT PROBLEM WITH APPLICATIONS TO PHASE SPACE OBSERVABLES

ANATOLIJ DVUREČENSKIJ, PEKKA LAHTI, AND KARI YLINEN

Abstract. We apply the theory of holomorphic functions of several complex variables to prove a multidimensional variant of an exponential boundedness criterion [6] for the classical moment problem and we extend a theorem of Petersen [21] concerning the relation between the multidimensional moment problem and the one-dimensional moment problem for half-lines and compact subsets of the real line $\mathbb{R}$. We use these results to solve the moment problem for the quantum phase space observables generated by the number states.

1. Introduction and notations

The need to regard quantum observables as positive normalized operator measures, as opposed to the more traditional spectral measure approach, motivates the study of the moment operators of such observables, and in particular raises the question of the uniqueness of the observable given its moment operators. The spectral theorem for self-adjoint operators suffices to exhaust these problems in the case of spectral measures; in particular, the first moment of a spectral measure already determines it uniquely.

An important class of quantum observables that are not spectral measures consists of certain phase space observables. These have proved highly useful in several branches of quantum physics, including quantum communication and information theory, quantum optics and quantum measurement theory. Especially the possibility of experimental implementation of such observables by modern technology has drawn a lot of attention to their study.

The original motivation for the research reported in this paper came from the desire to shed light on the problem of the moment operators of phase space observables. This is intimately connected with the general multidimensional moment problem, whose study in our presentation occupies Sections 2 and 3. The choice of material in this part is basically dictated by applications to phase space observables, though not all the results are strictly needed in the sequel.

We denote as usual $||x|| = (x_1^2+\cdots+x_n^2)^{1/2}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. If K is a nonempty Borel subset of $\mathbb{R}^n$, we let $\mathcal{B}(K)$ denote its Borel σ-algebra and let $\mathcal{M}_n^*(K)$ be the set of all measures $\mu : \mathcal{B}(K) \to [0, \infty)$ satisfying $\int_K ||x||^{2k} d\mu(x) < \infty$ for $k = 0, 1, 2, \ldots$. We write $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$. For $\mu \in \mathcal{M}_n^*(K)$ and $k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n$, we define the *moment* $c_k(\mu)$ as follows:

$$c_k(\mu) = c_{k_1\ldots k_n}(\mu) := \int_K x^k d\mu(x) = \int_K x_1^{k_1} \cdots x_n^{k_n} d\mu(x_1, \ldots, x_n).$$

The multidimensional moment problem is to find conditions on a (multi)-sequence $(c_k)_{k\in\mathbb{N}_0^n}$, under which there exists a measure $\mu : \mathcal{B}(K) \to [0, \infty)$ such that $c_k = c_k(\mu)$ for all $k \in \mathbb{N}_0^n$.

It is known that a measure μ need not be uniquely determined by its moment sequence $(c_k)_{k\in\mathbb{N}_0^n}$.

For $\mu \in \mathcal{M}_n^*(K)$, we denote

$$V[K, \mu] := \{\nu \in \mathcal{M}_n^*(K) : c_k(\nu) = c_k(\mu) \text{ for all } k \in \mathbb{N}_0^n\}.$$

We say that μ is *determined on* K *by its moment sequence* if $V[K, \mu]$ is a singleton. In this situation we also say that μ is *determinate* (on K).

As usual, for $p \geq 1$, $L^p(K, \mu)$ will denote the space of all (equivalence classes of) Lebesgue measurable functions $f : K \to \mathbb{R}$ satisfying $\int_K |f(x)|^p d\mu(x) < \infty$. Let $\mathcal{P}_n$ denote the set of all polynomials in $x_1, \ldots, x_n$ (or also the set of their restrictions to a subset of $\mathbb{R}^n$ clear from the context).

In Section 2 we prove a multidimensional generalization of an exponential boundedness criterion of the classical moment problem of [6, Theorem 6]. Section 3 expands the results of Petersen [21] on the relation between the multidimensional and the one-dimensional moment problems. Section 4 introduces the phase space observables, Section 5 investigates their moment operators, and Section 6 shows the uniqueness of the number state generated phase space observables in view of their moment sequences. The final sections obtain the same result using the Cartesian margins (Section 7) and the polar margins (Section 8) of the phase space observables.

2. Uniqueness in the multidimensional moment problem: exponentially bounded measures

The theorem of this section is a multidimensional generalization of [6], Theorem 6. Our proof also resembles that of [6] (which according to the authors is inspired by [19]), but in our multidimensional case the theory of holomorphic functions of several complex variables is used.

It is worth noting that even before [19], a closely related proof was given in 1950 in the Russian original of [3, 4], see [3, p. 25–26]. We call *exponentially bounded* the type of measures appearing in the next result.

Theorem 2.1. *Let $\mu : \mathcal{B}(\mathbb{R}^n) \to [0, \infty)$ be a measure such that*

$$\int_{\mathbb{R}^n} e^{a\|x\|}\, d\mu(x) < \infty \tag{1}$$

for some $a > 0$. Then for any $p \geq 1$ the set $\mathcal{P}_n$ of real polynomials in n variables is dense in $L^p(\mathbb{R}^n, \mu)$.

Proof. Since $L^q(\mathbb{R}^n, \mu)$ for $\frac{1}{p} + \frac{1}{q} = 1$ is the dual of $L^p(\mathbb{R}^n, \mu)$, in view of the Hahn-Banach theorem it suffices to show that if $f \in L^q(\mathbb{R}^n, \mu)$ is such that

$$\int_{\mathbb{R}^n} x^k f(x)\, d\mu(x) = 0 \tag{2}$$

for every multi-index $k \in \mathbb{N}_0^n$, then $f(x) = 0$ a.e. (Note that by (1) and the Hölder inequality, the integral in (2) exists.) We denote

$$A = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |\operatorname{Im} z_j| < \frac{a}{2\sqrt{np}} \text{ for all } j = 1, \dots, n\}.$$

If $(z_1, \dots, z_n) \in A$, using the Schwarz inequality we get

$$\begin{aligned} |\exp(-i\sum_{j=1}^n z_j x_j)|^p [\exp(\frac{a}{2p} \parallel x \parallel)]^p &= [\exp(\sum_{j=1}^n \operatorname{Im} z_j x_j) \exp(\frac{a}{2p} \parallel x \parallel)]^p \\ &\leq [\exp(\sqrt{n}\frac{a}{2\sqrt{np}} \parallel x \parallel) \exp(\frac{a}{2p} \parallel x \parallel)]^p = e^{a\|x\|}, \end{aligned}$$

and so by the Hölder inequality

$$\int_{\mathbb{R}^n} |\exp(-i\sum_{j=1}^n z_j x_j) f(x)|\, e^{\frac{a}{2p}\|x\|}\, d\mu(x) < \infty. \tag{3}$$

We may thus define $F : A \to \mathbb{C}$ by the formula

$$F(z_1, \dots, z_2) = \int_{\mathbb{R}^n} \exp(-i\sum_{j=1}^n z_j x_j) f(x)\, d\mu(x), \tag{4}$$

and using (3) we also find by induction that

(5)

$$\frac{\partial^k}{\partial z_{j_1} \cdots \partial z_{j_k}} F(z_1, \dots, z_n) = \int_{\mathbb{R}^n} (-i)^k x_{j_1} \cdots x_{j_k} \exp(-i\sum_{j=1}^n z_j x_j) f(x)\, d\mu(x)$$

for all $(z_1, \dots, z_n) \in A$. Since F is in each variable separately complex differentiable, it is in A a holomorphic function of n complex variables. (This is so by Hartog's theorem but the more elementary Osgood's lemma in [18, p. 2] suffices here, as it is easy to show that by the dominated convergence theorem F is continuous.) Since each $\frac{\partial^k}{\partial z_{j_1}\cdots\partial z_{j_k}} F(0, \dots, 0) = 0$ by assumption, the coefficients of the power series expansion of F about the origin vanish [18, p. 3], and so by the identity theorem [18, p. 6], F is identically zero in A. In particular, for the Fourier–Stieltjes transform of the complex measure $f \cdot \mu$ we have

$$\int_{\mathbb{R}^n} \exp(-i \sum_{j=1}^{n} u_j x_j) f(x)\, d\mu(x) = 0$$

for all $(u_1, \dots, u_n) \in \mathbb{R}^n$, and so $f \cdot \mu = 0$, that is, f vanishes $\mu - a.e.$ □

Corollary 2.2. *An exponentially bounded measure $\mu : \mathcal{B}(\mathbb{R}^n) :\to [0, \infty)$ is ultradeterminate in the sense of* [16] *and thus determinate.*

Proof. See [16, pp. 61, 58] □

3. Marginal measures and the uniqueness question

In view of the importance of the final conclusion of Corollary 2.2 for our applications, we develop in this section an alternative proof, which does not depend on the theory of holomorphic functions of several complex variables. Some steps on the way have independent interest.

Lemma 3.1. *Let μ be a Borel measure which is determinate on $K \subseteq \mathbb{R}$, where K is one of the following possibilities:*

(i) *$K = [a, \infty)$, $a \in \mathbb{R}$;*
(ii) *$K = (-\infty, b]$, $b \in \mathbb{R}$;*
(iii) *K is a compact subset in $\mathbb{R}$.*

Then the set of all polynomials $\mathcal{P}_1$ is dense in $L^2(K, \mu)$.

Proof. (i) If $K = [0, \infty)$, then by [7, Corollary 3.9], $\mathcal{P}_1$ is dense in $L^2([0, \infty), \mu)$. Let now $K = [a, \infty)$, and let $T_a : [0, \infty) \to [a, \infty)$ be the translation defined by $T(u) := u + a$. Put $\mu_a(X) := \mu \circ T_a(X)$ for any Borel subset X of $[0, \infty)$. Then μ_a is a Borel measure on $[0, \infty)$. Let $c_k := \int_a^\infty t^k d\mu(t)$ and $\tilde{c}_k := \int_0^\infty u^k d\mu_a(u)$. Then $c_k = \sum_{i=0}^k \binom{k}{i} a^{k-i} \tilde{c}_i$ and $\tilde{c}_k := \sum_{i=0}^k \binom{k}{i} (-1)^{n-i} a^{k-i} c_i$. Therefore, μ is determinate on $[a, \infty)$ if and only if μ_a is determinate on $[0, \infty)$.

(ii) This case can be proved by modifying the previous argument.

(iii) Let K be a compact set in $\mathbb{R}$. Then in view of the Weierstrass theorem, $\mathcal{P}_1$ is dense in $L^2(K, \mu)$. □

The following result generalizes [21], Theorem 3.

Theorem 3.2. *Let $K = K_1 \times \cdots \times K_n$, where each K_i is either $\mathbb{R}$ or a nonempty subset of $\mathbb{R}$ satisfying one of the conditions* (i)–(iii) *in Lemma* 3.1. *Let $\pi_i : K \to K_i$ be the i-th projection of K onto K_i, $i = 1, \dots, n$. Then a measure $\mu \in \mathcal{M}_n^*(K)$ is determinate whenever all the projection measures $\mu \circ \pi_i^{-1}$ are determinate on K_i for $i = 1, \dots, n$.*

Proof. We only sketch the proof, the omitted details being essentially the same as in the proof of Theorem 3 in [21]. Let $\nu \in V[K, \mu]$. Since by assumption the measures $X \mapsto \mu(\pi_i^{-1}(X))$ are determinate, we get $\nu \circ \pi_i^{-1} = \mu \circ \pi_i^{-1}$. For any closed set $K \subset \mathbb{R}$, let $C_c(K)$ denote the space of continuous real functions on K with compact support. For any real functions f_i on K_i, $i = 1, \dots, n$, let $f_1 \otimes \cdots \otimes f_n$ be defined by $f_1 \otimes \cdots \otimes f_n(x_1, \dots, x_n) := f_1(x_1) \cdots f_n(x_n)$. Using the density results of Lemma 3.1 we may show, following the argument of [21] referred to above, that, given $\epsilon > 0$, for each $f_i \in C_c(K_i)$, $i = 1, \dots, n$ we can find n polynomials $p_i \in \mathcal{P}_1$, $i = 1, \dots, n$, such that

$$\int_K |f_1 \otimes \cdots \otimes f_n - p_1 \otimes \cdots \otimes p_n| d\nu < \epsilon. \tag{2.3}$$

For $f : K \to \mathbb{R}$, define the extension f^K of f to $\mathbb{R}^n$ via $f^K(x) = f(x)$ if $x \in K$ otherwise $f^K(x) = 0$. Let μ^K be the extension of μ to $\mathcal{B}(\mathbb{R}^n)$ given by $\mu^K(E) := \mu(E \cap K), E \in \mathcal{B}(R^n)$. Then $f \in L^1(K, \mu)$ if and only if $f^K \in L^1(\mathbb{R}^n, \mu^K)$, and if this is the case, then $\int_K f d\mu = \int_{\mathbb{R}^n} f d\mu^K$. It is known that the set $\{f_1 \otimes \cdots \otimes f_n : f_i \in C_c(\mathbb{R})\}$ is dense in $L^1(\mathbb{R}^n, \mu^K)$. Since if $f \in C_c(\mathbb{R})$, the restriction $f|_{K_i}$ of f to K_i gives a function in $C_c(K_i)$ for any $i = 1, \dots, n$. Hence if now $f \in L^1(K, \nu)$, then $f^K \in L^1(\mathbb{R}^n, \nu^K)$, and given $\eta > 0$ then we can find $f_1, \dots, f_n \in C_c(\mathbb{R})$ such that $\int_{\mathbb{R}^n} |f^K - f_1 \otimes \cdots \otimes f_n| d\nu^K < \eta$. Then $\int_K |f - f_1 \otimes \cdots \otimes f_n \chi_K| d\nu = \int_{\mathbb{R}^n} |f^K - f_1 \otimes \cdots \otimes f_n| d\nu^K < \eta$, which proves that the set $\{f_1 \chi_{K_1} \otimes \cdots \otimes f_n \chi_{K_n} : f_i \in C_c(\mathbb{R})\} = \{f_1 \otimes \cdots \otimes f_n : f_i \in C_c(K_i), i = 1, \dots, n\}$ is dense in $L^1(K, \nu)$. From (2.3) it easily follows that $\mathcal{P}_n$ is dense in $L^1(K, \nu)$, and therefore by a theorem of Douglas [12], ν is an extremal point of the convex set $V[K, \mu]$. Since this is true for any $\nu \in V[K, \mu]$, the set $V[K, \mu]$ has to be a singleton. $\square$

Corollary 3.3. *Let μ be an exponentially bounded Borel measures on $K = K_1 \times \cdots \times K_n$, with K_i as in Theorem 3.2, that is,*

$$\int_K e^{a||x||} d\mu(x) < \infty, \tag{6}$$

for some $a > 0$. Then all the moments $c_k(\mu)$, $k \in \mathbb{N}_0^n$, exist and are finite, and the measure μ is determinate.

Proof. By assumption, it is clear that all the multidimensional moments $\int_K x^k d\mu(x)$ exist and are finite. Fix $i = 1, \dots, n$. Then

$$\begin{aligned}\int_{\mathbb{K}_i} e^{a|t|} d\mu \circ \pi_i^{-1}(t) &= \int_K e^{a|x_i|} d\mu(x_1, \dots, x_n) \\ &\leq \int_K e^{a\|x\|} d\mu(x_1, \dots, x_n) < \infty.\end{aligned}$$

Using [15, Theorem II.5.2], or [6, Theorem 6], we see that $\mu \circ \pi_i^{-1}$ is determinate for $i = 1, \dots, n$. By Theorem 3.2, μ is determinate. □

4. Phase space observables

Consider a phase space observable A^T, defined by a state operator T, a positive trace one operator, by means of the weakly defined integral

$$A^T(Z) := \frac{1}{\pi} \int_Z D_z T D_z^* \, d\lambda(z), \;\; Z \in \mathcal{B}(\mathbb{C}),$$

where λ is the Lebesgue measure on the complex plane $\mathbb{C}$, and $D_z = e^{za^* - \overline{z}a}$, $z \in \mathbb{C}$, is the unitary shift operator associated with the ladder operators $a = \sum_{n\geq 0} \sqrt{n+1} |n\,\rangle\langle\, n{+}1|$ and $a^* = \sum_{n\geq 0} \sqrt{n+1} |n{+}1\,\rangle\langle\, n|$ of an orthonormal basis $(|\, n\rangle)_{n=0}^{\infty}$, called the number basis, of a complex separable Hilbert space $\mathcal{H}$. Let $A^T_{\psi,\varphi}$ denote the complex measure $Z \mapsto \langle\, \psi \,|\, A^T(Z)\varphi\,\rangle$ defined by the (positive normalized) operator measure A^T and the vectors $\psi, \varphi \in \mathcal{H}$, and let $\mathcal{L}(\mathcal{H})$ be the set of bounded operators on $\mathcal{H}$.

The *moment operators* of the operator measure A^T are the linear operators

$$A^T[m,n] := \int_{\mathbb{C}} z^m \overline{z}^n \, dA^T(z),$$

each defined on the linear subspaces

$$\mathcal{D}[m,n] = \{\varphi \in \mathcal{H} \mid z \mapsto z^m \overline{z}^n \text{ is integrable w.r.t. } A^T_{\psi,\varphi} \text{ for all } \psi \in \mathcal{H}\},$$

and satisfying, for any $\varphi \in \mathcal{D}[m,n], \psi \in \mathcal{H}$,

$$\langle\, \psi \,|\, A^T[m,n]\varphi\,\rangle = \int_{\mathbb{C}} z^m \overline{z}^n \, dA^T_{\psi,\varphi}(z).$$

We say that the operator measure A^T is *determinate* if it is uniquely determined by its moment operators $A^T[m,n], m, n \geq 0$.

In a previous article [14] we have investigated the moment problem for the polar coordinate ($\mathbb{C} \ni z = |z|e^{i\theta}, |z| \in [0,\infty), \theta \in [0, 2\pi)$)

marginal measures of the phase space observables $A^{|s\rangle}$ associated with the number states $|\, s\rangle$, $s \in \mathbb{N}_0$. The operator measures

$$\mathcal{B}([0,\infty)) \ni R \mapsto A^{|s\rangle}(R \times [0,2\pi)) \in \mathcal{L}(\mathcal{H}),$$
$$\mathcal{B}([0,2\pi)) \ni X \mapsto A^{|s\rangle}([0,\infty) \times X) \in \mathcal{L}(\mathcal{H})$$

were shown to be determinate. Here we investigate the moment problem for the phase space observables $A^{|s\rangle}$.

Remark 4.1. The complex moment problem of the measures $A^{|s\rangle}_{\varphi,\varphi} : \mathcal{B}(\mathbb{C}) \to [0,1]$ is here interpreted as the $\mathbb{R}^2$-moment problem of $A^{|s\rangle}_{\varphi,\varphi} : \mathcal{B}(\mathbb{R}^2) \to [0,1]$ via the identification $z = x + iy$. The equivalence of the moment sequences $\int_{\mathbb{C}} z^m \overline{z}^n \, A^{|s\rangle}_{\varphi,\varphi}(d\lambda(z))$ and $\int_{\mathbb{R}^2} x^m y^n \, A^{|s\rangle}_{\varphi,\varphi}(dxdy)$, with $z = x+iy$, is obvious. In [22, Appendix] the one-to-one correspondence of the complex and the two-dimensional moment sequences has been demonstrated.

5. On the moment operators of $A^{|s\rangle}$

To determine the moment operators $A^{|s\rangle}[m,n], m,n \geq 0$ of a phase space observable $A^{|s\rangle}$ defined by a number state $|\, s\rangle$, we first observe that for any m,n, and for any number states $|\, k\rangle, |\, l\rangle$, the integral

$$\langle\, k \,|\, A^{|s\rangle}[m,n] | l \,\rangle = \int_{\mathbb{C}} z^m \overline{z}^n \, dA^{|s\rangle}_{|k\rangle,|l\rangle}(z)$$

exists and is finite. Indeed, by a direct computation one gets

$$(\!\!\!\!\!\int_{\mathbb{C}}\!) \, |z|^{m+n} \, dA^{|s\rangle}_{|k\rangle,|l\rangle}(z) = \frac{1}{\pi} \int_{\mathbb{C}} |z|^{m+n} \, \langle\, k \,|\, D_z | s \,\rangle \langle\, s \,|\, D_z^* | l \,\rangle \, d\lambda(z) =$$
$$\frac{\delta_{k,l}}{s!} \sum_{r=0}^{[k,s]} \sum_{r'=0}^{[k,s]} a(s,k,r) a(s,k,r') \int_0^\infty e^{-|z|^2} |z|^{m+n+2(s+k-r-r')} \, 2|z| d|z| < \infty,$$

where

$$a(s,k,r) = (-1)^{s-r} \binom{s}{r} \sqrt{k!}/(k-r)!,$$

and where $[k,s]$ denotes the minumum of k and s [20].

We recall from [20, Lemma A.2] that by the positivity of the operator measure A^T, that is, by the fact that any $A^T(Z), Z \in \mathcal{B}(\mathbb{C})$, is a positive operator, the domain $\mathcal{D}[m,n]$ of $A^T[m,n]$ contains as a subspace the set

$$\widetilde{\mathcal{D}}[m,n] = \{\varphi \in \mathcal{H} \,|\, z \mapsto |z|^{2(m+n)} \text{ is integrable w.r.t. } A^T_{\varphi,\varphi}\}.$$

Since A^T is *not* projection valued, the set $\widetilde{\mathcal{D}}[m,n]$ could be a proper subset of $\mathcal{D}[m,n]$.

The above result (7) shows that for any $A^{|s\rangle}$

$$\mathrm{lin}\,\{|\,k\rangle \mid k\in\mathbb{N}_0\}\subset\widetilde{\mathcal{D}}[m,n]\subset\mathcal{D}[m,n],$$

meaning that all the moment operators $A^{|s\rangle}[m,n]$ are densely defined. Denoting

$$\overline{A^{|s\rangle}[m,n]}:=\int_{\mathbb{C}}\overline{z^m\overline{z}^n}\,dA^{|s\rangle}(z)$$

we observe that $\overline{A^{|s\rangle}[m,n]}=A^{|s\rangle}[n,m]$, as well as $\overline{A^{|s\rangle}[n,m]}=A^{|s\rangle}[m,n]$. Therefore, using [20, Lemma A4], we see that the adjoint of $A^{|s\rangle}[m,n]$, resp. $A^{|s\rangle}[n,m]$, is an extension of $\overline{A^{|s\rangle}[m,n]}$, resp. $\overline{A^{|s\rangle}[n,m]}$, that is,

$$A^{|s\rangle}[m,n]\subseteq A^{|s\rangle}[n,m]^*, \tag{8}$$

$$A^{|s\rangle}[n,m]\subseteq A^{|s\rangle}[m,n]^*. \tag{9}$$

The matrix elements of the operators $A^{|s\rangle}[m,n]$ in the number basis can easily be computed, and we get [20]:

$$\langle\,k\mid A^{|s\rangle}[m,n]|l\,\rangle=\frac{1}{\pi}\frac{1}{s!}\sum_{r=0}^{[k,s]}\sum_{r'=0}^{[l,s]}a(s,k,r)a(s,l,r')\,I(m,n,s,k,l,r,r')$$

where

$$\begin{aligned}I(m,n,s,k,l,r,r') &= \int_{\mathbb{C}}e^{-|z|^2}\,z^{m+k+s-r-r'}\overline{z}^{n+l+s-r-r'}\,d\lambda(z)\\ &= 0,\quad\text{whenever}\quad k+m\neq l+n,\\ &= \pi\,(m+s+k-r-r')!,\quad\text{for}\quad k+m=l+n.\end{aligned}$$

Therefore,

$$\begin{aligned}\langle\,k\mid A^{|s\rangle}[m,n]|l\,\rangle &= 0,\quad\text{for}\quad k+m\neq l+n,\\ &= \frac{1}{s!}\sum_{r=0}^{[k,s]}\sum_{r'=0}^{[m-n+k,s]}a(s,k,r)a(s,m-n+k,r')(m+s+k-r-r')!,\\ &\qquad\text{for}\quad k+m=l+n.\end{aligned}$$

It seems difficult to determine the explicit form of the operators $A^{|s\rangle}[m,n]$. However, it is known [20] that

$$A^{|s\rangle}[n,n]=\sum_{i,j=0}^{n}a_{ij}s^{n-j}N^i,\ \ a_{ij}\ \text{integers},\ \ \mathcal{D}(A^{|s\rangle}[n,n])=\mathcal{D}(N^n),\ \ N=a^*a,$$

$$A^s[n,0]=a^n,\ \ A^s[0,n]=(a^*)^n,\ \ \mathcal{D}(A^s[n,0])=\mathcal{D}(A^s[0,n])=\mathcal{D}(a^n).$$

For $|\,s\rangle=|\,0\rangle$ one may quickly confirm that

$$\langle\,k\mid A^{|0\rangle}[m,n]|l\,\rangle=\langle\,k\,|\,a^m(a^*)^n|l\,\rangle$$

for any $m,n\in\mathbb{N}_0$, and for any number states $|\,k\rangle, |\,l\rangle$. Moreover, one easily shows that $\mathcal{D}(a^m(a^*)^n)\subseteq\widetilde{\mathcal{D}}[m,n]$ and that, actually, $A^{|0\rangle}[m,n]$ extends the operator $a^m(a^*)^n$, which, together with the above relations (8–9) shows that

$$A^{|0\rangle}[m,n]=a^m(a^*)^n.$$

The possibility of obtaining the operators $a^m(a^*)^n$ from the "diagonal coherent state representation" $\frac{1}{\pi}\int_{\mathbb{C}} z^m\overline{z}^n|z\,\rangle\langle\, z|\,d\lambda(z)$ was, perhaps, first noticed by Sudarshan [23]. The papers [10, 1] are further elaborations on the related 'phase space quantization methods'. From the point of view of the theory of operator integrals these pioneering papers amounted to showing that $a^m(a^*)^n\subset A^{|0\rangle}[m,n]$.

6. The uniqueness of $A^{|s\rangle}$

We show next that the phase space observable $A^{|s\rangle}$ is uniquely determined by its moment operators $A^{|s\rangle}[m,n]$, $m,n\in\mathbb{N}_0$. In other words, if $E:\mathcal{B}(\mathbb{C})\to\mathcal{L}(\mathcal{H})$ is another normalised positive operator measure such that its moment operators equal those of $A^{|s\rangle}$, that is, $E[m,n]=A^{|s\rangle}[m,n]$ for all $m,n\in\mathbb{N}_0$, then $E=A^{|s\rangle}$. Actually, the equality $E=A^{|s\rangle}$ already follows if the moment operators of E agree with those of $A^{|s\rangle}$ on a dense subset.

Let $|\,k\rangle, |\,l\rangle$ be any two number states and consider the complex measure $A^{|s\rangle}_{|k\rangle,|l\rangle}$. Its values are

$$A^{|s\rangle}_{|k\rangle,|l\rangle}(Z)=\frac{1}{\pi}\int_Z\langle\, k\,|\,D_z|s\,\rangle\,\langle\, s\,|\,D_z^*|l\,\rangle\;d\lambda(z)=$$
$$\frac{1}{\pi}\frac{1}{s!}\sum_{r=0}^{[k,s]}\sum_{r'=0}^{[l,s]}a(s,k,r)a(s,l,r')\int_Z e^{-|z|^2}z^{s+k-r-r'}\overline{z}^{s+l-r-r'}\;d\lambda(z).$$

In particular, for each $|\,k\rangle$ the probability measure $A^{|s\rangle}_{|k\rangle,|k\rangle}$ has the density

$$\frac{1}{\pi}\frac{1}{s!}\sum_{r=0}^{[k,s]}\sum_{r'=0}^{[k,s]}a(s,k,r)a(s,k,r')e^{-|z|^2}|z|^{2(s+k-r-r')}$$

with respect to the Lebesgue measure λ. But then for any $a \in \mathbb{R}$,

$$\begin{aligned}
\int_{\mathbb{C}} e^{a|z|}\, dA^{|s\rangle}_{|k\rangle,|k\rangle} &= \frac{1}{\pi}\frac{1}{s!}\sum_{r=0}^{[k,s]}\sum_{r'=0}^{[k,s]} a(s,k,r)a(s,k,r')\int_{\mathbb{C}} e^{a|z|}e^{-|z|^2}|z|^{2(s+k-r-r')}\, d\lambda(z) \\
&= \frac{1}{s!}\sum_{r=0}^{[k,s]}\sum_{r'=0}^{[k,s]} a(s,k,r)a(s,k,r')\int_0^\infty e^{a|z|}e^{-|z|^2}|z|^{2(s+k-r-r')}\, 2|z|d|z| \\
&= \frac{1}{s!}\sum_{r=0}^{[k,s]}\sum_{r'=0}^{[k,s]} a(s,k,r)a(s,k,r')e^{(a/2)^2}\int_0^\infty e^{-(|z|-a/2)^2}|z|^{2(s+k-r-r')}\, 2|z|d|z| \\
&< \infty.
\end{aligned}$$

By Corollary 2.2 each $A^{|s\rangle}_{|k\rangle,|k\rangle}$ is determinate, that is, $|V(\mathbb{C}, A^{|s\rangle}_{|k\rangle,|k\rangle})| = 1$.

For any $|\,k\rangle, |\,l\rangle$, $k \neq l$, $c \in \mathbb{C}$, $|c| = 1$, we also have

$$\int_{\mathbb{C}} e^{a|z|}\, dA^{|s\rangle}_{|l\rangle+c|k\rangle,|l\rangle+c|k\rangle} = \int_{\mathbb{C}} e^{a|z|}\, dA^{|s\rangle}_{|l\rangle,|l\rangle} + \int_{\mathbb{C}} e^{a|z|}\, dA^{|s\rangle}_{|k\rangle,|k\rangle} < \infty,$$

since, for instance,

$$\int_{\mathbb{C}} e^{a|z|}\, dA^{|s\rangle}_{|k\rangle,|l\rangle} = 0.$$

Thus, all the measures $A^{|s\rangle}_{|l\rangle+c|k\rangle,|l\rangle+c|k\rangle}$ are determinate.

Assume now that $E : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$ is another operator measure for which $E[m,n] = A^{|s\rangle}[m,n]$ on $\mathrm{lin}\,\{|\,k\rangle \mid k \in \mathbb{N}_0\}$. Using the polarization identity we get for all number states $|\,k\rangle$ and $|\,l\rangle$,

$$\begin{aligned}
E_{|k\rangle,|l\rangle} &= \frac{1}{4}\sum_{r=0}^{3} i^r E_{|l\rangle+i^r|k\rangle,|l\rangle+i^r|k\rangle} \\
&= \frac{1}{4}\sum_{r=0}^{3} i^r A^{|s\rangle}_{|l\rangle+i^r|k\rangle,|l\rangle+i^r|k\rangle} = A^{|s\rangle}_{|k\rangle,|l\rangle}.
\end{aligned}$$

This shows that $E = A^{|s\rangle}$, that is, the phase space observable $A^{|s\rangle}$ defined by the number state $|\,s\rangle$, $s \in \mathbb{N}_0$, is determinate.

7. The uniqueness of $A^{|s\rangle}$ through its Cartesian margins

Using Theorem 3 of Petersen [21], the uniqueness $A^{|s\rangle}$ may also be obtained from the determinacy of its Cartesian marginal measures. We shall demonstrate this result next.

To facilitate the calculations, we pass to the $L^2(\mathbb{R})$-realization of the phase space observables $A^{|s\rangle}$. Let $W : \mathcal{H} \to L^2(\mathbb{R})$ be the unitary mapping for which $W(|\,n\rangle) = f_n$, $n \in \mathbb{N}_0$, where f_n is the n-th Hermite function,

$$
\begin{aligned}
f_n(x) &= N_n\, e^{-x^2/2}\, H_n(x),\ x \in \mathbb{R},\\
N_n &= (\sqrt{\pi}\, 2^n n!)^{-1/2},\\
H_n(X) &= (-1)^n e^{x^2} \frac{d^n}{dx^n}\, e^{-x^2},\ x \in \mathbb{R}.
\end{aligned}
$$

When we identify $\mathbb{C}$ with $\mathbb{R}^2$, and write $z = \frac{q+ip}{\sqrt{2}}$, the phase space observable $A^{|s\rangle}$, defined by f_s, gets the form

$$
A^{|s\rangle}(Z) = \frac{1}{2\pi} \int_Z |e^{-iqP+ipQ} f_s\,\rangle\langle\, e^{-iqP+ipQ} f_s|\, dqdp,
$$

with (Q, P) being the Schrödinger pair on $L^2(\mathbb{R})$ [17]. The Cartesian marginal measures of $A^{|s\rangle}$ are known to be the unsharp position $E^{(Q,s)}$ and the unsharp momentum $E^{(P,s)}$, with

$$
\begin{aligned}
E^{(Q,s)}(X) &= (\chi_X * |f_s|^2)(Q),\ X \in \mathcal{B}(\mathbb{R}),\\
E^{(P,s)}(Y) &= (\chi_Y * |\hat{f}_s|^2)(P),\ Y \in \mathcal{B}(\mathbb{R}),
\end{aligned}
$$

respectively, where $\chi_X * |f_s|^2$ is the convolution of the characteristic function χ_X with the density function $|f_s|^2$, and $\hat{f}_s$ is the Fourier transform of f_s, see, e.g. [11, Theorem 3.4.1].

Let $\varphi \in L^2(\mathbb{R})$ be a unit vector, and consider the probability measure $A^{|s\rangle}_{\varphi,\varphi}$. Its Cartesian marginal probability measures are $E^{(Q,s)}_{\varphi,\varphi}$ and $E^{(P,s)}_{\varphi,\varphi}$, respectively. Clearly, they are absolutely continuous with respect to the Lebesgue measure of $\mathbb{R}$. Let $g^{(Q,s)}_{\varphi,\varphi}$ be the Radon-Nikodym derivative of $E^{(Q,s)}_{\varphi,\varphi}$ with respect to dq. We assume now that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$ so that we may take

$$
g^{(Q,s)}_{\varphi,\varphi}(x) = \int_{\mathbb{R}} |f_s(x-q)|^2 |\varphi(q)|^2\, dq.
$$

Let $\operatorname{supp} g \subseteq [a,b]$, $M \in [0,\infty)$, be such that $|\varphi(q)|^2 \leq M$ for all $x \in [a,b]$, and let $|q| \leq C$, $a \leq q \leq b$. Then

$$
\begin{aligned}
g^{(Q,s)}_{\varphi,\varphi}(x) &= N_s^2 \int_a^b e^{-(x-q)^2} H_s(x-q)^2 |\varphi(q)|^2\, dq\\
&\leq M\, N_s^2\, e^{-x^2} \int_a^b e^{-q^2+2qx} H_s(x-q)^2\, dq\\
&\leq M\, N_s^2\, e^{-x^2} e^{2C|x|} p_{2s}(x),
\end{aligned}
$$

where p_{2s} is a polynomial of x of degree $2s$. But then for any $a > 0$,

$$
\int_{\mathbb{R}} e^{a|x|} e^{-x^2} e^{2C|x|} p_{2s}(x)\, dx < \infty,
$$

which shows that the probability measure $E^{(Q,s)}_{\varphi,\varphi}$ is exponentially bounded. Therefore, all the moments of the probability measure $E^{(Q,s)}_{\varphi,\varphi}$ are finite and the measure $E^{(Q,s)}_{\varphi,\varphi}$ is determinate for each unit vector $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$. Similarly, any probability measure $E^{(P,s)}_{\varphi,\varphi}$, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\| \varphi \|= 1$, is determinate, so that, by [21, Theorem 3], or by Theorem 3.2, any phase space probability measure $A^{|s\rangle}_{\varphi,\varphi}$, $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\| \varphi \|= 1$, is determinate.

Remark 7.1. Not all the probability measures $E^{(Q,s)}_{\varphi,\varphi}$, resp. $E^{(P,s)}_{\varphi,\varphi}$, $f \in L^2(\mathbb{R}), ||f|| = 1$, can be determinate since the moment operators of $E^{(Q,s)}$ are unbounded operators. Since the phase space observable $A^{|s\rangle}$ is known to be informationally complete [5, 8] (that is, for any two state operators T, U, if $A^{|s\rangle}_T = A^{|s\rangle}_U$, then $T = U$), it also follows that if ψ and φ are two different vector states such that $E^{(Q,s)}_{\psi,\psi} = E^{(Q,s)}_{\varphi,\varphi}$ and $E^{(P,s)}_{\psi,\psi} = E^{(P,s)}_{\varphi,\varphi}$, then the measures $E^{(Q,s)}_{\psi,\psi}$ and $E^{(P,s)}_{\psi,\psi}$ cannot both be determinate, since otherwise also $A^{|s\rangle}_{\psi,\psi}$ (as well as $A^{|s\rangle}_{\varphi,\varphi}$) would be determinate, with the implication that the states $|\psi\rangle\langle\psi|$ and $|\varphi\rangle\langle\varphi|$ would be the same, which need not be the case, see, e.g. [9, Sect. 2.3].

Assume now that $E : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(L^2(\mathbb{R}))$ is another positive operator measure such that $E_{\varphi,\varphi} = A^{|s\rangle}_{\varphi,\varphi}$ for all $\varphi \in \mathcal{C}_0^\infty(\mathbb{R})$, $\| \varphi \|= 1$. Let ψ be any unit vector of $L^2(\mathbb{R})$. Since $\mathcal{C}_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, ψ is either in $\mathcal{C}_0^\infty(\mathbb{R})$, or is a limit of a sequence of vectors $\varphi_n \in \mathcal{C}_0^\infty(\mathbb{R})$. Let $\psi = \lim \varphi_n$. Then $\lim E_{\varphi_n,\varphi_n}(Z) = E_{\psi,\psi}(Z)$ as well as $\lim E_{\varphi_n,\varphi_n}(Z) = A^{|s\rangle}_{\psi,\psi}(Z)$ uniformly for $Z \in \mathcal{B}(\mathbb{C})$, which implies that $E_{\varphi_n,\varphi_n} \to E_{\psi,\psi}$ and $E_{\varphi_n,\varphi_n} \to A^{|s\rangle}_{\psi,\psi}$ in the total variation norm [13, p. 97]. Therefore, $E_{\psi,\psi} = A^{|s\rangle}_{\psi,\psi}$ for any unit vector $\psi \in L^2(\mathbb{R})$. By the polarization identity, the operator measures E and $A^{|s\rangle}$ are the same. To conclude, we have established the following result.

Corollary 7.2. *Let $A^{|s\rangle}$ be the phase space observable defined by the number state $| s\rangle = W^{-1}f_s$, $s \in \mathbb{N}_0$. The moment operators $A^{|s\rangle}[m,n]$ are densely defined, $W^{-1}(C_0^\infty(\mathbb{R})) \subset \widetilde{\mathcal{D}}[m,n] \subseteq \mathcal{D}[m,n]$, and the observable $A^{|s\rangle}$ is uniquely determined by the restrictions of its moment operators to $W^{-1}(C_0^\infty(\mathbb{R}))$.*

8. The uniqueness of $A^{|s\rangle}$ in terms of its polar coordinate margins

In addition to the complex moments - which, as we have seen, essentially amount to the real moments in terms of the Cartesian representation - of a phase space observable, it is illuminating to consider the real moments in terms of the polar coordinate representation. This we do next making use of the generalization of Petersen's result expounded in Theorem 3.2.

Consider the phase space observable $A^{|s\rangle}$ and its polar coordinate moment operators $\int_0^\infty \int_0^{2\pi} r^n \theta^m \, dA^{|s\rangle}(re^{i\theta})$. The polar coordinate marginal measures are

$$\begin{aligned}\mathcal{B}([0,\infty)) \ni R &\mapsto A^{|s\rangle}(R \times [0,2\pi)) \in \mathcal{L}(\mathcal{H}),\\ \mathcal{B}([0,2\pi)) \ni X &\mapsto A^{|s\rangle}([0,\infty) \times X) \in \mathcal{L}(\mathcal{H}),\end{aligned}$$

the second of them being compactly supported and thus determinate. In [14, Section 5] it was shown that also the radial margin of $A^{|s\rangle}$ is uniquely determined by its (unbounded self-adjoint) moment operators. Therefore, by Theorem 3.2, we may conclude that the phase space observable $A^{|s\rangle}$ is uniquely determined also by its polar coordinate moment operators $\int_0^\infty \int_0^{2\pi} r^n \theta^m \, dA^{|s\rangle}(re^{i\theta}), n, m \in \mathbb{N}_0$. The same conclusion can also be obtained from [7, Theorem 3.6] concerning rotation invariant moment problem. We do not pursue to determine the moment operators $\int_0^\infty \int_0^{2\pi} r^n \theta^m \, dA^{|s\rangle}(re^{i\theta})$, since their physical relevance is less direct.

Acknowledgement. One of the authors (AD) is thankful to the Academy of Finland for organizing his stay at the University of Turku, in October 2000. The paper was partially supported by the grant VEGA 2/7193/00, Slovakia.

References

[1] G.S. Agarwal, E. Wolf, Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics, I, *Phys. Review D* **2** (1970), 2161–2186.

[2] N. Akhiezer, *The Classical Moment Problem*, Edinburg, Oliver and Boyd, 1965.

[3] N.I. Akhiezer, I.M. Glazman, *Theory of Linear Operators in Hilbert Space, Vol. 1*, Fredrik Ungar, New York, 1961.

[4] N.I. Akhiezer, I.M. Glazman, *Theory of Linear Operators in Hilbert Space, Vol. 2*, Fredrik Ungar, New York, 1963.

[5] S.T. Ali, E. Progovecki, *Physics* **89A** (1977), 501–521.

[6] C. Berg, J.P.R. Christensen, Density questions in the classical theory of moments, *Ann. Inst. Fourier, Grenoble* **31** (1981), 99–114.

[7] C. Berg, M. Thill, Rotation invariant moment problems, *Acta Math.* **167** (1991), 207–227.

[8] P. Busch, G. Cassinelli, P. Lahti, Probability structures for quantum state spaces, *Rev. Math. Phys.* **7** (1995), 1105–1121.

[9] P. Busch, P. Lahti, The determination of the past and future of a physical system in quantum mechanics, *Found. Phys.* **19** (1989), 633–678.
[10] K.E. Cahill, R.J. Glauber, Ordered expansions in boson amplitude operators, *Phys. Review* **177** (1969), 1857–1881.
[11] E.B. Davies, *The Quantum Theory of Open Systems*, Academic Press, 1976.
[12] R.D. Douglas, On extremal measures and subspace density, *Mich. Math. J.* **11** (1964), 243–246.
[13] N. Dunford, J.T. Schwartz, *Linear Operators. Part II*, Interscience Publishers, John Wiley & Sons, 1963.
[14] A. Dvurečensij, P. Lahti, K. Ylinen, Positive operator measures determined by their moment operators, *Rep. Math. Phys.* **45** (2000), 139–146.
[15] G. Freud, *Orthogonal Polynomials*, Akadémiai Kiadó, Budapest, 1971.
[16] B. Fuglede, The multidimensional moment problem, *Expos. Math.* **1** (1983), 47–65.
[17] J.C. Garrison, J. Wong, Canonically conjugate pairs, uncertainty relations, and phase operators, *J. Math. Phys.* **11** (1970) 2242-2249.
[18] R.C. Gunning, H. Rossi, *Analytic Functions of Several Complex Variable*, Prentice-Hall, 1965.
[19] E. Hewitt, Remark on orthonormal sets in $\mathcal{L}_2(a,b)$, *Amer. Math. Monthly* **61** (1954), 249–250.
[20] P. Lahti, M. Mączyński, K. Ylinen, The moment operators of phase space observables and their number margins, *Rep. Math. Phys.* **41** (1998), 319–331.
[21] L.C. Petersen, On the relation between the multidimensional moment problem and the one-dimensional moment problem, *Math. Scand.* **51** (1982), 361–366.
[22] J. Stochel, F.H. Szafraniec, The complex moment problem and subnormality: a polar decomposition approach, *J. Funct. Anal.* **159** (1998), 432–491.
[23] E.C.G. Sudarshan, Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams, *Phys. Review Letters* **10** (1963), 277–279.

Anatolij Dvurečenskij, Mathematical Institute, Slovak Academy of Sciences, SK–814 73 Bratislava, Slovakia
E-mail address: dvurecen@mau.savba.sk

Pekka Lahti, Department of Physics, University of Turku, FIN-20014 Turku, Finland
E-mail address: pekka.lahti@utu.fi

Kari Ylinen, Department of Mathematics, University of Turku, FIN-20014 Turku, Finland
E-mail address: ylinen@utu.fi

This preprint series was founded in 1996. Its purpose is to present manuscripts of submitted or unpublished papers and reports of fellows of the Mathematical Institute of the Slovak Academy of Sciences in Bratislava and its Košice branch. The authors are fully responsible for the content of the preprints. Postal addresses:

Mathematical Institute
Slovak Academy of Sciences
Štefánikova 49
SK–814 73 Bratislava
SLOVAKIA

Tel./Fax (+421 7) 5249 7316

Mathematical Institute
Slovak Academy of Sciences
Grešákova 6
SK–040 01 Košice
SLOVAKIA

Tel./Fax (+421 95) 6228291

PostScript files of preprints are available at WWW address:
`http://www.mat.savba.sk/preprints`

Latest preprints of the series:

4/2001 *Anatolij Dvurečenskij, Pekka Lahti, Kari Ylinen*: The uniqueness question in the multidimensional moment problem with applications to phase space observables

3/2001 *Anatolij Dvurečenskij, Pavel Pták*: Quantum logics with the Riesz interpolation property

2/2001 *Olga Nánásiová, Sylvia Pulmannová*: Abelian extensions of difference sets

1/2001 *Ján Jakubík*: Cantor-Bernstein theorem for lattices

23/2000 *Anatolij Dvurečenskij, Thomas Vetterlein*: Algebras in the positive cone of *po*-groups

22/2000 *Anatolij Dvurečenskij, Thomas Vetterlein*: Generalized pseudo-effect algebras

21/2000 *Martin Kochol*: Tension polynomials of graphs

20/2000 *Martin Kochol*: Sublinear defect principle in graph theory

19/2000 *Martin Kochol*: Polynomials associated with nowhere-zero flows

18/2000 *Martin Kochol*: Superposition and constructions of graphs without nowhere-zero k-flows

17/2000 *Sylvia Pulmannová*: Divisible effect algebras and interval effect algebras

16/2000 *Gejza Jenča, Sylvia Pulmannová*: Quotients of partial abelian monoids and the Riesz decomposition property

15/2000 *Pekka Lahti, Sylvia Pulmannová*: Coexistence vs. functional coexistence of quantum observables

14/2000 *Beloslav Riečan*: Almost everywhere convergence in MV-algebras with product

13/2000 *Ján Jakubík*: Strong subdirect products of MV-algebras

12/2000 *Roman Frič*: Convergence and duality

11/2000 *Ján Jakubík*: Higher degrees of distributivity in MV-algebras

10/2000 *Beloslav Riečan*: On the probability on BL algebras

9/2000 *Beloslav Riečan*: On the Yager type entropy of dynamical systems

8/2000 *Anatolij Dvurečenskij*: States on pseudo MV-algebras

7/2000 *Anatolij Dvurečenskij, Thomas Vetterlein*: Pseudo-effect algebras. II. group representations

6/2000 *Anatolij Dvurečenskij, Thomas Vetterlein*: Pseudo-effect algebras. I. basic properties

5/2000 *Ján Jakubík*: On free MV-algebras

(continued on inside back cover)

4/2000 *Anatolij Dvurečenskij*: Pseudo MV-algebras are intervals in l-groups

3/2000 *Anna Jenčová*: Geometry of quantum states: dual connections and divergence functions

2/2000 *Anatolij Dvurečenskij*: MV-observables and MV-algebras

1/2000 *Michal Fečkan*: Transversal bounded solutions for difference equations

21/1999 *Roman Frič*: Lukasiewicz tribes are absolutely sequentially closed bold algebras

20/1999 *Miroslav Repický*: Spaces not distinguishing convergences

19/1999 *Beloslav Riečan*: On the probability theory on MV algebras

18/1999 *Beloslav Riečan, Jozefína Petrovičová*: Kolmogorov-Sinai theorem for fuzzy T-dynamical systems

17/1999 *Beloslav Riečan*: On a Banach space valued observables

16/1999 *Michal Fečkan*: Criteria on the nonexistence of invariant submanifolds for dynamical systems

15/1999 *Anatolij Dvurečenskij*: On Loomis-Sikorski's theorem for MV-algebras and BCK-algebras

14/1999 *Otakar Grošek, Ladislav Satko, Karol Nemoga*: ZKDWCNDPDC...

13/1999 *Otakar Grošek, Karol Nemoga, Ladislav Satko*: A remark to Luby-Rackoff and Ueli M. Maurer pseudorandom generators

12/1999 *Beloslav Riečan*: On the sum of observables in product MV algebras

11/1999 *Peter Eliaš*: A hierarchy of thin sets related to the boundedness of trigonometric series

10/1999 *Miloslav Duchoň, Peter Maličký*: On Herglotz theorem in partially ordered vector spaces

9/1999 *Ján Jakubík*: Torsion classes of specker lattice ordered groups

8/1999 *Ján Jakubík*: Graph automorphisms and cells of lattices

7/1999 *Miloslav Duchoň, Peter Maličký*: Positive linear maps on trigonometric polynomials

6/1999 *Miloslav Duchoň, Anna Kolesárová*: Fourier transforms of vector-valued measures on a certain compact semigroup

5/1999 *Miroslav Ploščica*: Separation properties in congruence lattices of lattices

4/1999 *Georges Chevalier, Anatolij Dvurečenskij, Karl Svozil*: Piron's and Bell's geometric lemmas and Gleason's theorem

3/1999 *Jarmila Hedlíková*: Relatively orthomodular lattices

2/1999 *Anatolij Dvurečenskij, Pekka Lahti, Kari Ylinen*: Some remarks on the moment problem

1/1999 *Anatolij Dvurečenskij*: Loomis-Sikorski theorem for σ-complete MV-algebras and ℓ-groups

27/1998 *Martin Kochol*: Cubic graphs without a Petersen minor have nowhere-zero 5-flows

26/1998 *Anatolij Dvurečenskij, Ferdinand Chovanec, Eva Rybáriková*: D-homomorphisms and atomic σ-complete Boolean D-posets

25/1998 *Anatolij Dvurečenskij*: Commutative BCK-algebras with product

24/1998 *Anatolij Dvurečenskij, Antonio Di Nola*: Product MV-algebras

23/1998 *Anatolij Dvurečenskij, Maria Gabriella Graziano*: Dedekind complete commutative BCK-algebras

22/1998 *Anatolij Dvurečenskij*: Ideals of commutative BCK-algebras

21/1998 *Anatolij Dvurečenskij, Beloslav Riečan*: Weakly divisible MV-algebras and product

20/1998 *Ivan Korec*: Definability of addition and multiplication from Pascal's triangle modulo n and number-theoretical functions