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COPRODUCTS OF D-POSETS AND THEIR APPLICATION TO PROBABILITY

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Abstract. D-posets introduced by F. Chovanec and F. Kôpka ten years ago provide a suitable algebraic structure to model events in probability theory. Generalizing analogous results for fields of sets and bold algebras, we describe a duality between certain coproducts of D-posets and generalized measurable spaces. An important role in the duality is played by sequential convergence. We mention some applications to the foundations of probability.

1. Introduction

As shown in NOVÁK [1958], [1962], [1965], [1968], topological methods and sequential structures are natural and useful tools for studying fundamental notions of the probability theory. Indeed (cf. FRIČ [1997], [2000b], [200x]), the σ-additivity of a finitely additive probability measure is equivalent to the sequential continuity with respect to a natural sequential convergence of events, the extension of probability measures from a field of sets to the generated σ-field, or from a bold algebra (of fuzzy sets) to the generated tribe, is from the topological and categorical viewpoint of the same nature as the Čech-Stone compactification and the Hewitt realcompactification, and the relationships between observables and random variables can be described as a categorical duality (i.e. a natural equivalence of certain categories).

In the present paper we generalize some earlier results concerning fields of sets and bold algebras (cf. FRIČ [1999], [2000a], [2000b]) to coproducts in the category of D-posets. The motivation comes from probability, in particular, from modelling events of a quantum nature (some events are not compatible). In each section we recall some fundamental notions and give appropriate references. Further information can be found, e.g., in FRIČ [1997], [2000a], RIEČAN and NEUBRUNN [1997], HERRLICH and STRECKER [1976], PORST and THOLEN [1991], DVUREČENSKIJ and PULMANNOVÁ [2000], and in the references cited therein.

Recall that if X is a set, then by a sequential convergence on X we understand a set of pairs $(< x_n >, x) \in X^N \times X$, specifying which sequence converges to which limit, and we assume the usual axioms of convergence: each sequence converges to

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at most one limit, each constant sequence converges to its value, each subsequence of a convergent sequence converges to the same limit, and if $< x_n >$ is a sequence and x is a point such that for each subsequence $< x'_n >$ of $< x_n >$ there is a subsequence $< x''_n >$ of $< x'_n >$ which converges to x, then $< x_n >$ converges to x. A map is sequentially continuous if it preserves the convergence of sequences.

2. Coproducts

In this section we recall some basic notions and extend some results concerning coproducts of fields of sets in the category of D-posets, contained in FRIČ [2000a], to bold algebras.

Recall (cf. KÔPKA and CHOVANEC [1994], CHOVANEC and KÔPKA [2000]) that a *D-poset* is a quintuple $(X, \leq, \ominus, 0, 1)$, where X is a set, $\leq$ is a partial order, 0 is the least element, 1 is the greatest element, $\ominus$ is a partial operation on X such that $a \ominus b$ is defined iff $b \leq a$, and the following axioms are assumed:

(D1) $a \ominus b = a$ for each $a \in X$;

(D2) If $c \leq b \leq a$, then $a \ominus b \leq a \ominus c$ and $(a \ominus c) \ominus (a \ominus b) = b \ominus c$.

If no confusion can arise, then the quintuple $(X, \leq, \ominus, 0, 1)$ is condensed to X.

A map h of a D-poset X into a D-poset Y which preserves the D-poset structure is said to be a *D-homomorphism*. Let $\mathcal{D}$ be the resulting concrete category.

It is known that D-posets are equivalent to the so-called effect algebras introduced by FOULIS and BENNETT [1994]. Interesting results about effect algebras and D-posets can be found in DVUREČENSKIJ and PULMANNOVÁ [2000] and in a series of papers by Z. Riečanová, e.g. RIEČANOVÁ [2000].

Each MV-algebra (cf. CIGNOLI, D'OTTAVIANO and MUNDICI [2000]), in particular each bold algebra, is a D-poset. For the reader's convenience we repeat the definition of a bold algebra and some related notions. They will play an important role in our duality and its applications. Further information is contained e.g. in RIEČAN and NEUBRUNN [1997]. Information concerning sequential convergence on MV-algebras can be found in JAKUBÍK [1995], FRIČ [1997].

Let I be the unit interval [0,1] carrying the Łukasiewicz operations $x \oplus y = \min\{1, x + y\}$, $x^c = 1 - x$, $x \odot y = \max\{0, x + y - 1\}$, the usual order and the usual convergence of sequences. Let X be a set and let I^X be the set of all functions on X ranging in I carrying the pointwise Łukasiewicz operations, order and convergence. A *bold algebra* is a subset $\mathcal{X}$ of I^X closed with respect to the inherited Łukasiewicz operations, order, and convergence. An *MV-algebra homomorphism* of a bold algebra $\mathcal{X} \subseteq I^X$ into a bold algebra $\mathcal{Y} \subseteq I^Y$ is a map of $\mathcal{X}$ into $\mathcal{Y}$ preserving the Łukasiewicz operations. If $\mathcal{X} \subseteq I^X$ is a bold algebra, then each $x \in X$ represents a sequentially continuous MV-algebra homomorphism ev_x of $\mathcal{X}$ into I defined by $ev_x(f) = f(x)$, $f \in \mathcal{X}$. If each sequentially continuous MV-algebra homomorphism of $\mathcal{X} \subseteq I^X$ into is represented by (a unique) point $x \in X$, then $\mathcal{X}$ is said to be *sober*. If $\mathcal{X}$ is closed with respect to the pointwise limits of sequences in $\mathcal{X}$, then $\mathcal{X}$ is said to be a (Łukasiewicz) *tribe*. Clearly, if $\mathcal{X}$ ranges in $\{0,1\}^X \subseteq I^X$, then the bold algebra $\mathcal{X}$ becomes a field of subsets of X (hence a boolean algebra). Each bold algebra is, in fact, a D-poset: $f \ominus g$ is defined whenever $g(x) \leq f(x)$ for each $x \in X$ and then $f \ominus g = f - g$. Similarly, each MV-algebra homomorphism (hence each boolean homomorphism) is a D-homomorphism.

Let $\{(X_t, \leqq_t, \ominus_t, 0_t, 1_t);\ t \in T\}$ be a family of D-posets. Recall that a D-poset $(X, \leqq_X, \ominus_X, 0_X, 1_X)$ together with D-homomorphisms $\{\kappa_t : X_t \to X;\ t \in T\}$, called *coprojections*, is the *coproduct* of $\{(X_t, \leqq_t, \ominus_t, 0_t, 1_t);\ t \in T\}$ if whenever $(U, \leqq_U, \ominus_U, 0_U, 1_U)$ is a D-poset and $\{\varphi_t\colon X_t \to U;\ t \in T\}$ are D-homomorphisms, then there is a unique D-morphism $\varphi : X \to U$ such that $\varphi \circ \kappa_t = \varphi_t$ for each $t \in T$. The coproduct exists and it is uniquely determined (up to an isomorphism). Having in mind applications to probability, for bold algebras considered as D-posets we describe the coproduct more explicitly.

CONSTRUCTION 1. Let $\{\mathcal{X}_t \subseteq I^{X_t};\ t \in T\}$ be a family of bold algebras. For each $t \in T$, put $X(t) = X_t \times \{t\}$ and let $\mathcal{X}(t) = \{f \in I^{X(t)};\ f(x,t) = g(x)$ for some $g \in \mathcal{X}_t\}$; then $\mathcal{X}_t$ and $\mathcal{X}(t)$ are isomorphic bold algebras and the sets $\mathcal{X}(t)$ and $\mathcal{X}(s)$ are disjoint for $t \neq s$. Let $X = \bigcup_{t \in T} X(t)$. For each $t \in T$, define $\mathcal{A}(t) \subseteq I^X$ as follows: $f \in \mathcal{A}(t)$ whenever there exists $g \in \mathcal{X}_t$, $g \neq 0_{X_t}$, $g \neq 1_{X_t}$, such that $f(x,t) = g(x)$ for all $(x,t) \in \mathcal{X}(t)$ and $f(x,t) = 0$ for $s \in T$, $s \neq t$. Let $\mathcal{X} = \{0_X, 1_X\} \cup (\bigcup_{t \in T} \mathcal{A}(t))$. Then $\mathcal{X} \subseteq I^X$ carries a natural partial order $\leq_{\mathcal{X}}$. Define a partial operation $\ominus_{\mathcal{X}}$ as follows:

(i) Put $1_X \ominus_{\mathcal{X}} 1_X = 0_X$, $1_X \ominus_{\mathcal{X}} 0_X = 1_X$, and $0_X \ominus_{\mathcal{X}} 0_X$;
(ii) For each $f \in \mathcal{A}(t), t \in T$, put $(1_X \ominus_{\mathcal{X}} f)(x,t) = 1 - f(x,t)$ for all $(x,t) \in \mathcal{X}(t)$ and put $(1_X \ominus_{\mathcal{X}} f)(x,s) = 0$ for $s \in T, s \neq t$, and put $f \ominus_{\mathcal{X}} 0_X = f$;
(iii) For $f, g \in \mathcal{A}(t), t \in T, g \leq_{\mathcal{X}} f$, put $f \ominus_{\mathcal{X}} g = f - g$.

For each $t \in T$, define the coprojection $\kappa_t : \mathcal{X}_t \longrightarrow \mathcal{X}$ as follows: $\kappa_t(0_{X_t}) = 0_X$, $\kappa_t(1_{X_t}) = 1_X$, and, for $g \in \mathcal{X}_t$, $g \neq 0_{X_t}$, $g \neq 1_{X_t}$, put $(\kappa_t(g))(x,t) = g(x)$ if $x \in X_t$ and put $(\kappa_t(g))(x,s) = 0$ whenever $s \in T$, $s \neq t$, and $x \in \bigcup_{t \in T} X_t$. Then $\mathcal{X} \subseteq I^X$ together with the coprojections $\{\kappa_t : \mathcal{X}_t \longrightarrow \mathcal{X};\ t \in T\}$ is the coproduct in $\mathcal{D}$.

Definition 2.1. Let $\{\mathcal{X}_t \subseteq I^{X_t};\ t \in T\}$ be a family of bold algebras and let $\mathcal{X} \subseteq I^X$ together with the coprojections $\{\kappa_t : \mathcal{X}_t \longrightarrow \mathcal{Y}_t;\ t \in T\}$ be their coproduct in $\mathcal{D}$. Then $\mathcal{X}$ is said to be the Δ-*sum* of $\{\mathcal{X}_t \subseteq I^{X_t};\ t \in T\}$, in symbols $\mathcal{X} = \Delta_{t \in T}\mathcal{X}_t$.

Theorem 2.2. *Let $\{\mathcal{X}_t \subseteq I^{X_t};\ t \in T\}$ and $\{\mathcal{Y}_t \subseteq I^{Y_t};\ t \in T\}$ be families of bold algebras and let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ together with the coprojections $\{\kappa_t : \mathcal{X}_t \longrightarrow \mathcal{X};\ t \in T\}$ and $\{\lambda_t : \mathcal{Y}_t \longrightarrow \mathcal{Y};\ t \in T\}$ be their coproducts in $\mathcal{D}$.*

(i) *Let $\{h_t : \mathcal{X}_t \longrightarrow \mathcal{Y}_t;\ t \in T\}$ be a family of MV-algebra homomorphisms. Then there exists a unique D-homomorphism $h : \mathcal{X} \longrightarrow \mathcal{Y}$ such that $h \circ \kappa_t = \lambda_t \circ h_t$ for each $t \in T$.*
(ii) *Let $\{p_t : \mathcal{X}_t \longrightarrow I;\ t \in T\}$ be a family of finitely additive probability measures. Then there exists a unique D-homomorphism $p : \mathcal{X} \longrightarrow I$ such that $p \circ \kappa_t = p_t$ for each $t \in T$.*

Proof. Both assertions follow from the construction of coproduct. We leave out details.

Definition2.3. Under the assumptions of Theorem 2.1,

(i) $h : \mathcal{X} \longrightarrow \mathcal{Y}$ is said to be the Δ-*sum* of $\{h_t : \mathcal{X}_t \longrightarrow \mathcal{Y}_t;\ t \in T\}$, in symbols $h = \Delta_{t \in T} h_t$;
(ii) $p : \mathcal{X} \longrightarrow I$ is said to be the Δ-*sum* of $\{p_t : \mathcal{X}_t \longrightarrow I;\ t \in T\}$, in symbols $p = \Delta_{t \in T} p_t$.

Our next step is to equip each Δ-sum of bold algebras with a suitable sequential convergence (initial with respect to the coprojections).

CONSTRUCTION 2. Let $\{\mathcal{X}_t \subseteq I^{X_t};\ t \in T\}$ be a family of bold algebras and let $\mathcal{X} \subseteq I^X$ together with the coprojections $\{\kappa_t : \mathcal{X}_t \longrightarrow \mathcal{X};\ t \in T\}$ be their coproduct in $\mathcal{D}$, i.e. their Δ-sum. Define a sequential convergence on $\mathcal{X}$ as follows: a sequence $< f_n >$ converges to f iff, for some $t \in T$, there is a sequence $< g_n >$ converging in $\mathcal{X}_t$ to $g \in \mathcal{X}_t$ such that $\kappa_t(g_n) = f_n$ and $\kappa_t(g) = f$. Since $\mathcal{X}(t)$ and $\mathcal{X}(s)$ are disjoint whenever $t \neq s$ and each κ_t is one-to-one, the resulting convergence satisfies all usual axioms of convergence and it is the finest sequential convergence on $\mathcal{X}$ making all coprojections κ_t sequentially continuous. In fact, each image $\kappa_t(\mathcal{X})$ is a bold algebra isomorphic to $\mathcal{X}(t)$.

Corollary 2.4. *Under the assumptions of Theorem 2.2,*

(i) *if each $h_t, t \in T$, is sequentially continuous, then $h : \mathcal{X} \longrightarrow \mathcal{Y}$ is sequentially continuous;*
(ii) *if each p_t is sequentially continuous, then $p : \mathcal{X} \longrightarrow I$ is sequentially continuous.*

Proof. A straightforward proof is left out.

Let T be a set. Let DB_T be the category whose objects are Δ-sums of families $\{\mathcal{X}_t \subseteq I^{X_t}; t \in T\}$ of bold algebras and whose morphisms are sequentially continuous Δ-sums $h : \mathcal{X} \longrightarrow \mathcal{Y}$ of families $\{h_t : \mathcal{X}_t \longrightarrow \mathcal{Y}_t; t \in T\}$ of sequentially continuous MV-algebra homomorphisms, where $\mathcal{X}$ and $\mathcal{Y}$ are the Δ-sums of families of bold algebras $\{\mathcal{X}_t \subseteq I^{X_t}; t \in T\}$ and $\{\mathcal{Y}_t \subseteq I^{Y_t}; t \in T\}$, respectively. Let SDB_T be the subcategory of sober objects.

3. Duality

In the classical Kolmogorovian probability theory, each random variable, as a measurable function on the original probability space $(\Omega, \mathbb{S}, p)$ ranging in the real line R, induces a boolean homomorphism of the measurable subsets $\mathbb{B}$ of R into $\mathbb{S}$. This is one side of the Stone-type duality between certain boolean homomorphisms, called observables, and random variables. The other side is based on some special properties of $\mathbb{B}$. Unlike the Stone duality (based on compactness which implies that each finitely additive probability measure is countably additive - certainly an undesirable assumption), the duality described in FRIČ [1997], [2000], [200y] (based on sequential convergence) provides a more flexible tool for the study of dualities between generalized observables and generalized random variables.

In this section we describe a duality for certain coproducts of bold algebras in $\mathcal{D}$.

Let X be a set and let $\mathbb{S}$ be a field of subsets of X. Then $(X, \mathbb{S})$ is said to be a *measurable space*. If $(X, \mathbb{S})$ and $(Y, \mathbb{T})$ are measurable spaces, then a map f of Y into X is said to be *measurable*, more exactly $(\mathbb{S}, \mathbb{T})$-*measurable*, if the preimage $f^{\leftarrow}(S) = \{y \in Y; f(y) \in S\}$ belongs to $\mathbb{T}$ whenever S belongs to $\mathbb{S}$. This classical definition is equivalent to the following one (more natural from the point of view of category theory): for each characteristic function $\chi_S : X \longrightarrow \{0, 1\}$, $S \in \mathbb{S}$, the composition $\chi_S \circ f : Y \longrightarrow \{0, 1\}$ is a characteristic function of some $T \in \mathbb{T}$ (cf. FRIČ [200y], [2000b]). The interested reader is referred to FRIČ [200y] for more categorical approach to measurable maps.

Accordingly, if $\mathcal{X} \subseteq I^X$ is a bold algebra, then $(X, \mathcal{X})$ is said to be a *measurable space* and if $(X, \mathcal{X})$, $(Y, \mathcal{Y})$ are measurable spaces and f is a map of Y into X such that for each $g \in \mathcal{X}$ the composition $g \circ f$ belongs to $\mathcal{Y}$, then f is said to be $(\mathcal{X}, \mathcal{Y})$-*measurable.*

Definition 3.1. Let T be a set. Let $\{\mathcal{X}_t \subseteq I^{X_t}; t \in T\}$ be a family of bold algebras and let $\mathcal{X} \subseteq I^X$ be their Δ-sum. Then the pair $(X, \mathcal{X})$ is said to be a *measurable space*. If $\mathcal{X}$ is sober, then $(X, \mathcal{X})$ is said to be *sober*. Let $\{\mathcal{Y}_t \subseteq I^{Y_t}; t \in T\}$ be another family of bold algebras and let $\mathcal{Y} \subseteq I^Y$ be their Δ-sums. Let $\{f_t : Y_t \longrightarrow X_t; t \in T\}$ be a family of $(\mathcal{X}_t, \mathcal{Y}_t)$-measurable maps. Define a map $f : Y \longrightarrow X$ as follows: for $(y,t) \in Y(t) = Y_t \times \{t\}$, $t \in T$, put $f(y,t) = (f_t(y), t) \in X_t$. Then f is said to be the Δ-*sum* of $\{f_t : Y_t \longrightarrow X_t; t \in T\}$, in symbols $f = \Delta_{t\in T} f_t$; each Δ-sum is said to be $(\mathcal{X}, \mathcal{Y})$-*measurable*.

Let $\mathcal{X} = \Delta_{t\in T}\mathcal{X}_t$, $\mathcal{Y} = \Delta_{t\in T}\mathcal{Y}_t$,, and let f be an $(\mathcal{X}, \mathcal{Y})$-measurable map. Recall (cf. CONSTRUCTION 1), that $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ are special maps of the disjoint unions $X = \bigcup_{t\in T} X_t \times \{t\}$, $Y = \bigcup_{t\in T} Y_t \times \{t\}$ into I. Then for each $g \in \mathcal{X}$, the composition $g \circ f$ belongs to $\mathcal{Y}$. This induces a map $f^{\triangleleft}$ of $\mathcal{X}$ into $\mathcal{Y}$. Indeed, $0_{\mathcal{X}}$ and $1_{\mathcal{X}}$ are constant maps on X. Hence $f^{\triangleleft}(0_{\mathcal{X}}) = 0_{\mathcal{Y}}$ and $f^{\triangleleft}(1_{\mathcal{X}}) = 1_{\mathcal{Y}}$. If $g \in \mathcal{X}$, $g \neq 0_{\mathcal{X}}$, $g \neq 1_{\mathcal{X}}$, then there exists $t \in T$ and $h \in \mathcal{X}$, $h \neq 0_{\mathcal{X}}$, $h \neq 1_{\mathcal{X}}$, such that $g(x,t) = h(x)$ for all $x \in X_t$ and $g(x,r) = 0$ for $r \in T$, $r \neq t$. Since each $f_t, t \in T$, is $(\mathcal{X}_t, \mathcal{Y}_t)$-measurable, necessarily $g \circ f \in \mathcal{Y}$

Lemma 3.2. *Let f be an $(\mathcal{X}, \mathcal{Y})$-measurable map. Then the induced map $f^{\triangleleft}$ is a sequentially continuous D-homomorphism.*

Proof. Since each $f_t : Y_t \longrightarrow X_t, t \in T$ induces an MV-homomorphism of $\mathcal{X}_t$ into $\mathcal{Y}_t$ which is sequentially continuous with respect to the pointwise convergence (cf. Lemma 3.1 in FRIČ [200y]), straightforward calculations show that $f^{\triangleleft}$ is a sequentially continuous D-homomorphism of $\mathcal{X}$ into $\mathcal{Y}$.

Theorem 3.3. *Let $\{\mathcal{X}_t \subseteq I^{X_t}; t \in T\}$ and $\{\mathcal{Y}_t \subseteq I^{Y_t}; t \in T\}$ be families of bold algebras and let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ together with the coprojections $\{\kappa_t : \mathcal{X}_t \longrightarrow \mathcal{X}; t \in T\}$ and $\{\lambda_t : \mathcal{Y}_t \longrightarrow \mathcal{Y}; t \in T\}$ be their Δ-sums. Let $\{h_t : \mathcal{X}_t \longrightarrow \mathcal{Y}_t; t \in T\}$ be a family of MV-algebra homomorphisms and let $h : \mathcal{X} \longrightarrow \mathcal{Y}$ be their Δ-sum. If each $\mathcal{X}_t$, $t \in T$, is sober, then there is a unique family of maps $\{f_t : Y_t \longrightarrow X_t; t \in T\}$ such that each f_t is $(\mathcal{X}_t, \mathcal{Y}_t)$-measurable and their Δ-sum $f : Y \longrightarrow X$ induces h, in symbols $h = f^{\triangleleft}$.*

Proof. Since each $\mathcal{X}_t$, $t \in T$, $t \in T$, is sober, it follows from Theorem 2.3 in FRIČ [200y] that there exists a unique $(\mathcal{X}_t, \mathcal{Y}_t)$-measurable map $f_t : Y_t \longrightarrow X_t$ such that $h_t = f_t^{\triangleleft}$. It follows from the costruction of Δ-sums that $h = f^{\triangleleft}$.

Let T be a set. Let DM_T be the category whose objects are measurable spaces $(X, \mathcal{X})$, where $\mathcal{X} \subseteq I^X$ is a Δ-sum of a family $\{\mathcal{X}_t \subseteq I^{X_t}; t \in T\}$ of bold algebras and whose morphisms are Δ-sums $f = \Delta_{t\in T} f_t$, where $\{\mathcal{X}_t \subseteq I^{X_t}; t \in T\}$ and $\{\mathcal{Y}_t \subseteq I^{Y_t}; t \in T\}$ are families of bold algebras, $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are measurable spaces, $\mathcal{X} = \Delta_{t\in t}\mathcal{X}_t$, $\mathcal{Y} = \Delta_{t\in t}\mathcal{Y}_t$, and $\{f_t : Y_t \longrightarrow X_t; t \in T\}$ are families of $(\mathcal{X}_t, \mathcal{Y}_t)$-measurable maps, $t \in T$. Let SDM_T be the subcategory of sober objects.

Theorem 3.4. *The categories SDB_T and SDM_T are dually isomorphic.*

Proof. Passing from a sober measurable space $(X, \mathcal{X})$ as an object of SDM_T to $\mathcal{X}$ as a Δ-sum and back, we get a one-to-one correspondence and, in fact a pair of contravariant functors between the two categories. It is easy to see (cf. Lemma 3.2 and Theorem 3.3) that they constitute the desired dual isomorphism.

Theorem 3.5. *The categories SDB_T and DB_T are naturally equivalent.*

Proof. The assertion follows from the construction of Δ-sum and the fact that the categories BD of bold algebras and $SOBD$ of sober bold algebras are naturally equivalent (cf. FRIČ [200y]).

Corollary 3.6. *The categories DB_T and SDM_T are dual.*

Observe that while each semisimple MV-algebra is isomorphic to a bold algebra, for each MV-algebra M there is an ultrafilter u such that M can be represented via functions into the ultrapower I_u of I (cf. DI NOLA [1991]). Using the duality between such "nonstandard" bold algebras and "nonstandard" measurable spaces based on approximation (instead of convergence) described in FRIČ [2000b], a duality for the corresponding Δ-sums can be constructed.

4. Applications

In this section we hint the motivation and some application of the coproducts in the category of D-posets to the foundations of probability.

Recent results concerning probability on MV-algebras can be found in MUNDICI and RIEČAN [200z], FRIČ [2000b].

Let $(\Omega, \mathbb{S}, p)$ be a probability space and let f be a random variable,i.e., a $(\mathbb{B}, \mathbb{S})$-measurable map of Ω into the real line R, where $\mathbb{B}$ is the set of all measuable subsets of R. Then $f^{\triangleleft}$ is a sequentially continuous boolean hommorphism of $\mathbb{B}$ into $\mathbb{S}$ and $p \circ f^{\triangleleft}$ is a probability measure on $\mathbb{B}$ (called the distribution induced by f). The classical case can be generalized as follows: we replace $\mathbb{S}$ and $\mathbb{B}$ with suitable bold algebras $\mathcal{X}$ and $\mathcal{B}$ and, as shown in FRIČ [200x], [2000b], then we can generalize $f, f^{\triangleleft}, p$ so that f becomes a $(\mathcal{B}, \mathcal{X})$-measurable map, $f^{\triangleleft}$ becomes a sequentially continuous MV-algebra homomorphism, and p and $p \circ f^{\triangleleft}$ become generalized probability measures. This model has a fuzzy nature.

As proposed in FRIČ [2000a], it is possible to generalize the classical model via coproducts in the category of D-posets so that the events have a "sharp" (or boolean) character and some events are not compatible. The motivation comes from "partial" experiments. Given $(\Omega, \mathbb{S}, p)$, we choose a family $\{(\Omega_t, \mathbb{S}_t, p_t);\ t \in T\}$, where each Ω_t belongs to $\mathbb{S}$, $p(\Omega_t) \neq 0$, each $\mathbb{S}_t$ is, e.g., the trace of $\mathbb{S}$ to Ω_t (all sets of the form $\Omega_t \cap S$, $S \in \mathbb{S}$), and each p_t is, e.g., the conditional probability $p(A) = p(A \cap \Omega_t)/p(\Omega_t$, $A \in \mathbb{S}_t$, given by Ω_t. Further, given a random variable f, we can consider a family $\{f_t = f \restriction \Omega_t;\ t \in T\}$ of "partial" random variables. Since the partial "sure" events Ω_t need not be mutually disjoint, the same event can be considered in different partial original probability spaces $(\Omega_t, \mathbb{S}_t, p_t)$ and events in different $\mathbb{S}_t$ become incompatible. Hence the model has a quantum nature. Formally, we work with the family $\{(\Omega_t, \mathbb{S}_t, p_t);\ t \in T\}$ as with a Δ-sum.

The results of the previous sections allow us to generalize the Kolmogorovian model one step further: we start with a suitable family $\{(X_t, \mathcal{X}_t, p_t);\ t \in T\}$ of "fuzzy" probability spaces (e.g. induced by "partial" experiments on a single "fuzzy" probability space $(X, \mathcal{X}, p)$) and formally work with the Δ-sum of the family.

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