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# Reduction of the 5-flow conjecture to cyclically 6-edge-connected snarks

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# REDUCTION OF THE 5-FLOW CONJECTURE TO CYCLICALLY 6-EDGE-CONNECTED SNARKS

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Abstract. We show that the smallest possible counterexample to the 5-Flow Conjecture of Tutte (every bridgeless graph has a nowhere-zero 5-flow) must be a cyclically 6-edge-connected cubic graph.

## 1. Introduction

A graph admits a *nowhere-zero $k$-flow* ($k$ is an integer $\geq 2$) if its edges can be oriented and assigned numbers $\pm 1, \dots, \pm(k-1)$ so that for every vertex, the sum of the incoming values equals the sum of the outcoming ones. It is well-known that a graph with a bridge (1-edge-cut) does not have a nowhere-zero $k$-flow for any $k \geq 2$ (see, e. g., [J], [K6], [Z]). The famous 5-*Flow Conjecture* of Tutte [T1] is that every bridgeless graph has a nowhere-zero 5-flow.

An edge cut of a graph is called *cyclic* if deleting its edges results in a graph having at least two cyclic components. A graph is called *cyclically $k$-edge-connected* if it has no cyclic cut of cardinality smaller than $k$.

It is well-known (see cf. Jaeger [J] and Zhang [Z]) that the smallest counterexample to the 5-flow conjecture must be a *snark* which is a cyclically 4-edge-connected cubic graph without an edge-3-coloring and with girth (the length of the shortest cycle) at least 5. Furthermore by Celmins [C], this counterexample must be cyclically 5-edge-connected and with girth at least 7 (see also [J], [K4]). This result was very interesting because until recently, no snarks with girth at least 7 had been know. Furthermore, Jaeger and Swart [JS] conjectured that such snarks do not exist. In [K2] we disproved this conjecture constructing cyclically 5-edge-connected snarks with arbitrary large girth (see also [K6]).

In this paper we prove that the smallest counterexample to the 5-flow conjecture must be cyclically 6-edge-connected. Since all cyclically 6-edge-connected snarks known until now have girth 6 (see [I], [K1], [K3], [K6]), this counterexample belongs to a class of graphs for which we do not know whether it is nonempty.

## 2. Preliminaries

The graphs considered in this paper are all finite and unoriented. Multiple edges and loops are allowed. If $G$ is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of $G$, respectively. By a *multi-terminal network*, briefly a *network*, we mean a pair $(G, U)$ where $G$ is a graph and $U = (u_1, \dots, u_n)$ is an ordered set of pairwise distinct vertices of $G$. If no confusion can occur, we denote by $U$ also the set $\{u_1, \dots, u_n\}$. The vertices from $U$ and $V(G) \setminus U$ are called the *outer* and *inner* vertices of the network $(G, U)$, respectively. We allow $n = 0$, i. e., $U = \emptyset$.

We associate with each edge of $G$ two distinct arcs, distinct for distinct edges (see also [K6]). If one of the arcs corresponding to an edge is denoted by $x$, the other is denoted by $x^{-1}$. If the ends of an edge $e$ are the vertices $u$ and $v$, one of the arcs corresponding to $e$ is said to be *directed from $u$ to $v$* (and the other from $v$ to $u$). In particular, a loop corresponds to two distinct arcs both directed from a vertex to itself. Let $D(G)$ denote the set of arcs on $G$. Then $|D(G)| = 2|E(G)|$. If $v \in V(G)$, then $\omega_G(v)$ denotes the set of arcs of $G$ directed from $v$ to $V(G) \setminus \{v\}$.

If $G$ is a graph and $A$ is an additive Abelian group, then an *$A$-chain* in $G$ is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that

$$\partial\varphi(v) \;=\; \sum_{x \in \omega_G(v)} \varphi(x) \qquad (v \in V(G))$$

is called the *boundary* of $\varphi$. An $A$-chain $\varphi$ in $G$ is called *nowhere-zero* if $\varphi(x) \neq 0$ for every $x \in D(G)$. If $(G, U)$ is a network, then an $A$-chain $\varphi$ in $G$ is called an *$A$-flow* in $(G, U)$ if $\partial\varphi(v) = 0$ for every inner vertex $v$ of $(G, U)$. The following statement is proved in [K5], [K6].

**Lemma 1.** *If $\varphi$ is an $A$-flow in a network $(G, U)$, then $\sum_{u \in U} \partial\varphi(u) = 0$.*

By a (*nowhere-zero*) *$A$-flow* in a graph $G$ we mean a (nowhere-zero) $A$-flow in the network $(G, \emptyset)$. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in Jaeger [J] and Zhang [Z]. By Tutte [T1], [T2], a graph has a nowhere-zero $k$-flow if and only if it has a nowhere-zero $A$-flow for any Abelian group $A$ of order $k$. Thus the study of nowhere-zero 5-flows is in certain sense equivalent with the study of nowhere-zero $\mathbb{Z}_5$-flows. We shall use this fact and deal only with $\mathbb{Z}_5$-flows because they are easier to handle than integral flows.

## 3. Simple networks

A network $(G, U)$, $U = (u_1, \ldots, u_n)$, is called *simple* if the vertices $u_1, \ldots, u_n$ have valency 1. If $\varphi$ is a nowhere-zero $\mathbb{Z}_5$-flow in $(G, U)$, then denote by $\partial\varphi(U)$ the $n$-tuple $(\partial\varphi(u_1), \ldots, \partial\varphi(u_n))$. By Lemma 1, $\partial\varphi(U)$ belongs to the set

$$S_n = \{(s_1, \ldots, s_n);\ s_1, \ldots, s_n \in \mathbb{Z}_5 - 0,\ s_1 + \cdots + s_n = 0\}.$$

Let $P = \{Q_1, \ldots, Q_r\}$ be a partition of the set $\{1, \ldots, n\}$. If one of $Q_1, \ldots, Q_r$ is a singleton, $P$ is called *trivial*, otherwise it is called *nontrivial*. Let $\mathcal{P}_n$ denote the set of nontrivial partitions of $\{1, \ldots, n\}$.

If $s = (s_1, \ldots, s_n) \in S_n$, $P = \{Q_1, \ldots, Q_r\} \in \mathcal{P}_n$, and $\sum_{i \in Q_j} s_i = 0$ for $j = 1, \ldots, r$, then we say that $P$ and $s$ are *compatible.* (For example, $\{\{1, 2\}, \{3, 4, 5\}\} \in \mathcal{P}_5$ is compatible with $(1, 4, 1, 2, 2) \in S_5$.)

If $|\mathcal{P}_n| = m$, then $\mathcal{P}_n$ can be considered as an $m$-tuple $(P_{n,1}, \ldots, P_{n,m})$. For any $s \in S_n$, denote by $\chi_n(s)$ the integral vector $(c_1, \ldots, c_m)$ so that $c_i = 1$ $(c_i = 0)$ if $P_{n,i}$ is (is not) compatible with $s$, $i = 1, \ldots, m$.

Let $T_n$ be the set of mappings from $S_n$ to the set of rationals $\mathbb{Q}$. If $t \in T_n$, then define

$$\chi_n(t) = \sum_{s \in S_n} t(s) \cdot \chi_n(s). \tag{1}$$

Let $(G, U)$, $U = (u_1, \ldots, u_n)$, be a simple network. For every $s \in S_n$, denote by $F_{G,U}(s)$ the number of nowhere-zero $\mathbb{Z}_5$-flows $\varphi$ in $(G, U)$ satisfying $\partial\varphi(U) = s$. If $t \in T_n$, then define

$$F_{G,U}(t) = \sum_{s \in S_n} t(s) \cdot F_{G,U}(s). \tag{2}$$

**Lemma 2.** *Let $(G, U)$, $U = (u_1, \ldots, u_n)$, be a simple network and $t, t' \in T_n$ satisfy $\chi_n(t) = \chi_n(t')$. Then $F_{G,U}(t) = F_{G,U}(t')$.*

*Proof.* We can assume that no two outer vertices of $(G, U)$ are joined by an edge (otherwise subdivide each such an edge by a new vertex of valency 2) and $G$ has no isolated vertex. For such networks we apply induction on $|E(G)| \geq n$.

If $|E(G)| = n$, then $G$ is bipartite with partitions $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_r\}$. Thus there exists a partition $P = \{Q_1, \ldots, Q_r\}$ of $\{1, \ldots, n\}$ such that $i \in Q_j$ iff $u_i$ is adjacent to $v_j$ ($i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, r\}$). If $P$ is trivial (i. e., $(G, U)$ has an inner vertex of valency 1), then $F_{G,U}(s) = 0$ for every $s \in S_n$, whence by (2), $F_{G,U}(t) = F_{G,U}(t') = 0$.

If $P \in \mathcal{P}_n$, then there exists $j \in \{1, \ldots, m\}$ such that $P = P_{n,j}$. Let $z$ and $z'$ be mappings from $S_n$ to the set of nonnegative integers satisfying $\chi_n(z) = \chi_n(z')$. Then consider the multiset $\Phi$ ($\Phi'$) of elements from $S_n$ so that every $s \in S_n$ is contained in it exactly $z(s)$ ($z'(s)$) times. Let $\Psi$ ($\Psi'$) be the multiset of elements of $\Phi$ ($\Phi'$) which are compatible with $P_{n,j}$ or, in other words, which are equal to $\partial\varphi(U)$ for a nowhere-zero $\mathbb{Z}_5$-flow $\varphi$ in $(G, U)$. Thus by (2), $|\Psi| = F_{G,U}(z)$ ($|\Psi'| = F_{G,U}(z')$). Furthermore,

$$\sum_{s \in \Phi} \chi_n(s) = \sum_{s \in S_n} z(s) \cdot \chi_n(s) = \chi_n(z) = \chi_n(z') = \sum_{s \in S_n} z'(s) \cdot \chi_n(s) = \sum_{s \in \Phi'} \chi_n(s).$$

Let $(c_1, \ldots, c_m) = \sum_{s \in \Phi} \chi_n(s)$ ($(c'_1, \ldots, c'_m) = \sum_{s \in \Phi'} \chi_n(s)$). For $i = 1, \ldots, m$, $c_i$ ($c'_i$) is the number how many times the permutation $P_{n,i}$ is compatible with elements from $\Phi$ ($\Phi'$). We have $c_i = c'_i$ for $i = 1, \ldots, m$ and in particular $|\Psi| = c_j = c'_j = |\Psi'|$ i. e., $F_{G,U}(z) = F_{G,U}(z')$. Since this holds for any $z, z'$ with the properties mentioned above, using (1) and (2) we can check that $F_{G,U}(t) = F_{G,U}(t')$ for every $t, t' \in T_n$ satisfying $\chi_n(t) = \chi_n(t')$.

If $|E(G)| > n$, then there exists an edge $e$ of $G$ with ends $v_1, v_2 \neq u_1, \ldots, u_n$. Let $G - e$ and $G/e$ be the graphs arising from $G$ after deleting and contracting $e$, respectively. We claim that for every $s \in S_n$,

$$F_{G,U}(s) = F_{G/e,U}(s) - F_{G-e,U}(s). \tag{3}$$

Let $\varphi$ be a nowhere-zero $\mathbb{Z}_5$-flow in $(G/e, U)$ so that $\partial\varphi(U) = s$. Then $\varphi$ is a nowhere-zero $\mathbb{Z}_5$-flow in the network $(G - e, (v_1, v_2, u_1, \ldots, u_n))$, whence by Lemma 1, $\partial\varphi(v_1) + \partial\varphi(v_2) = 0$. If $\partial\varphi(v_1) = \partial\varphi(v_2) = 0$, $\varphi$ is a nowhere-zero $\mathbb{Z}_5$-flow in $(G - e, U)$. Otherwise $\varphi$ can be extended to exactly one nowhere-zero $\mathbb{Z}_5$-flow in $(G, U)$. This proves (3). Hence by (2), $F_{G,U}(t) = F_{G/e,U}(t) - F_{G-e,U}(t)$ for every $t \in T_n$. This implies the statement by the induction hypothesis. □

**Lemma 3.** *Suppose that $\{b_1, \dots, b_r\}$ is a base of the linear hull in $\mathbb{Q}^m$ of the set of vectors $\{\chi_n(s); s \in S_n\}$. Then for every simple network $(G, U)$ with $n$ outer vertices there exist rationals $x_1, \dots, x_r$ such that for every $s \in S_n$, $F_{G,U}(s) = \sum_{i=1}^r c_i \cdot x_i$ where $c_1, \dots, c_r$ are rationals satisfying $\chi_n(s) = \sum_{i=1}^r c_i \cdot b_i$.*

*Proof.* For every $i = 1, \dots, r$, there exists $t_i \in T_n$ such that $\chi_n(t_i) = b_i$. Let $x_i = F_{G,U}(t_i)$. Then $\chi_n(s) = \sum_{i=1}^r c_i \cdot \chi_n(t_i) = \chi_n(\sum_{i=1}^r c_i \cdot t_i)$, whence by (2) and Lemma 2, $F_{G,U}(s) = F_{G,U}(\sum_{i=1}^r c_i \cdot t_i) = \sum_{i=1}^r c_i \cdot F_{G,U}(t_i) = \sum_{i=1}^r c_i \cdot x_i$. $\square$

Let $\mathcal{A}$ denote the automorphism group of $\mathbb{Z}_5$. The elements of $\mathcal{A}$ are $\alpha_0 = \mathrm{id}$, $\alpha_1 = (1, 2, 4, 3)$, $\alpha_2 = (1, 4)(2, 3)$ and $\alpha_3 = (1, 3, 4, 2)$. If $s = (s_1, \dots, s_n) \in S_n$ and $\alpha \in \mathcal{A}$, then denote $\alpha(s) = (\alpha(s_1), \dots, \alpha(s_n)) \in S_n$. Clearly, $\chi_n(s) = \chi_n(\alpha(s))$, whence by Lemma 2, $F_{G,U}(s) = F_{G,U}(\alpha(s))$ for every simple network $(G, U)$ with $n$ outer vertices.

## 4. The main result

$\mathcal{P}_5$ contains the following partitions (considering all sums mod 5):

$$\begin{aligned} P_{5,i} &= \{\{i, i+1\}, \{i+2, i+3, i+4\}\}, \quad (i = 1, \dots, 5), \\ P_{5,5+i} &= \{\{i, i+2\}, \{i+1, i+3, i+4\}\}, \quad (i = 1, \dots, 5), \\ P_{5,11} &= \{1, 2, 3, 4, 5\}. \end{aligned}$$

If $s = (s_1, \dots, s_5) \in S_5$, then denote by $\pi(s) = (s_5, s_1, s_2, s_3, s_4)$. Let $e_i$ denote the unit vector from $\mathbb{R}^{11}$ with the $i$th coordinate 1, $i = 1, \dots, 11$.

**Theorem 1.** *Let $G$ be a counterexample to the 5-flow conjecture of the smallest possible order. Then $G$ is a cyclically 6-edge-connected snark with girth $\geq 7$.*

*Proof.* By Celmins [C], $G$ is a cyclically 5-edge-connected snark with girth $\geq 7$. Suppose that $G$ has a cyclic 5-edge cut $\{e_1, \dots, e_5\}$. Then $G - \{e_1, \dots, e_5\}$ has exactly two components $G'$ and $G''$, which are cyclic and bridgeless. For $i = 1, \dots, 5$, let $v_i'$ and $v_i''$ be the end of $e_i$ contained in $G'$ and $G''$, respectively. Add to $G'$ ($G''$) five vertices $u_1, \dots, u_5$ ($w_1, \dots, w_5$) and, for $i = 1, \dots, 5$, join $u_i$ with $v_i'$ ($w_i$ with $v_i''$). We get from $G'$ ($G''$) a new graph $X$ ($Y$). Consider the simple networks $(X, U)$, $U = (u_1, \dots, u_5)$, and $(Y, W)$, $W = (w_1, \dots, w_5)$.

$F_{X,U}(s)$, $F_{Y,W}(s) \geq 0$ for every $s \in S_5$. If there exists $s \in S_5$ such that $F_{X,U}(s)$, $F_{Y,W}(s) > 0$, then $(X, U)$ and $(Y, W)$ have nowhere-zero $\mathbb{Z}_5$-flows $\varphi_1$ and $\varphi_2$, respectively, such that $\partial\varphi_1(U) = s$ and $\partial\varphi_2(W) = \alpha_2(s)$ (because $F_{Y,W}(\alpha_2(s)) = F_{Y,W}(s) > 0$) which can be "pieced together" into a nowhere-zero $\mathbb{Z}_5$-flow in $G$, a contradiction. Thus $F_{X,U}(s) \cdot F_{Y,W}(s) = 0$ for every $s \in S_5$.

Let $S$ (resp., $S'$) be the set of all possible permutations of elements $1, 1, 1, 3, 4$ (resp., $1, 1, 2, 2, 4$). Clearly, $S, S' \subseteq S_5$. If $s \in S$ ($s \in S'$), then $\chi_5(s) = e_a + e_b + e_c + e_{11}$ ($\chi_5(s) = e_a + e_b + e_{11}$) where $a, b, c$ ($a, b$) are pairwise distinct elements from $\{1, \dots, 10\}$ and the partitions $P_{5,a}, P_{5,b}, P_{5,c}$ ($P_{5,a}, P_{5,b}$) are of the form $\{\{i_1, i_2\}, \{i_3, i_4, i_5\}\}$ where the $i_1$th and $i_2$th components of $s$ are 4 and 1. Thus for $i = 1, \dots, 5$, we have

$$\begin{aligned} \chi_5(\pi^{i-1}(4, 1, 3, 1, 1)) - \chi_5(\pi^{i-1}(4, 2, 2, 1, 1)) &= e_i, \\ \chi_5(\pi^{i-1}(4, 3, 1, 1, 1)) - \chi_5(\pi^{i-1}(4, 2, 2, 1, 1)) &= e_{5+i}. \end{aligned}$$

Moreover $\chi_5(1,1,1,1,1) = e_{11}$. Thus $\mathbb{Q}^{11}$ is the linear hull of vectors $\chi_5(s)$, $s \in S_5$, in $\mathbb{Q}^{11}$ and $\{e_1, \dots, e_{11}\}$ is its base. Hence by Lemma 3, there exist rationals $x_i$ and $y_i$, $i = 1, \dots, 11$, such that for every $s \in S_5$, $F_{X,U}(s) = \sum_{i=1}^{11} c_i x_i$ and $F_{Y,W}(s) = \sum_{i=1}^{11} c_i y_i$ where $\chi_5(s) = (c_1, \dots, c_{11}) = \sum_{i=1}^{11} c_i \cdot e_i$.

$F_{X,U}(1,1,1,1,1) = x_{11}$ and $F_{Y,W}(1,1,1,1,1) = y_{11}$. Thus $x_{11}, y_{11} \geq 0$ and $x_{11} \cdot y_{11} = 0$. Without loss of generality we can assume that $x_{11} = 0$.

Suppose that $x_i \geq 0$ for every $i = 1, \dots, 10$. If $x_1 = \cdots = x_{11} = 0$, then $F_{X,U}(s) = 0$ for every $s \in S_5$. Identify $u_1$ with $u_2$ and $u_3$ with $u_4, u_5$ in $X$ and suppress the vertex of valency 2. The resulting cubic graph is bridgeless, has order smaller than $G$, and does not have a nowhere-zero 5-flow (otherwise $F_{X,U}(s) > 0$ for some $s \in S_5$), which contradicts the minimality of $G$. Hence at least one $x_i$ must be positive. We can choose the notation of edges $e_1, \dots, e_5$ so that $x_1 > 0$. Thus $F_{X,U}(s) > 0$ and $F_{Y,W}(s) = 0$ if $s \in S_5$ and $\chi_5(s)$ has first coordinate 1. Then identifying $w_1$ with $w_2$ and $w_3$ with $w_4, w_5$ in $Y$ and suppressing the vertex of valency 2 we get a bridgeless cubic graph without a nowhere-zero 5-flow and of order smaller than $G$, a contradiction.

Therefore at least one $x_i$ is negative. We can choose the notation of edges $e_1, \dots, e_5$ so that $x_1 < 0$. Consider

$$\begin{aligned}
p_1 &= (1,4,1,2,2), & \chi_5(p_1) &= e_{11} + e_1 + e_2,\\
p_2 &= (1,4,2,1,2), & \chi_5(p_2) &= e_{11} + e_1 + e_7,\\
p_3 &= (1,4,2,2,1), & \chi_5(p_3) &= e_{11} + e_1 + e_{10},\\
p_4 &= (4,1,1,2,2), & \chi_5(p_4) &= e_{11} + e_1 + e_6,\\
p_5 &= (4,1,2,1,2), & \chi_5(p_5) &= e_{11} + e_1 + e_9,\\
p_6 &= (4,1,2,2,1), & \chi_5(p_6) &= e_{11} + e_1 + e_5.
\end{aligned}$$

Since $F_{X,U}(p_i) \geq 0$ for $i = 1, \dots, 6$, we have $x_2, x_5, x_6, x_7, x_9, x_{10} \geq -x_1 - x_{11} = -x_1 > 0$. Thus $F_{X,U}(1,1,2,2,4) = x_5 + x_{10} + x_{11} > 0$.

Let $s = (s_1, \dots, s_5) \in S$. Then $F_{X,U}(s) = x_a + x_b + x_c$ where $a, b, c$ are pairwise distinct elements from $\{1, \dots, 10\}$. If $a = 1$, i. e., $s_1 + s_2 = s_3 + s_4 + s_5 = 0$, then $b, c \neq 3, 4, 8$ (otherwise at least one of the sums $s_3 + s_4$, $s_4 + s_5$, $s_3 + s_5$ equals 0, whence at least one of $s_5, s_3, s_4$ is 0), and since $x_2, x_5, x_6, x_7, x_9, x_{10} \geq -x_1 > 0$, we have $x_a + x_b + x_c > 0$. From now on let $a, b, c \neq 1$. If $x_3, x_4, x_8 \geq 0$, then $x_a + x_b + x_c$ can be 0 only if $x_3 = x_4 = x_8 = 0$ and $\{a, b, c\} = \{3, 4, 8\}$, whence $s_3 + s_4 = s_4 + s_5 = s_3 + s_5 = 0$ and $s_3 = s_4 = s_5 = 0$; thus $x_a + x_b + x_c > 0$ in this case. If $x_3 < 0$, then $x_2, x_4, x_6, x_7, x_8, x_9 \geq -x_3 > 0$ (because $F_{G,U}(\pi^2(p_i)) \geq 0$ for $i = 1, \dots, 6$), $\{3,5\}, \{3,10\} \not\subseteq \{a, b, c\}$ (otherwise either $s_3 + s_4 = s_5 + s_1 = 0$ or $s_3 + s_4 = s_5 + s_2 = 0$, whence either $s_2 = 0$ or $s_1 = 0$), and since $a, b, c \neq 1$, $x_5, x_{10} > 0$, we have $x_a + x_b + x_c > 0$ in this case. If $x_4 < 0$ (resp., $x_8 < 0$), then interchanging the indices 3 and 5 (resp., 4 and 5) of edges $e_1, \dots, e_5$ we get the previous case $x_1, x_3 < 0$. Thus $F_{X,U}(s) > 0$.

We can conclude that if $x_1 < 0$, then $F_{X,U}(s) > 0$ and $F_{Y,W}(s) = 0$ for every $s \in S$ or $s = (1,1,2,2,4)$. Consider

$$\begin{aligned}
p_7 &= (4,3,1,1,1), & \chi_5(p_7) &= e_{11} + e_5 + e_6 + e_9,\\
p_8 &= (4,1,3,1,1), & \chi_5(p_8) &= e_{11} + e_5 + e_1 + e_9,\\
p_9 &= (4,1,1,3,1), & \chi_5(p_9) &= e_{11} + e_5 + e_1 + e_6.
\end{aligned}$$

Since $p_7, p_8, p_9 \in S$, we have

$$\begin{aligned} 0 &= F_{Y,W}(p_7) - F_{Y,W}(p_8) = y_6 - y_1, \\ 0 &= F_{Y,W}(p_7) - F_{Y,W}(p_9) = y_9 - y_1, \\ 0 &= F_{Y,W}(p_8) - F_{Y,W}(p_9) = y_9 - y_6. \end{aligned} \tag{5}$$

Therefore $y_1 = y_6 = y_9$. For $i = 1, 2, 3, 4$, replacing $p_7$, $p_8$, $p_9$ with $\pi^i(p_7)$, $\pi^i(p_8)$, $\pi^i(p_9)$, respectively, in (5) we get $y_2 = y_7 = y_{10}$, $y_3 = y_8 = y_6$, $y_4 = y_9 = y_7$, $y_5 = y_{10} = y_8$, whence $y_1 = \cdots = y_{10}$. Since $0 = F_{Y,W}(1,1,1,3,4) - F_{Y,W}(1,1,2,2,4) = y_8$ and $0 = F_{Y,W}(1,1,2,2,4) = y_5 + y_{10} + y_{11}$, we have $y_1 = \cdots = y_{11} = 0$. Thus we get a smaller counterexample in a similar way as in the case $x_1 = \cdots = x_{11} = 0$. This proves the statement. $\square$

## 5. Concluding remarks

If $n = 2$ or $3$, then $\mathcal{P}_n$ contains exactly one partition. Thus, by Lemma 2, $F_{G,U}(s) = F_{G,U}(s')$ for any $s, s' \in S_2$ ($S_3$) and any simple network $(G, U)$ with two (three) outer vertices. This implies that the smallest counterexample to the 5-flow conjecture must be cyclically 4-edge-connected (see also Jaeger [J]).

Since the results of Celmins [C] are not published, we also sketch a proof of the statement that the smallest counterexample to the 5-flow conjecture has no cyclic 4-edge cut. Otherwise in a similar way as in Theorem 1 construct simple networks $(X, U)$, $U = (u_1, \ldots, u_4)$, and $(Y, W)$, $W = (w_1, \ldots, w_4)$, which satisfy $F_{X,U}(s)$, $F_{Y,W}(s) \geq 0$ and $F_{X,U}(s) \cdot F_{Y,W}(s) = 0$ for every $s \in S_4$. $\mathcal{P}_4$ contains partitions $P_{4,1} = \{\{1,2\},\{3,4\}\}$, $P_{4,2} = \{\{2,3\},\{4,1\}\}$, $P_{4,3} = \{\{1,3\},\{2,4\}\}$, and $P_{4,4} = \{\{1,2,3,4\}\}$. Vectors $\chi_4(1,1,1,2) = (0,0,0,1)$, $\chi_4(1,4,2,3) = (1,0,0,1)$, $\chi_4(1,2,3,4) = (0,1,0,1)$, $\chi_4(1,2,4,3) = (0,0,1,1)$ generate $\mathbb{Q}^4$. Thus the linear hull of vectors $\chi_4(s)$, $s \in S_4$, is $\mathbb{Q}^4$ and applying Lemma 3 for the standard base in $\mathbb{Q}^4$ we get that there exist rationals $x_1, \ldots, x_4$ and $y_1, \ldots, y_4$ such that for every $s \in S_4$, $F_{X,U}(s) = \sum_{i=1}^4 c_i x_i$ and $F_{Y,W}(s) = \sum_{i=1}^4 c_i y_i$ where $(c_1, \ldots, c_4) = \chi_4(s)$. Since $F_{X,U}(1,1,1,2) = x_4$ and $F_{Y,W}(1,1,1,2) = y_4$, we have $x_4, y_4 \geq 0$ and $x_4 \cdot y_4 = 0$. Suppose that $x_4 = 0$. Then $F_{G,U}(1,4,2,3) = x_1 \geq 0$, $F_{G,U}(1,2,3,4) = x_2 \geq 0$, $F_{G,U}(1,2,4,3) = x_3 \geq 0$. If $x_1 = \cdots = x_4 = 0$, then identifying $u_1$ ($u_3$) with $u_2$ ($u_4$) in $X$ and suppressing the vertices of valency 2, we get a smaller counterexample. Therefore, at least one $x_i$ must be positive and without loss of generality we can assume that $x_1 > 0$. Then $F_{X,U}(s) > 0$ and $F_{Y,W}(s) = 0$ if $\chi_4(s)$ has first coordinate 1. Thus identifying $w_1$ ($w_3$) with $w_2$ ($w_4$) in $Y$ and suppressing the vertices of valency 2 we get a smaller counterexample, concluding the proof.

## References


[C] U. A. Celmins, *On cubic graphs that do not have an edge-3-colouring*, Ph.D. Thesis, Dep. of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1984.

[I] R. Isaacs, *Infinite families of nontrivial trivalent graphs which are not Tait colorable*, Amer. Math. Monthly **82** (1975), 221-239.

[J] F. Jaeger, *Nowhere-zero flow problems*, Selected topics in Graph Theory 3 (L. W. Beineke and R. J. Wilson, eds.), Academic Press, New York, 1988, pp. 71-95.

[JS] F. Jaeger and T. Swart, *Conjecture* 1, Combinatorics 79 (M. Deza and I. G. Rosenberg, eds.), Annals of Discrete Mathematics, Vol 9, North-Holland, Amsterdam, 1980, p. 305.

[K1] M. Kochol, *Constructions of cyclically 6-edge-connected snarks*, Tech. Rep. TR-II-SAS-07/93-05, Institute for Informatics, Slovak Academy of Sciences, Bratislava, Slovakia, 1993.

[K2] M. Kochol, *Snarks without small cycles*, J. Combin. Theory Ser. B **67** (1996), 34-47.
[K3] M. Kochol, *A cyclically* 6*-edge-connected snark of order* 118, Discrete Math. **161** (1996), 297-300.
[K4] M. Kochol, *Cubic graphs without a Petersen minor have nowhere-zero* 5*-flows*, Acta Math. Univ. Comenian. (N.S.) **68** (1999), 249-252.
[K5] M. Kochol, *Hypothetical complexity of the nowhere-zero* 5*-flow problem*, J. Graph Theory **28** (1998), 1-11.
[K6] M. Kochol, *Superposition and constructions of graphs without nowhere-zero $k$-flows*, European J. Combin. (to appear).
[T1] W. T. Tutte, *A contribution to the theory of chromatic polynomials*, Canad. J. Math. **6** (1954), 80-91.
[T2] W. T. Tutte, *A class of Abelian groups*, Canad. J. Math. **8** (1956), 13-28.
[Z] C.-Q. Zhang, *Integral Flows and Cycle Covers of Graphs*, Dekker, New York, 1997.

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