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FORBIDDEN SUBGRAPHS FOR THE SMALLEST COUNTEREXAMPLE TO THE 5-FLOW CONJECTURE

Martin Kochol

Abstract. Using counting arguments we characterize some forbidden subgraphs for the smallest counterexample to the 5-Flow Conjecture of Tutte (every bridgeless graph has a nowhere-zero 5-flow). In particular, we show that this counterexample has girth at lest 9.

1. Introduction

A graph admits a *nowhere-zero k-flow* if its edges can be oriented and assigned numbers $\pm 1, \dots, \pm(k-1)$ so that for every vertex, the sum of the values on incoming edges equals the sum on the outcoming ones. It is well-known that a graph with a bridge (1-edge-cut) does not have a nowhere-zero k-flow for any $k \geq 2$ (see, e. g., [J], [K6], [Z]). The famous 5-*Flow Conjecture* of Tutte [T1] is that every bridgeless graph has a nowhere-zero 5-flow.

Let G be a counterexample to the 5-flow conjecture of the smallest possible order. It is well-known (see cf. Jaeger [J] and Zhang [Z]) that G must be a *snark* which is a cyclically 4-edge-connected cubic graph without an edge-3-coloring and with girth (the length of the shortest cycle) at least 5. Note that a graph is *cyclically k-edge-connected* if deleting fewer than k edges does not result in a graph having at least two cyclic components. By [K7], G must be cyclically 6-edge-connected and by Celmins [C], G has no circuit of length ≤ 6 (see also [K4]).

Suppose that H is a graph with exactly n (≥ 2) vertices of valency 2 and all other vertices of valency 3. We show that if a matrix M_H, characterized by H, has the same valency as another specified matrix M_n, then H cannot be a subgraph of G. In particular, using computer we prove that if H is a circuit of order $2 \leq n \leq 8$, then M_H and M_n have the same rank. Thus G has girth at least 9, which improves the result of Celmins [C]. Note that it has a sense to find the lower bounds for the girth of G, because we do not know whether there exists a cyclically 6-edge-connected snark with girth more than 6 (see [I], [K1], [K2], [K3], [K6]).

2. Preliminaries

The graphs considered in this paper are all finite and unoriented. Multiple edges and loops are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. By a *multi-terminal network*, briefly a *network*, we mean a pair (G, U) where G is a graph and $U = (u_1, \dots, u_n)$ is an

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ordered set of pairwise distinct vertices of G. The vertices $u_1, \ldots, u_n$ are called the *outer* vertices of (G, U) and the others are called the *inner* vertices of (G, U).

We associate with each edge of G two distinct arcs, distinct for distinct edges (see also [K6]). If one of the arcs corresponding to an edge is denoted by x, the other is denoted by x^{-1}. If the ends of an edge e are the vertices u and v, one of the arcs corresponding to e is said to be *directed from u to v* (and the other from v to u). In particular, a loop corresponds to two distinct arcs both directed from a vertex to itself. Let $D(G)$ denote the set of arcs on G. Then $|D(G)| = 2|E(G)|$. If $v \in V(G)$, then $\omega_G(v)$ denotes the set of arcs of G directed from v to $V(G) \setminus \{v\}$.

If G is a graph and A is an additive Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ such that $\partial\varphi(v) = \sum_{x \in \omega_G(v)} \varphi(x)$ $(v \in V(G))$ is called the *boundary* of φ. An A-chain φ in G is called *nowhere-zero* if $\varphi(x) \neq 0$ for every $x \in D(G)$. If (G, U) is a network, then an A-chain φ in G is called an *A-flow* in (G, U) if $\partial\varphi(v) = 0$ for every inner vertex v of (G, U).

By a (*nowhere-zero*) *A-flow* in a graph G we mean a (nowhere-zero) A-flow in the network $(G, \emptyset)$. Our concept of nowhere-zero flows in graphs coincides with the usual definition of nowhere-zero flows as presented in Jaeger [J] and Zhang [Z]. By Tutte [T1], [T2], a graph has a nowhere-zero k-flow if and only if it has a nowhere-zero A-flow for any Abelian group A of order k. Thus the study of nowhere-zero 5-flows is in certain sense equivalent with the study of nowhere-zero $\mathbb{Z}_5$-flows. We use this fact and deal only with $\mathbb{Z}_5$-flows because they are easier to handle than integral flows.

A network (G, U), $U = (u_1, \ldots, u_n)$, is called *simple* if the vertices $u_1, \ldots, u_n$ have valency 1. If φ is a nowhere-zero $\mathbb{Z}_5$-flow in (G, U), then denote by $\partial\varphi(U)$ the n-tuple $(\partial\varphi(u_1), \ldots, \partial\varphi(u_n))$. By [K5], [K6], $\sum_{i=1}^n \partial\varphi(u_i) = 0$. Furthermore, $\partial\varphi(u_i) \neq 0$ because u_i has valency 1 $(i = 1, \ldots, n)$. Thus $\partial\varphi(U)$ belongs to the set

$$S_n = \{(s_1, \ldots, s_n);\ s_1, \ldots, s_n \in \mathbb{Z}_5 - 0,\ s_1 + \cdots + s_n = 0\}.$$

Let $P = \{Q_1, \ldots, Q_r\}$ be a partition of the set $\{1, \ldots, n\}$, $n \geq 2$ (i. e., $Q_1, \ldots, Q_r$ are pairwise disjoint nonempty sets whose union is $\{1, \ldots, n\}$). If one of $Q_1, \ldots, Q_r$ is a singleton, P is called *trivial*, otherwise it is called *nontrivial*. Let $\mathcal{P}_n$ denote the set of nontrivial partitions of $\{1, \ldots, n\}$ and $p_n = |\mathcal{P}_n|$. If $s = (s_1, \ldots, s_n) \in S_n$, $P = \{Q_1, \ldots, Q_r\} \in \mathcal{P}_n$, and $\sum_{i \in Q_j} s_i = 0$ for $j = 1, \ldots, r$, then we say that P and s are *compatible*. (For example, $\{\{1, 2\}, \{3, 4, 5\}\} \in \mathcal{P}_5$ is compatible with $(1, 4, 1, 2, 2) \in S_5$.) In this paper, we consider $\mathcal{P}_n$ as an p_n-tuple $(P_{n,1}, \ldots, P_{n,p_n})$. For any $s \in S_n$, denote by $\chi_n(s)$ the integral vector $(c_1, \ldots, c_{p_n})$ so that $c_i = 1$ $(c_i = 0)$ if $P_{n,i}$ is (is not) compatible with s, $i = 1, \ldots, p_n$.

Let V_n be the linear hull of $\{\chi_n(s); s \in S_n\}$ in $\mathbb{Q}^{p_n}$.

If (G, U), $U = (u_1, \ldots, u_n)$, is a simple network and $s \in S_n$, then denote by $F_{G,U}(s)$ the number of nowhere-zero $\mathbb{Z}_5$-flows φ in (G, U) satisfying $\partial\varphi(U) = s$. In [K7] is proved the following statement.

Lemma 1. *Suppose that $\{\beta_1, \ldots, \beta_r\}$ is a base of V_n, $n \geq 2$. Then for every simple network (G, U) with n outer vertices there exist rationals $x_1, \ldots, x_r$ such that for every $s \in S_n$, $F_{G,U}(s) = \sum_{i=1}^r d_i \cdot x_i$ where $d_1, \ldots, d_r$ are the rationals satisfying $\chi_n(s) = \sum_{i=1}^r d_i \cdot \beta_i$.*

3. The main result

Let H be a graph having exactly n (≥ 2) vertices of valency 2, namely $v_1, \ldots, v_n$, and all other vertices of valency 3. Add to H vertices $u_1, \ldots, u_n$ and edges $v_1u_1, \ldots, v_nu_n$. Denote the resulting graph by H'. Consider the simple network (H', U), $U = (u_1, \ldots, u_n)$. Let $S_H = \{s \in S_n; F_{H',U}(s) > 0\}$ and V_H be the linear hull of $\{\chi_n(s); s \in S_H\}$ in $\mathbb{Q}^{p_n}$. Then $V_H \subseteq V_n$.

Lemma 2. *If $V_H = V_n$, then H cannot be a subgraph of the smallest counterexample to the 5-flow conjecture.*

Proof. Let G be a counterexample to the 5-flow conjecture of the smallest possible order. As pointed out before, G is a cyclically 4-edge-connected cubic graph. Suppose that H is a subgraph of G.

Let $e_1, \ldots, e_n$ be the edges from $E(G) \setminus E(H)$ incident with $v_1, \ldots, v_n$, respectively, and I be the graph arising from G after deleting all vertices from $V(H)$. Let $v_1', \ldots, v_n'$ be the ends of $e_1, \ldots, e_n$, respectively, in I and I' be the graph arising from I after adding vertices $w_1, \ldots, w_n$ and edges $w_1v_1', \ldots, w_nv_n'$. Consider the simple network (I', W), $W = (w_1, \ldots, w_n)$.

$F_{H',U}(s)$, $F_{I',W}(s) \geq 0$ for every $s \in S_n$. If there exists $s \in S_n$ such that $F_{H',U}(s)$, $F_{I',W}(s) > 0$, then (H', U) and (I', W) have nowhere-zero $\mathbb{Z}_5$-flows φ_1 and φ_2, respectively, such that $\partial\varphi_1(U) = \partial\varphi_2(W) = s$ and the flows φ_1 and $-\varphi_2$ can be "pieced together" into a nowhere-zero $\mathbb{Z}_5$-flow in G, a contradiction. Thus $F_{H',U}(s) \cdot F_{I',W}(s) = 0$ for every $s \in S_n$. Since $S_H = \{s \in S_n; F_{H',U}(s) > 0\}$, we have $F_{I',W}(s) = 0$ for every $s \in S_H$.

Take n-tuples $t_1, \ldots, t_r \in S_H$ so that $\chi_n(t_1), \ldots, \chi_n(t_r)$ form a base in V_H. By the assumptions, $V_H = V_n$. Hence by Lemma 1, there exist rationals $x_1, \ldots, x_r$ such that for any $s \in S_n$, $F_{I',W}(s) = \sum_{i=1}^{r} d_i \cdot x_i$ where $d_1, \ldots, d_r$ are the rationals satisfying $\chi_n(s) = \sum_{i=1}^{r} d_i \cdot \chi_n(t_i)$. Thus $x_i = F_{I',W}(t_i)$ and since $t_i \in S_H$, we have $x_i = 0$, $i = 1, \ldots, r$. Hence $F_{I',W}(s) = 0$ for every $s \in S_n$.

Choose an edge e of H and consider the cubic graph G' homeomorphic with $G-e$. G' is bridgeless (because G is cyclically 4-edge-connected) and has a nowhere-zero $\mathbb{Z}_5$-flow (because G is the smallest counterexample to the 5-flow conjecture). This implies that $F_{I',W}(s) > 0$ for some $s \in S_n$, a contradiction. Thus H cannot be an induced subgraph of G. $\square$

We give a recursive formula for computing p_n. If $\{Q_1, \ldots, Q_r\} \in \mathcal{P}_n$ and $n \in Q_j$, then $|Q_j| \in \{2, \ldots, n-2, n\}$. For $i = 2, \ldots, n-2$ (resp. $i = n$), $\mathcal{P}_n$ contains exactly $\binom{n-1}{i-1} p_{n-i}$ (resp. 1) partitions such that the element n is contained in an i-element subset. Hence for any $n \geq 2$, we have

$$p_n = 1 + \sum_{i=2}^{n-2} \binom{n-1}{i-1} p_{n-i}. \tag{1}$$

Let $\mathcal{A}$ denote the automorphism group of $\mathbb{Z}_5$. The elements of $\mathcal{A}$ are $\alpha_0 = \mathrm{id}$, $\alpha_1 = (1, 2, 4, 3)$, $\alpha_2 = (1, 4)(2, 3)$ and $\alpha_3 = (1, 3, 4, 2)$. If $s = (s_1, \ldots, s_n) \in S_n$ and $\alpha \in \mathcal{A}$, then denote $\alpha(s) = (\alpha(s_1), \ldots, \alpha(s_n)) \in S_n$. We say that s and $\alpha(s)$ are σ_n-equivalent. Clearly, $\chi_n(s) = \chi_n(\alpha(s))$ and $F_{U,G}(s) = F_{U,G}(\alpha(s))$ for any simple network (G, U) with n outer vertices (because if φ is a nowhere-zero $\mathbb{Z}_5$-flow in (G, U), then so is $\alpha(\varphi)$).

Let $a_n = |S_n|/4$ be the number of σ_n-classes and $t_1, \dots, t_{a_n}$ be pairwise non σ_n-equivalent elements of S_n. Let M_n be the $a_n \times p_n$ matrix so that the i-th row of M_n is $\chi_n(t_i)$, $i = 1, \dots, a_n$. Note that M_n does not depend on the choice of $t_1, \dots, t_{a_n}$.

Let $a_H = |S_H|/4$. Without abuse of generality we can suppose that $t_1, \dots, t_{a_H}$ are elements of S_H. Let M_H be the $a_H \times p_n$ submatrix of M_n so that the i-th row of M_H is $\chi_n(t_i)$, $i = 1, \dots, a_H$. Note that M_H does not depend on the choice of $t_1, \dots, t_{a_H}$. Clearly, $V_H = V_n$ if and only if M_H and M_n have the same rank. Thus the following statement follows immediately from Lemma 2.

Lemma 3. *If the matrices M_H and M_n have the same rank, then H cannot be a subgraph of the smallest counterexample to the 5-flow conjecture.*

Clearly $a_1 = 0$ and $a_2 = 1$. Furthermore, S_n has exactly $4|S_{n-2}|$ $(3|S_{n-1}|)$ n-tuples so that the sum of the last two elements is zero (nonzero). Thus $|S_n| = 4|S_{n-2}| + 3|S_{n-1}|$, whence $a_n = 4a_{n-2} + 3a_{n-1}$ for any $n \geq 3$. Therefore $a_5 = 51$, $a_6 = 205$, $a_7 = 819$, $a_8 = 3277$. By (1), we get that $p_5 = 11$, $p_6 = 41$, $p_7 = 162$, $p_8 = 715$.

Let C_n be the circuit (connected 2-regular graph) of order $n \geq 2$. It is easy to check that $S_{C_n} = S_n$, whence $V_{C_n} = V_n$ for any $n \leq 4$ (see, e. g., [K5]). Using computer we can check that $a_{C_5} = 45$, $a_{C_6} = 151$, $a_{C_7} = 483$, $a_{C_8} = 1513$ and that the rank of M_{C_n} equals the rank of M_n for $n = 5, \dots, 8$. More precisely, the rank of M_n is 11, 40, 147, 568 if $n = 5, 6, 7, 8$, respectively. Thus, by Lemma 3, the following statement holds true.

Theorem 1. *The smallest counterexample of the 5-flow conjecture has girth at least 9.*

In order to compute the ranks of M_{C_n} and M_n, it suffices to apply the Gauss elimination method for M_n and check whether the nonzero rows correspond with the rows of M_{C_n}. We have computed this problem on a personal computer using Maple 5 programming language. Note that the program runs 376 seconds for $n = 7$, but more that 5 days for $n = 8$ (M_7 and M_8 are matrices of orders 819×162 and 3277×715, respectively). We hope that a similar program can be applied also for $n \geq 9$ if we use better computers.

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