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MV-test Spaces versus MV-algebras

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Abstract

In analogy with effect algebras, we introduce the tests and the test spaces as well as the MV-test spaces. Roughly speaking, a test corresponds to a hypothesis on the propositional system, or equivalently to a partition of unity. We show that there is a one-to-one correspondence between MV-algebras and MV-test spaces.

1 Introduction

The event structure of a logical system under investigation (e.g. quantum physical system) is identified with a quantum logic [BLM] or an effect algebra [FoBe] in contrast to classical mechanics when it is assumed to be a Boolean algebra. MV-algebras are important structures which entered mathematics in fifties by Chang [Cha]. For them there is a famous Mundici's representation theorem [Mun], [CDM], saying that MV-algebras are practically intervals in unital Abelian ℓ-groups. If we drop the lattice structure of unital po-groups,

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we have the intervals which correspond always to effect algebras, but not, in general, vice-versa.

We recall that a partial algebra $E = (E; +, 0, 1)$ is said to be an *effect algebra* if, for all $a, b, c \in E$,

(E1) $a+b$ is defined in E iff $b+a$ is defined, and in such the case $a+b = b+a$;

(E2) $a+b, (a+b)+c$ are defined iff $b+c$ and $a+(b+c)$ are defined, and in such the case $(a+b)+c = a+(b+c)$;

(E3) for any $a \in E$, there exists a unique element $a' \in E$ such that $a+a' = 1$;

(E4) if $a+1$ is defined in E, then $a = 0$.

If we define $a \le b$ iff there exists an element $c \in E$ such that $a+c = b$, then $\le$ is a partial ordering, and we write $c := b - a$.

For example, if (G, u) is an Abelian unital po-group with a strong unit u, and if $\Gamma(G, u) := \{g \in G : \ 0 \le g \le u\}$ is endowed with the restriction of the group addition $+$, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra.

We recall that an *MV-algebra* is an algebra $M := (M; \oplus, \odot, ^*, 0, 1)$ of type (2,2,1,0,0) such that, for all $a, b, c \in M$, we have $a \odot b = (a^* \oplus b^*)^*$ and

(M1) $a \oplus b = b \oplus a$;

(M2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(M3) $a \oplus 0 = a$;

(M4) $a \oplus 1 = 1$;

(M5) $(a^*)^* = a$;

(M6) $a \oplus a^* = 1$;

(M7) $0^* = 1$;

(M8) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

If we define a partial operation $+$ on M in such a way that $a+b$ is defined in M iff $a \le b^*$, and then $a+b := a \oplus b$, then $(M; +, 0, 1)$ is an effect algebra.

For two elements a and b of an effect algebra E (or an MV-algebra) we write $a \perp b$ iff $a+b$ is defined.

For a finite sequence $F = \{a_i\}_{i=1}^n$ of elements in E, we write $\bigoplus F := a_1 + \cdots + a_n$ supposing that the element $a_1 + \cdots + a_n$ is defined in E, in such a case we say also that F is $\bigoplus$-orthogonal.

A finite sequence $F = \{a_i\}_{i=1}^n$ of non-zero elements in E is said to be a *partition of unity* if $a_1 + \cdots + a_n = 1$. We recall that our definition of partition of unity is different of that of Mundici [Mun 2], while e.g., $\{a, a*\}$ is a partition of unity ($0 \neq a \neq 1$) but it is not a partition in the sense of Mundici.

It seems that MV-algebras play an analogical role as Boolean algebras within quantum logics [DvPu 1]. Recently it was observed by Riečanová [Rie] that every lattice effect algebra E can be covered by MV-subalgebras of E.

Foulis and Randall [FoRa], in 1972, gave a new mathematical foundation of an operational probability theory and statistics based upon a generalization of the conventional notion of a sample space in the sense of Kolmogorov [Kol], which was done in 1933. They have introduced tests and tests spaces which in a "Kolmogorovian fashion" entail a propositional system of the quantum mechanical system. The main notion is a test space with tests, being defined on the set of outcomes as a natural generalization of hypotheses which can be verified on outcomes. Introducing an appropriate equivalence, we obtain an algebraic structures describing the quantum mechanical system.

This was generalized in [DvPu] (see also [DvPu 1] for D-posets, or equivalently for effect algebras).

Inspiring by this approach and intimate relations among effect algebras and MV-algebras, we introduce two kinds of tests spaces, and MV-test spaces which are in a one-to-one correspondence with MV-algebras. In addition, we show that there is a one-to-one correspondence between states on MV-algebras and weights on MV-test spaces.

2 Test Spaces I

In the present Section, we introduce test spaces which are motivated by [DvPu 1], [DvPu]. Another equivalent notion will be introduced in Section 4.

Let X be a nonempty set, elements of X are called *outcomes*. A function $F : I \to X$ is said to be of *finite multiplicity* if, for any $x \in X$, $F^{-1}(x)$ is of finite cardinality. For arbitrary sets I and J and functions $F \in X^I$ and $G \in X^J$ define $F \preccurlyeq G$ iff there is an injection $\sigma : I \to J$ such that $F = G \circ \sigma$, i.e. $F(i) = G(\sigma(i))$ for all $i \in I$. If $F \preccurlyeq G$ and $\sigma : I \to J$ is a bijection, we say that F and G are *equivalent*, in symbols $F \sim G$, and in what follows, we shall identify functions which are equivalent.[2] Therefore, we can define unambiguously a function $F\dot{\cup}G$ as follows: Let $K = I' \cup J'$, where $I' \cap J' = \emptyset$ and $\phi : I \to I'$, $\psi : J \to J'$ are bijections. Then $F\dot{\cup}G$ is an element of X^K such that

$$F\dot{\cup}G(k) = \begin{cases} F(i), & \text{if } i \in I,\ k = \phi(i), \\ G(j), & \text{if } j \in J,\ k = \psi(j). \end{cases}$$

We note that the associativity $(F\dot{\cup}G)\dot{\cup}H = F\dot{\cup}(G\dot{\cup}H)$ holds, and therefore for both sides we write $F\dot{\cup}G\dot{\cup}H$. It can be easily checked that $\preccurlyeq$ is a preorder which becomes a partial order on equivalence classes with respect to $\sim$.

Let $\mathcal{R}(F) := \{F(i) : i \in I\}$ denote the range of $F \in X^I$.

Definition 2.1 *Let $\mathcal{T} = \{F \in X^I : I \in \mathcal{I}\}$, where $X \neq \emptyset$, and $\mathcal{I}$ is a nonvoid family of index sets. We say that the pair $(X, \mathcal{T})$ is a test space (of the first type) if and only if the following three conditions are satisfied:*

(i) *Any $T \in \mathcal{T}$ is of finite multiplicity.*

(ii) *For every $x \in X$ there is a $T \in \mathcal{T}$ such that $x \in \mathcal{R}(T)$.*

(iii) *If $S, T \in \mathcal{T}$ and $S \preccurlyeq T$, then $S \sim T$.*

Any element of $\mathcal{T}$ is said to be a test.

Lemma 2.2 *If $F \in X^I$ is a test, then $I \neq \emptyset$.*

Proof. Let $F = \emptyset \in \mathcal{T}$. Then, for any $T \in \mathcal{T}$, $F \preccurlyeq T$ implies $F \sim T$ and, by (iii) of Definition 2.1, $T = \emptyset$ which is in contradiction with the condition (ii) of Definition 2.1. $\square$

Definition 2.3 *Let $(X, \mathcal{T})$ be a test space. Let J be arbitrary and let $G \in X^J$. We say that G is an event if and only if there is a test $T \in \mathcal{T}$ such that $G \preccurlyeq T$. Let us denote the set of all events in $\mathcal{T}$ by $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$.*

Clearly, $\emptyset \in \mathcal{E}$.

Definition 2.4 *Let $(X, \mathcal{T})$ be a test space. We say that two events F and G are*

(i) *orthogonal to each other, in symbols $F \perp G$, if and only if there is a test $T \in \mathcal{T}$ such that $F\dot{\cup}G \preccurlyeq T$;*

[2] Observe that if $I = \emptyset$, then $X^I = \{\emptyset\}$.

(ii) *local complements of each other, in symbols* $F\, loc\, G$, *if and only if there is a test* $T \in \mathcal{T}$ *such that* $F \dot{\cup} G \sim T$;

(iii) *perspective with axis* H, *in symbols* $F \approx_H G$, *if and only if they share a common local complement* H.

We write $F \approx_H G$ *or* $F \approx G$ *if the axis is not emphasized.*

Lemma 2.5 (i) *If* $\emptyset \approx G$, *then* $\emptyset \sim G$.
(ii) *If* $T, S \in \mathcal{T}$ *are tests, then* $T \approx S$ *with axis* $\emptyset$.

Proof. (i) Every local complement of $\emptyset$ is a test. Therefore, if $\emptyset \approx G$, there is a test T such that $T \dot{\cup} G$ is a test. But then $T \preccurlyeq T \dot{\cup} G$ implies $T \sim T \dot{\cup} G$, which implies $G \sim \emptyset$.

(ii) Observe that the unique local complement of every test is $\emptyset$. □

Definition 2.6 *A test space* $(X, \mathcal{T})$ *is algebraic if and only if, for* $F, G, H \in \mathcal{E}$, $F \approx G$ *and* $F \perp H$ *entail* $G \perp H$.

For simplicity, we usually refer to X, rather than to $(X, \mathcal{T})$, as a test space. In what follows, let X be a test space.

Lemma 2.7 *A test space* X *is algebraic if and only if, for* $F, G, H \in \mathcal{E}$,

$$F \approx G,\ \ F\, loc\, H\ \ \Rightarrow G\, loc\, H.$$

Proof. Let X be algebraic and let $F, G, H \in \mathcal{E}$, $F \approx G$, $F\, loc\, H$. The algebraicity implies that $G \perp H$. Let Q be a local complement of $G \dot{\cup} H$. Then $H \dot{\cup} Q\, loc\, G$ implies $H \dot{\cup} Q \perp F$. But since $H\, loc\, F$, $H \dot{\cup} F \preccurlyeq H \dot{\cup} F \dot{\cup} Q$ implies $H \dot{\cup} F \sim H \dot{\cup} F \dot{\cup} Q$, we get $Q = \emptyset$.

The converse implication follows from the fact that if two events are orthogonal, then either one of them can be enlarged to a local complement of the other. □

Lemma 2.8 *If* X *is algebraic, then* $\approx$ *is transitive on* $\mathcal{E}$, *hence* $\approx$ *is an equivalence relation.*

Proof. Suppose $F, G, H \in \mathcal{E}$ with $F \approx G$ and $G \approx H$. Let K be the axis for $G \approx H$. By Lemma 2.7, K is a local complement of F, hence $F \approx H$ with axis K.

□

3 MV-test Spaces

In the present Section, we show how MV-algebras can be identified with test spaces. Here by tests and test spaces we will understand the corresponding notions defined in Sections 2; in an analogical way we can use also the notions which will be defined in Section 4.

We say that a test space $(X, \mathcal{T})$ satisfies the *strong Riesz property* if, given four events E_1, E_2, F_1, F_2 with $E_1 \perp E_2$ and $F_1 \perp F_2$ such that $E_1 \dot{\cup} E_2 \approx F_1 \dot{\cup} F_2$, there exist four events $C_{11}, C_{12}, C_{21}, C_{22}$ such that $E_1 \approx C_{11} \dot{\cup} C_{12}, E_2 \approx C_{21} \dot{\cup} C_{22}, F_1 \approx C_{11} \dot{\cup} C_{21}, F_2 \approx C_{21} \dot{\cup} C_{22}$, and if there exists an event C such that $C \preccurlyeq C_{12}$, and $C \preccurlyeq C_{21}$, then $C \approx \emptyset$.

In what follows, we show that any algebraic test space satisfying the strong Riesz property will entail an MV-algebra, and conversely, each MV-algebra gives us an algebraic test space satisfying the strong Riesz property. Hence, any algebraic test space satisfying the strong Riesz property is said to be an *MV-test space*.

Definition 3.1 *Let X be an algebraic test space. If $F \in \mathcal{E}$, we define $\pi(F) := \{G \in \mathcal{E} : G \approx F\}$ and we refer to $\pi(F)$ as the proposition affiliated with F. The set*

$$\Pi = \Pi(X) := \{\pi(F) : F \in \mathcal{E}\} \tag{3.1}$$

is called the logic of the test space X.

We define $0, 1 \in \Pi$ by

$$0 = \pi(\emptyset), \quad 1 = \pi(T), \tag{3.2}$$

where T is any test.

The following two theorems are key-results of the section.

Theorem 3.2 *Let $(X, \mathcal{T})$ be an MV-test space. Then $\Pi(X)$ with $0 := \pi(\emptyset)$, $1 = \pi(T)$, where T is a test, can be organized into an MV-algebra.*

Proof. We define a partial binary operation $+$ on $\Pi(X)$ as follows. For two elements $a = \pi(A)$ and $b = \pi(B)$, $a + b$ is defined in $\Pi(X)$ and equal $c = \pi(C)$ iff there exists an event $C' \in \pi(C)$ such that for some $A' \in \pi(A)$ and $B' \in \pi(B)$ with $A' \dot{\cup} B' \approx C'$. The algebraicity of the test space yields that our partial binary operation $+$ is well-defined. In addition, it is commutative associative, with neutral element 0. In addition, $M := \Pi(X)$ is a pseudo-effect algebra. For, any $a = \pi(A)$ we define $a^* = \pi(A')$, where A' is a local complement of A.

The partial addition $+$ implies a partial binary relation $\leq$ on $\Pi(X)$ defined via $a \leq b$ iff there exists an event $c \in M$ such that $a + c = b$. In such a case, we define $c = b - a$.

Suppose now that $a_1 + a_2 = b_1 + b_2$ in M. Our hypotheses stress that there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $a_1 = c_{11} + c_{12}, a_2 = c_{21} + c_{22}, b_1 = c_{11} + c_{21}, b_2 = c_{12} + c_{22}$ and $c_{12} \wedge c_{21} = 0$. Using this property, we can prove according to [DvVe I, Prop. 3.3] that our effect algebra is a lattice. Indeed, since $a + a^* = b + b^* = 1$ for all $a, b \in M$, there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $a = c_{11} + c_{12}, a^* = c_{21} + c_{22}, b = c_{11} + c_{21}, b^* = c_{12} + c_{22}$ and $c_{12} \wedge c_{21} = 0$. Then $c_{12} \vee c_{21} = c_{12} + c_{21}$, and $c_{11} + (c_{12} \vee c_{21}) = (c_{11} + c_{12}) \vee (c_{11} + c_{21}) = a \vee b$.

In addition, it is possible to show, [DvVe II, Prop. 8.7] that, for all $a, b \in M$,

$$a - (a \wedge b) = (a \vee b) - b.$$

Hence, by [DvVe II, Thm 8.8], M can be organized into an MV-algebra $(M; \oplus, \odot, ^*, 0, 1)$ as follows

$$\begin{aligned} a \oplus b &:= (a^* - (a^* \wedge b))^*, \\ a \odot b &:= a - (a \wedge b^*). \end{aligned}$$

□

Theorem 3.3 *Let M be an MV-algebra. Then there exists an MV-test space $(X, \mathcal{T})$ such that M is isomorphic with $\Pi(X)$ as MV-algebras.*

Proof. Let $X = M \setminus \{0\}$ and let $\mathcal{T}$ be the set of all finite partitions of 1 in M consisting of nonzero elements. We claim that $(X, \mathcal{T})$ is a test space. Indeed, we have $\{1\} \in \mathcal{T}$, and if $a \neq 0$, $a \neq 1$, then $\{a, a^*\} \in \mathcal{T}$ and $a \in \mathcal{R}(\{a, a^*\})$ which proves that it is a test space. Moreover, $(X, \mathcal{T})$ is algebraic. Indeed, let F, G, H be events such that $F \approx G$ and $F \, loc \, H$. Then $F = \{a_1, \ldots, a_n\}$, $G = \{b_1, \ldots, b_m\}$, $H = \{c_1, \ldots, c_k\}$. Let $K = \{d_1, \ldots, d_l\}$ be the axis for $F \approx G$. Now $\bigoplus F + \bigoplus K = 1$, $\bigoplus G + \bigoplus K = 1$, $\bigoplus F + \bigoplus H = 1$, so that $\bigoplus H = \bigoplus K$ and $\bigoplus G + \bigoplus H = 1$, hence $G \, loc \, H$. Because every

MV-algebra satisfies the strong Riesz property, so satisfies also the test space, thus the test space is an MV-test space.

Let $\Pi(X)$ be the logic of X. By Theorem 3.2, $\Pi(X)$ is an MV-algebra. Define a mapping $\phi: M \to \Pi(X)$ by

$$\phi(a) = \begin{cases} \pi(\{a\}), & \text{if } a \neq 0, \\ \pi(\emptyset), & \text{if } a = 0. \end{cases} \tag{3.3}$$

The mapping ϕ preserves the partial addition $+$ from M onto $\Pi(X)$. Indeed, let $a + b \in M$ If one of a, b is 0, $\phi(a) \perp \phi(b)$ is evident. If $a \neq 0 \neq b$, then either $a = b^*$ or $a \neq b^*$, then $\{a, b\}$ or $\{a, b, (a+b)^*\}$, respectively, are partitions of 1, which implies that $\phi(a) \perp \phi(b)$, and $\phi(a+b) = \phi(a) + \phi(b)$.

Let now $\pi(\{a\}) \perp \pi(\{b\})$. There is a partition of 1 containing a and b, hence $a \perp b$ in M. This proves that ϕ is injective. To prove that ϕ is onto, let $A = \{a_1, \ldots, a_n\}$ be an event. Then A is $\bigoplus$-orthogonal, and put $a = \bigoplus_{i=1}^n a_i$. Then

$$\pi(A) = \pi(\{a_1, \ldots, a_n\}) = \bigoplus_{i=1}^{n} \pi(\{a_i\}) = \pi(\{a\}) = \phi(a),$$

hence ϕ is onto.

Moreover, we show that ϕ preserves $\vee$. It is clear that $\phi(a \vee b) \geq \phi(a), \phi(b)$. If $\phi(x) \geq \phi(a), \phi(b)$, then $x \geq a, b$, so that $x \geq a \vee b$, and $\phi(x) \geq \phi(a \vee b)$. According to the rest of the proof of Theorem 3.2, ϕ is an MV-isomorphism in question. $\square$

Let $\phi: X \to Y$ be a mapping. For $E \in X^I$ we denote by $\phi(E)$ the function

$$\phi(E) := \phi \circ E \in Y^I,$$

that is,

$$\phi(E)(i) = \phi(E(i)), \ i \in I. \tag{3.4}$$

A mapping $\phi: X \to Y$ is called a *homomorphism* of test spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ if $\phi(T) \in \mathcal{S}$ for every $T \in \mathcal{T}$. Evidently, if $E, F \in \mathcal{E}(X)$ and $E \perp F$, then $\phi(E),\ \phi(F) \in \mathcal{E}(Y)$, $\phi(E) \perp \phi(F)$ and $\phi(E \dot{\cup} F) = \phi(E) \dot{\cup} \phi(F)$ (we identify events related by $\sim$). Also, $E \approx F$ implies $\phi(E) \approx \phi(F)$.

Proposition 3.4 *Let ϕ be a homomorphism of two test spaces $(X, \mathcal{T})$ and $(Y, \mathcal{S})$. If $(X, \mathcal{T})$ is an MV-test space, then so is $(\phi(X), \phi(\mathcal{T}))$, where $\phi(\mathcal{T})$ stands for $\{\phi(T) : T \in \mathcal{T}\}$.*

Proof. For any $y \in \phi(X)$, there is an $x \in X$ with $y = \phi(x)$. If $T \in \mathcal{T}$ is such that $x \in \mathcal{R}(T)$, then $\phi(x) \in \mathcal{R}(\phi(T))$. This proves that $(\phi(X), \phi(\mathcal{T}))$ is a test space. If now $E \in X^I$, $F \in X^J$ are such that $\phi(E) \perp \phi(F)$, then $I \cap J = \emptyset$, so that $E \perp F$, and moreover, $\phi(E \dot\cup F) = \phi(E) \dot\cup \phi(F)$. If $\phi(E)\, loc\, \phi(F)$, then $E \perp F$ implies that there is a $G \in \mathcal{E}(X)$ such that $E \dot\cup G \dot\cup F \in \mathcal{T}$. But then $\phi(E) \dot\cup \phi(G) \dot\cup \phi(F) \in \phi(\mathcal{T})$, and also $\phi(E) \dot\cup \phi(F) \in \phi(\mathcal{T})$, which yields $\phi(G) = \emptyset$, hence $G = \emptyset$, and $E\, loc\, F$. This proves that $E\, loc\, F$ iff $\phi(E)\, loc\, \phi(F)$, from which we easily derive that $(\phi(X), \phi(\mathcal{T}))$ is algebraic iff $(X, \mathcal{T})$ is algebraic. In addition, we have easily that if $(X, \mathcal{T})$ is an MV-test space, then so is $(\phi(X), \phi(\mathcal{T}))$. $\square$

Proposition 3.5 *Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be MV-test spaces and let $\phi : X \to Y$ be a homomorphism. Then the mapping $\psi : \Pi(X) \to \Pi(Y)$ defined by*

$$\psi(\pi(E)) := \pi(\phi(E)) \quad (E \in \mathcal{E}(X)), \tag{3.5}$$

is a homomorphism of effect algebras.

Proof. Clearly, $\psi(1) = 1$. Further, $\pi(F) \perp \pi(G)$ implies $F \perp G$ and $\phi(F) \perp \phi(G)$, so that $\psi(\pi(F)) \perp \psi(\pi(G))$ and $\phi(F \dot\cup G) = \phi(F) \dot\cup \phi(G)$, which gives $\psi(\pi(F) + \pi(G)) = \psi(\pi(F)) + \psi(\pi(G))$. $\square$

We recall that if $h : M_1 \to M_2$ is a homomorphism of MV-algebras, then the restriction of h onto $M_1 \setminus \{0\}$ is a homomorphism of MV-test spaces $(M_1 \setminus \{0\}, \mathcal{T})$ and $(M_2 \setminus \{0\}, \mathcal{S})$, where $\mathcal{T}$ and $\mathcal{S}$ are the sets of all partitions in M_1 and M_2, iff h is injective. Hence we define the following two categories.

Let $\mathcal{MV}_0$ be the category of MV-algebras whose objects are MV-algebras and morphisms are injective homomorphisms of MV-algebras. By $\mathcal{MVT}$ we denote the category of MV-test spaces whose objects are MV-test spaces and morphisms are injective homomorphisms of MV-test spaces.

It would be interesting to exhibit whether are these categories equivalent.

4 Test Spaces II

In the present Section, we describe an alternative and equivalent way how to define test spaces; this approach is motivated by [Gud].

Let X be a nonempty set describing a system, elements of X are called *outcomes*, and let $\mathcal{T} \subseteq \{0, 1, 2, \ldots\}^X$. We call $(X, \mathcal{T})$ a *test space* (of the second type) if

(1) for any $x \in X$ there exists a $T \in \mathcal{T}$ such that $T(x) \neq 0$;

(2) if $S, T \in \mathcal{T}$ with $S \leq T$ (i.e., $S(x) \leq T(x)$ for all $x \in X$), then $S = T$.

The elements of $\mathcal{T}$ are said to be *tests*.

We call a function $F \in \{0, 1, 2, \ldots\}^X$ an *event* if $F \leq T$ for some $T \in \mathcal{T}$, and let $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$ be the set of all events. We say that $F, G \in \mathcal{E}$ are (i) *mutually exclusive* ($F \perp G$) if $F + G \in \mathcal{E}$; (ii) *local complements* of each other ($F \operatorname{loc} G$) if $F + G \in \mathcal{T}$, (iii) *perspective* ($F \approx G$) if they share a local complement. We say that a test space $\mathcal{T}$ is *algebraic* if, for $F, G, H \in \mathcal{E}$, $F \approx G$ and $H \perp F$ imply $H \perp G$.

For two events $F, G \in \mathcal{E}$ we write $F \preccurlyeq G$ iff there is a mutually exclusive event $H \perp F$ such that $F + H = G$.

Let $O : X \to \{0, 1, 2, \ldots\}$ be defined by $O(x) = 0$ for all $x \in X$. Then O is an event, while $O \leq T$ for any test T, i.e., $O \in \mathcal{E}$. In addition, any test $T \in \mathcal{E}$. We recall that if T is a test, then $T \neq O$. If not, since $O \leq S$ for any $S \in \mathcal{T}$, we would have by (2), $S = O$ which is a contradiction with (1) of the definition of the test space.

It is worthy to recalling that if $\mathcal{T} \subseteq \{0, 1\}^X$, then tests are crisp subsets of X because they are characteristic functions of subsets of X, and they correspond to two-valued reasoning, i.e. when the system is defined precisely. If $\mathcal{T} \subseteq \{0, 1, \ldots, n\}^X$, then tests correspond to the situation when the system is described by multivalued logic, e.g. by an $n + 1$-valued logic using fuzzy sets from

$$L_{n+1} := \{0, 1/n, 2/n, \ldots, n/n\}$$

as it is shown in the following examples.

Example 4.1 *Let B be a Boolean algebra of subsets of a nonvoid set Ω. Define $X = B \setminus \{\emptyset\}$. Let $A_1, \ldots, A_n$ be a decomposition of Ω, i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$, and $A_1 \cup \cdots \cup A_n = \Omega$. Define a function $F_{A_1,\ldots,A_n} : X \to \{0, 1\}$ by*

$$F_{A_1,\ldots,A_n}(A) = \begin{cases} 1 & \text{if } A = A_i \text{ for some } i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Then the system $\mathcal{T}$ of all such functions $F_{A_1,\ldots,A_n}$ is an algebraic test space.

The following two examples are connected with the Ulam game with one lie (n lies), [CDM], i.e. a game with two players where one player is asking the smallest possible number of question to find an unknown number from the known set of numbers, and the second player can only answer "yes" or "no", being allowed at most one lie (n lies).

Example 4.2 *Let us consider a three-valued logic, i.e., the Lukasiewicz logic connected with the truth values system* $L_3 = \{0, 1/2, 1\}$. *Assume that our system is described by a two-valued set* $X = \{x_1, x_2\}$. *Define two function* F_1 *and* F_2 *as mappings from* X *into* $\{0, 1, 2\}$ *by*

$$F_1(x) = \begin{cases} 2 & \text{if } x = x_1, \\ 0 & \text{if } x = x_2, \end{cases}$$

and

$$F_2(x) = \begin{cases} 0 & \text{if } x = x_1, \\ 1 & \text{if } x = x_2. \end{cases}$$

Then $\{F_1, F_2\}$ *is an algebraic test space connected with a three-valued reasoning.*

Example 4.3 *Let us consider an* $n+1$*-valued logic, i.e., the Lukasiewicz logic connected with the truth values system* $L_{n+1} = \{0, 1/n, 2/n, \ldots, 1\}$. *Assume that our system is described by an* n*-element set* $X = \{x_1, \ldots, x_n\}$. *Define for any* n*-tuple* $(k_1, \ldots, k_n)$ *of integers from the set* $\{0, 1, \ldots, n\}$ *with* $1k_1 + 2k_2 + \cdots nk_n = n$ *a function* $F_{k_1,\ldots,k_n} : X \to \{0, 1, \ldots, n\}$ *by*

$$F_{k_1,\ldots,k_n}(x) = k_i \quad \text{if } x = x_i \text{ for some } i = 1, \ldots, n.$$

Then the system $\mathcal{T}$ *of all such functions* $F_{k_1,\ldots,k_n}$ *is an algebraic test space connected with* $n+1$*-valued reasoning.*

It is interesting to recall that the test spaces are in fact all partitions of the true-value which are possible in the given system using two-valued or multivalued reasoning for the description of our logical system.

We recall that the following results have similar proofs as those in Section 2; we include them here only to be self-containing.

Proposition 4.4 *A test space* $(X, \mathcal{T})$ *is algebraic if and only if, for all* $F, G, H \in \mathcal{E}$

$$F \approx G,\ F \,loc\, H \ \Rightarrow\ G \,loc\, H.$$

Proof. Let $(X, \mathcal{T})$ be an algebraic test space, and let $F, G, H \in \mathcal{E}$, $F \approx G$, $F \operatorname{loc} H$. The algebraicity implies that $G \perp H$. Let Q be a local complement of $G + H$. Then $(H + Q) \operatorname{loc} G$ implies $(H + Q) \perp F$. But since $(H + F) \preccurlyeq H + F + Q$, we get $Q = O$, i.e. $G \operatorname{loc} H$.

Suppose the converse and let $F \approx G$ and $H \perp F$. Then $H + F \in \mathcal{E}$, and there is a local complement F' to $H + F$, i.e. $H + F + F' \in \mathcal{T}$. Hence $(H + F') \operatorname{loc} F$ and by the assumptions, $G \operatorname{loc} (H + F')$ so that $H \perp G$. $\square$

Proposition 4.5 *Let $(X, \mathcal{E})$ be an algebraic test space. Then the relation $\approx$ is an equivalence on the space of all events $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$.*

Proof. Let F be an event. Then $F \approx F$ while there is a test T such that $F \leq T$. Then $F' = T - F$ is a local complement of F, hence $F \approx F$.

It is evident that if $F \approx G$ then $G \approx F$.

Assume $F \approx G$ and $G \approx H$. If K is a local complement to H, by Proposition 4.4, so is for G and consequently for F. $\square$

We say that a test space $(X, \mathcal{T})$ satisfies the *strong Riesz property* if, given four events E_1, E_2, F_1, F_2 with $E_1 \perp E_2$ and $F_1 \perp F_2$ such that $E_1 + E_2 \approx F_1 + F_2$, there exist four events $C_{11}, C_{12}, C_{21}, C_{22}$ such that $E_1 \approx C_{11} + C_{12}, E_2 \approx C_{21} + C_{22}, F_1 \approx C_{11} + C_{21}, F_2 \approx C_{21} + C_{22}$, and if there exists an event C such that $C \preccurlyeq C_{12}$, and $C \preccurlyeq C_{21}$, then $C \approx \emptyset$.

As in Section 3, we define, for an algebraic test space $(X, \mathcal{T})$,

$$\pi(F) := \{G \in \mathcal{E} : \, G \approx F\}$$

and we refer to $\pi(F)$ as the *proposition* (or an *event*) affiliated with F. The set

$$\Pi = \Pi(X, \mathcal{T}) := \{\pi(F) : \, F \in \mathcal{E}\}$$

is called the *event structure* of the test space $(X, \mathcal{T})$.

We define $0, 1 \in \Pi$ by

$$0 = \pi(O), \quad 1 = \pi(T),$$

where T is any test.

We say that a test space $(X, \mathcal{T})$ (of the second type) satisfies the *strong Riesz property* if, given four events E_1, E_2, F_1, F_2 with $E_1 \perp E_2$ and $F_1 \perp F_2$ such that $E_1 + E_2 \approx F_1 + F_2$, there exist four events $C_{11}, C_{12}, C_{21}, C_{22}$ such that $E_1 \approx C_{11} + C_{12}, E_2 \approx C_{21} + C_{22}, F_1 \approx C_{11} + C_{21}, F_2 \approx C_{21} + C_{22}$, and if there exists an event C such that $C \preccurlyeq C_{12}$, and $C \preccurlyeq C_{21}$, then $C \approx \emptyset$.

An algebraic test space (of the second type) satisfying the strong Riesz property is said to be an *MV-test space* (of the second type). Comparing both types of MV-test spaces we will see that they are practically the same, as it follows from the next result because an analogical result to Theorem 3.2 and Theorem 3.3 is also true for this type of test spaces.

Theorem 4.6 *Let M be an MV-algebra. Then there exists an MV-test space $(X,\mathcal{T})$ such that $\Pi(X,\mathcal{T})$ is isomorphic with M as MV-algebras. Conversely, if $(X,\mathcal{T})$ is an MV-test space, then $\Pi(X)$ can be organized into an MV-algebra.*

Proof. The proof follows similar arguments as those in Theorems 3.2 and 3.3. We outline only the main lines.

Let M be an MV-algebra and let $(M;+,0,1)$ be the effect algebra derived from the MV-algebra M, where $+$ is the partial addition taken from $\oplus$ via $a+b := a\oplus b$ whenever $a\le b^*$. Set $X = E\setminus\{0\}$. We define a test space $\mathcal{T}$ as the system of all functions $F: X\to\{0,1,2,\ldots\}$ such that F is zero for all but finite numbers of X and

$$\sum_{a\in E} F(a)a = 1. \tag{4.1}$$

In other words, any non-zero value of $F(a)a$ is a multiplicity of the element a in the partition of 1.

We claim that $(X,\mathcal{T})$ is an algebraic test space. Indeed, let $a\in X$. Then either there is an integer $n\ge 1$ such that $na=1$, or there is no such an integer. In the first case we define F_a by $F_a(a)=n$ and $F_a(x)=0$ for $x\ne a$. In the second case, we put $F_a(a)=1$, $F_a(a')=1$, and $F_a(x)=0$ if $x\notin\{a,a'\}$. Then $F_a\in\mathcal{T}$ and $F_a(a)\ne 0$.

If now $F\le G$ for $F,G\in\mathcal{T}$, then

$$1=\sum_{a\in E}F(a)\,a\le\sum_{a\in E}G(a)\,a=1$$

which proves $F=G$.

It is evident that a function $F: X\to\{0,1,2,\ldots\}$ is an event iff F takes nonzero values only for finitely many elements of X, and

$$\sum_{a\in E}F(a)\,a\in E.$$

Hence $F'\in\mathcal{E}(X,\mathcal{T})$ is a local complement of F iff

$$\sum_{a\in E}F'(a)a+\sum_{a\in E}F(a)\,a=1.$$

Hence if $F\approx G$ and $H\operatorname{loc}F$, then $H\operatorname{loc}G$.

Let $\Pi(X,\mathcal{T})$ be the event structure affiliated with the algebraic test space $(X,\mathcal{T})$ which, analogically as in Theorem 3.3, is an effect algebra.

The function $F_a : X \to \{0,1,2,\ldots\}$ defined by

$$F_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{otherwise} \end{cases}$$

is an element of $\mathcal{E}$. Define a mapping $\phi : E \to \Pi(X,\mathcal{T})$ by

$$\phi(a) = \begin{cases} \pi(F_a) & \text{if } a \neq 0, \\ \pi(O) & \text{if } a = 0. \end{cases}$$

The mapping ϕ preserves $+$ in E. Indeed, if $a,b \in E$ and let $a+b \in E$. If one of them is 0, then $\phi(a)+\phi(b)$ is defined in $\Pi(X,\mathcal{T})$. If $a \neq 0 \neq b$, then either $a = b'$ or $a \neq b'$ and $a+b = 1$ or $a+b+(a+b)' = 1$, respectively. Hence $\phi(a)+\phi(b)$ is defined, and $\phi(a)+\phi(b) = \phi(a+b)$.

Let now $\phi(F_a) \perp \phi(F_b)$. There exists a partition of 1 containing a and b, hence $a \perp b$ in E. This proves that ϕ is injective and preserves $+$ in E and in $\Phi(X,\mathcal{T})$, respectively.

To show that ϕ is surjective, let F be an event in $\mathcal{E}$. Then F is nonzero for finitely many elements of E and $\sum_{a\in E} F(a)\, a \in E$. Hence

$$\phi(a) = \phi(E_a) = \phi(F)$$

proving that ϕ is onto.

Using now similar arguments as those in Theorem 3.3, we can prove that $\Pi(X)$ is an MV-algebra isomorphic with M.

The converse statement is now clear and it follows the main steps of the proof of Theorems 3.2 and 3.3. $\square$

5 States and Weights

A state on an MV-algebra M is a mapping $s : M \to [0,1]$ such that $s(1) = 1$ and $s(a \oplus b) = s(a) + s(b)$ whenever $a \leq b^*$.

States on MV-algebras were introduced by F. Chovanec [Cho] and by D. Mundici [Mun 1] with the intent of capturing the notion of "average degree of truth" of a proposition, see also [RiMu].

It is important to recall that every MV-algebra possesses at least one state, see [Mun 1].

Let $(X, \mathcal{T})$ be an (algebraic) test space (of the second type). A *weight* on X is a function $\omega : X \to [0,1]$ such that, for any test $T \in \mathcal{T}$,

$$\omega(T) := \sum_{x \in X} T(x)\, \omega(x) = 1. \tag{5.1}$$

In what follows, we show that there is a one-to-one correspondence between states on MV-algebras and weights in MV-test spaces.

Theorem 5.1 *Let ω be a weight on an MV-test space $(X, \mathcal{T})$. The mapping $s : \Pi(X, \mathcal{T}) \to [0,1]$ defined via*

$$s(\pi(F)) := \sum_{x \in X} F(x)\, \omega(x), \ \ \pi(F) \in \Pi(X, \mathcal{T}), \tag{4.2}$$

is a state in the MV-algebra $\Pi(X, \mathcal{T})$. Conversely, if s is a state on an MV-algebra M, then the function $\omega : M \setminus \{0\} \to [0,1]$ defined by $\omega(x) := s(x)$, $x \in M \setminus \{0\}$, is a weight on the MV-test space $(M \setminus \{0\}, \mathcal{T})$, where $\mathcal{T}$ is the system of all tests defined by (4.1).

Proof. Let $a = \phi(F) = \phi(F_1)$ and let F' be a local complement of F, so is of F_1. Hence the function s defined by (4.2) is well-defined in view of

$$\sum_{x \in X} F(x)\, \omega(x) + \sum_{x \in X} F'(x)\, \omega(x) = 1 = \sum_{x \in X} F_1(x)\, \omega(x) + \sum_{x \in X} F'(x)\, \omega(x).$$

The additivity of s is now clear.

Conversely, let s be a state on an effect algebra E. Using the proof of Theorem 4.6 and the algebraic test spaces (3.3), wee see that $\omega(x) := s(x)$, $x \neq 0$, gives a weight on $X = E \setminus \{0\}$ and $\mathcal{T}$. $\square$

If we take an MV-test space of the first kind, then a weight is defined as a function $\omega : X \to [0,1]$ for an MV-test space $(X, \mathcal{T})$ such that, for every $E \in \mathcal{T}$, $E \in X^I$, we have

$$\omega(E) := \sum_{i \in I} \omega(E(i)) = 1.$$

It is evident that we can establish an analogical assertion among weights on MV-test spaces (of the first kind) and states on MV-algebras as in Theorem 5.1.

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