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ARCHIMEDEANESS AND THE MACNEILLE COMPLETION OF PSEUDOEFFECT ALGEBRAS AND PO-GROUPS

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ABSTRACT. Pseudoeffect (PE-) algebras are partial additive structures differing from effect algebras in no longer being necessarily commutative. PE-algebras typically arise from intervals of *po*-groups; this applies in particular to all those which fulfil a certain Riesz property.

In this paper, we discuss the property of archimedeaness for PE-algebras on the one hand and for *po*-groups on the other hand. We prove that under the assumption of supremum-homogenicity, archimedeaness holds for a PE-algebra with Riesz property if and only if it holds for its representing group. The algebra is in that case commutative.

This result is established by using the technique of MacNeille completion. We give the exact condition for this completion to exist, and we clearly exhibit the role played by archimedeaness and by supremum-homogenicity.

1. INTRODUCTION

In quantum physics, the positive operators of a Hilbert space lying below identity correspond to the yes-no questions testable experimentally at some physical system; for that reason they are frequently called quantum effects. In 1994, Foulis and Bennett belonged to the first to examine the internal structure of the set of all quantum effects [FoBe]. To this end, effect algebras were introduced.

Effect algebras are partial additive structures, which are, motivated by the specific physical example, assumed to be commutative. Now, this kind of algebra was not long ago generalized to pseudoeffect algebras, or PE-algebras; the basic difference is that the axiom of commutativity is no longer assumed [DvVe1].

The similarity as well as distinctness of effect and pseudoeffect algebras is clearly seen as follows. Every interval between zero and some positive element in some partially ordered abelian group gives rise to an effect algebra; in particular, the standard effect algebra, whose ground set are the quantum effects, arises in this way. Now, the same is true for PE-algebras; the only difference is that the underlying *po*-group need not be abelian.

It is difficult to give exact algebraic conditions under which a PE-algebra is an interval of some *po*-group. A certain property of Riesz kind was proved to be sufficient

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in [DvVe2]. Actually, PE-algebras fulfilling this condition and unital *po*-groups fulfilling an analogous condition, together with their respective homomorphisms, were shown to be categorically equivalent.

The relation between such a pair of a PE-algebra of Riesz kind and its representing *po*-group was first studied in [DvVe2], where several finitary properties were considered – lattice order, certain Riesz properties, commutativity and linearity. Furthermore, in [DvVe5] some properties were taken into consideration that possibly involve infinitely many elements at a time – monotone σ-completeness, σ-completeness, completeness, countable Riesz interpolation property, orthogonal σ-completeness, and orthogonal completeness. All mentioned properties were shown to be preserved from a PE-algebra to its representing group and vice versa.

We turn in this paper to the more delicate property of archimedeaness. It is an open problem if this property is in general preserved from a PE-algebra to its representing group. But we are able to give an affirmative answer for the case that a condition which we call supremum-homogenicity holds. In this case, the PE-algebra is commutative, that is, an effect algebra.

To establish this result, it is necessary to consider the MacNeille completion of a given PE-algebra, that is, to embed a PE-algebra isomorphically into a complete one. Riečanová [Rie] has given an exact condition, called here Subset Closeness Property, or (SCP), for an effect algebra to possess a MacNeille completion; we show that her result is valid also in the non-commutative case. Furthermore, (SCP) naturally splits into two parts, one of which turns out to be the same as archimedeaness, and the second of which, when assuming the Riesz property (RDP), is shown to be the same as supremum-homogenicity. It is nicely seen how these two "halfs" of (SCP) are involved in the process of constructing a complete PE-algebra from a non-complete one.

2. PE-algebras and *po*-group intervals

Pseudoeffect algebras [DvVe1] are partial additive algebras whose axioms are rather similar to those of effect algebras [FoBe], which have become so well-known in recent years. They differ with respect to one basic property: Commutativity no longer necessarily holds. Commutative PE-algebras are the same as effect algebras.

We refer to [DvVe1] for the basic properties of PE-algebras. A representation theory was developed in [DvVe2], and more structure theory is contained in [DvVe3], [DvVe4], and [DvVe5].

Our axiom system is given as follows. A *pseudoeffect algebra*, or a PE-algebra for short, is a structure $(E;+,0,1)$, where $+$ is a partial binary operation and 0, 1 are constants, such that, for all $a,b,c \in E$, (PE1) $(a+b)+c$ is defined iff $a+(b+c)$ is defined, in which case these elements are equal, (PE2) there is exactly one $d \in E$ and exactly one $e \in E$ such that we have $a+d = e+a = 1$, (PE3) in the case that $a+b$ exists, there are $d,e \in E$ such that $a+b = d+a = b+e$, (PE4) in the case that $1+a$ or $a+1$ exists, we have $a = 0$.

Any PE-algebra E may be partially ordered by setting $a \leq b$ if for some $c \in E$ we have $a+c = b$. By (PE3), this order is two-sided.

For a PE-algebra as well as for any other partial additive algebra $(E;+,0,1)$, we call, for some $a \in E$, elements d and e such that $a+d = e+a = 1$ a *right* and a *left complement* of a, respectively. In case E is a PE-algebra, we actually may, by (PE2), associate to any $a \in E$ unambiguously the one right complement $a^\sim$ and the one left complement a^-; this means we have unary operations $^\sim$, $^-$ on E such that

$$a + a^\sim = a^- + a = 1. \tag{1}$$

Furthermore, since the law of cancellation holds [DvVe1, Lemma 1.4(v)], we may for a PE-algebra introduce two partial binary operations $/$, $\backslash$ in such a way that, for $a, b \in E$, $b \,/\, a$ is defined iff $b \setminus a$ is defined iff $a \leq b$, in which case we require

$$(b \,/\, a) + a = a + (b \setminus a) = b. \tag{2}$$

Note that in particular we have, for $a \in E$, $a^- = 1 \,/\, a$ and $a^\sim = 1 \setminus a$.

As mentioned, PE-algebras arise primarily from intervals in partially ordered groups. Namely, let a *po*-group G and a positive element $u \in G^+$ be given; in case that u is a strong unit, we call the pair (G, u) a *unital po-group*. Then $(\Gamma(G, u); +, 0, u)$ is a PE-algebra, where $\Gamma(G, u) \stackrel{\text{def}}{=} \{g \in G\colon\ 0 \leq g \leq u\}$, $+$ is the partial binary operation that is the restriction of the group addition to those pairs of elements of $\Gamma(G, u)$ whose sum lies again in $\Gamma(G, u)$ and 0 is the neutral element of G. Now, a PE-algebra $(\Gamma(G, u); +, 0, u)$, where G is some *po*-group and $u \in G^+$, is called an *interval PE-algebra.*

We are interested in the exact algebraic conditions under which a PE-algebra is an interval PE-algebra. This somewhat difficult problem gets treatable under a certain condition that may be compared to the Riesz properties known for *po*-groups. We recall from [DvVe1] the following definitions. We say that a PE-algebra fulfils

- (RDP_1) if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1 + d_2 = a_1,\ d_3 + d_4 = a_2,\ d_1 + d_3 = b_1,\ d_2 + d_4 = b_2$, and (ii) every $d_2' \leq d_2$ commutes with every $d_3' \leq d_3$;
- (RDP) if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1,\ d_3 + d_4 = a_2,\ d_1 + d_3 = b_1,\ d_2 + d_4 = b_2$;
- (RDP_0) if for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1,\ d_2 \leq b_2$ and $a = d_1 + d_2$;
- (RIP) if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in E$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

We have the following implications (cf. [DvVe1, Proposition 3.3(i)]):

$$(\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}). \tag{3}$$

Analogously, we may define properties (RDP_1), (RDP), (RDP_0), (RIP) for *po*-groups; see [DvVe1] for details. Also for them, (3) holds [DvVe1, Proposition 4.2(i)].

Now, the most important result for the sequel is the following [DvVe2, Theorem 5.7].

Theorem 2.1. *Any PE-algebra $(E; +, 0, 1)$ fulfilling (RDP_1) is an interval PE-algebra: For some unital po-group $(\mathcal{G}(E), u)$ we have $(\Gamma(\mathcal{G}(E), u); +, 0, u) \cong (E; +, 0, 1)$. The unital po-group $(\mathcal{G}(E), u)$ is determined uniquely up to isomorphism.*

For a PE-algebra E fulfilling (RDP_1), we will in the sequel identify the sets E and $\Gamma(\mathcal{G}(E), u)$, which in particular means $u = 1$, and we will refer to $(\mathcal{G}(E), 1)$ simply as the extension of E to its representing unital *po*-group.

Next, we will turn to homomorphisms involving PE-algebras, so as to clarify the terminology used in the sequel.

Definition 2.2. Let $(E;+,0,1)$ and $(F;+,0,1)$ be two PE-algebras. A mapping $\varphi\colon E \to F$ is called a *homomorphism*, if $\varphi(0)=0$, $\varphi(1)=1$ and whenever $a+b$ is defined for $a,b \in E$, also $\varphi(a)+\varphi(b)$ is defined and equals $\varphi(a+b)$.

A mapping $\varphi\colon E \to F$ is called an *isomorphic embedding* if it is an injective homomorphism with the following property: If, for any $a,b \in E$, $\varphi(a)+\varphi(b)$ is defined, already $a+b$ is defined.

Moreover, an isomorphic embedding $\varphi\colon E \to F$ is called *dense*, if for any $c \in F$ there are sets $A_l, A_u \subseteq E$ such that $c = \bigvee \varphi(A_l) = \bigwedge \varphi(A_u)$.

Now, we have the following strengthening of Theorem 2.1 [DvVe2, Theorem 7.4].

Theorem 2.3. *The map that associates with any PE-algebra E fulfilling (RDP_1) the extension $(\mathcal{G}(E),1)$ of E to its representing unital po-group, defines an equivalence between the category of PE-algebras fulfilling (RDP_1) together with their homomorphisms and the category of unital po-groups fulfilling (RDP_1) together with their homomorphisms.*

In particular, given such a pair E and $(\mathcal{G}(E),1)$, the group $(\mathcal{G}(E),1)$ has the universal property for E. Namely, let (H,v) be a unital po-group, and let $\varphi\colon E \to \Gamma(H,v)$ be a homomorphism. Then there exists an extension of φ to a homomorphism of unital po-groups $h_\varphi\colon \mathcal{G}(E) \to H$.

The relationship between a pair E and $\mathcal{G}(E)$ has been studied with respect to several order and Riesz properties in [DvVe2, Section 6] and [DvVe5]. The most basic result of this kind is contained in Theorem 2.3 which shows that the Riesz property (RDP_1) is preserved from a PE-algebra to its extension to a representing unital *po*-group.

Another result of this kind needed here is the following [DvVe5, Theorem 5.3]. A PE-algebra is said to be *monotone σ-complete* if every increasing sequence of elements has a supremum; a *po*-group is said to be *monotone σ-complete* if every increasing sequence of elements having an upper bound has a supremum.

Theorem 2.4. *Let $(E;+,0,1)$ be a PE-algebra fulfilling (RDP_1), and let $(\mathcal{G}(E),1)$ be the extension of E to its representing unital po-group. Then E is monotone σ-complete if and only if so is $\mathcal{G}(E)$.*

We conclude this section by noting that pseudo-MV algebras may be identified with the lattice-ordered PE-algebras fulfilling (RDP_1). We refer to [GeIo] for the definition and to [Dv] for the ℓ-group representation of pseudo-MV algebras.

Namely, given a PE-algebra $(E;+,0,1)$ of the mentioned kind, we may extend $+$ to the total operation $\oplus$ by defining $a \oplus b \stackrel{\mathrm{def}}{=} a + (a^\sim \wedge b)$ for $a,b \in E$; then $(E;\oplus,0,1,{}^-,{}^\sim)$ is a pseudo-MV algebra, and all pseudo-MV algebras arise in this way. From $(E;\oplus,0,1,{}^-,{}^\sim)$, we may recover the underlying PE-algebra by defining $a+b \stackrel{\mathrm{def}}{=} a \oplus b$ iff $b \leq a^\sim$; the operations ${}^-,{}^\sim$ then fulfil (1). See [DvVe2] for the details.

3. Archimedeaness and the Subset Closeness Property for PE-algebras

We now turn to a property much more difficult to handle than the ones discussed in our previous articles [DvVe2, DvVe5]: archimedeaness. This notion was defined in [DvVe2] in accordance to the case of effect algebras: It says that there are no infinitesimal elements, i.e. that there is no non-zero element with the property that any multiple of it exists (see Definition 3.8(i)).

If a PE-algebra arises from a unit interval of an archimedean *po*-group, then it is easily seen that the PE-algebra has the equally named property [DvVe2, Proposition 6.13].

But conversely, assume a PE-algebra to be archimedean and, by fulfilling (RDP_1), to be group representable. Our question is if then also the group is archimedean. Furthermore, an archimedean unital *po*-group is abelian; so the further question is natural if then the PE-algebra is actually commutative. We provide in this paper a partial solution to these problems.

Let us first outline how the situation is for unital *po*-groups. Their archimedeaness is actually equivalent to a seemingly very different property: It holds iff the group is isomorphically embeddable into a complete one. Furthermore, all complete unital *po*-groups, and thus all archimedean ones, are abelian, see e.g. [Fu, Chapter V].

It seems natural to check if similar results are also obtainable for PE-algebras. For if a PE-algebra fulfilling (RDP_1) is embeddable into a complete one, and if thereby the property of being group representable is preserved, then the group representing the completed algebra may give useful information about the group representing the original algebra – that is, information sufficient to prove its archimedeaness and so also its commutativity.

This is in fact roughly the way we will argue here. But the situation for PE-algebras is different from the one for *po*-groups; it has already been established by Riečanová [Rie] that archimedeaness alone is not sufficient for a completion of a PE-algebra to exist. The property needed is stronger and called strong D-continuity in [Rie]. We will establish in this section that the corresponding property for PE-algebras, to be called here Subset Closeness Property or (SCP), holds in a PE-algebra if and only if the algebra is completable.

Now, (SCP) contains by definition two directions of a certain equivalence. We show that one direction of this implication is just equivalent to archimedeaness; this generalizes the corresponding result of [Rie]. Furthermore, we prove that for PE-algebras fulfilling (RDP_1), the other direction is equivalent to what will be called supremum-homogenicity.

This means that it is the latter property which we must additionally assume if we want to argue in the outlined way. In the subsequent section, we in fact will see that under the assumption of supremum-homogenicity, archimedeaness may be shown to be preserved from a PE-algebra to its representing *po*-group, which then is abelian.

We will have to deal a lot with statements referring to subsets of some PE-algebra E. For notational convenience, we will use the following abbreviations. For any $a \in E$ and $A, B \subseteq E$, $a \leq B$ means $a \leq b$ for all $b \in B$, and $A \leq B$ means $a \leq b$ for all $a \in A$, $b \in B$; similarly $a \geq B$ and $A \geq B$ is defined. Furthermore, we define the (non-bold printed) operations $+$, $\setminus$, $/$ for pairs of an element and a set or for pairs of sets elementwise, provided that the involved operations are all defined. For instance, we let $A + B \stackrel{\text{def}}{=} \{a + b\colon\ a \in A,\ b \in B\}$ if for all $a \in A, b \in B$, the sum $a + b$ exists. Similarly, we define the (non-bold printed) operations $^-$, $^\sim$ for sets elementwise.

Definition 3.1. Let $(E;+,0,1)$ be a PE-algebra.

(i) For any $A \subseteq E$, let

$$U(A) \stackrel{\text{def}}{=} \{b \in E:\ b \geq A\} \text{ and}$$
$$L(A) \stackrel{\text{def}}{=} \{b \in E:\ b \leq A\}$$

the set of all upper and all lower bounds of A, respectively.

(ii) Let $A, B \subseteq E$ such that $A \leq B$. We say that A and B are *order-close* if

$$L(B) \leq U(A).$$

(iii) Let $A, B \subseteq E$ such that $A \leq B$. We say that A and B are *addition-close* if any $x \in E$ such that either $A + x$ exists and $A + x \leq B$, or $x + A$ exists and $x + A \leq B$, is equal to 0.

(iv) We say that E fulfils the *Subset Closeness Property*, or (SCP) for short, if for any $A, B \subseteq E$ such that $A \leq B$, A and B are order-close if and only if A and B are addition-close.

So subsets A and B of some PE-algebra are said to be order-close if any $a \in A$ lies below any $b \in B$, but the same time any upper bound of A lies above any lower bound of B.

Moreover, A and B are said to be addition-close, if A lies below B, but enlarging the whole A by a non-zero element from left or right, if possible, means that this condition no longer holds. Obviously, A and B, where $A \leq B$, are addition-close if and only if

$$\bigwedge (B \setminus A) = \bigwedge (B / A) = 0.$$

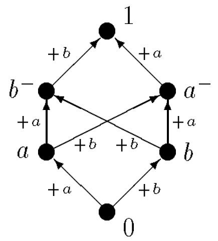
We note that (SCP) is for effect algebras the same as what was called strong D-continuity for D-posets in [Rie].

Now the condition for pairs of sets A, B, where $A \leq B$, to be order-close and to be addition-close look somewhat similar on first sight. But the following examples show that neither of the two conditions implies the other one. This means that the condition (SCP), requiring equivalence of both conditions, cannot be weakened so easily.

Example 3.2. The following example was proposed in [Rie].

Let $E = \{0, a, a^-, b, b^-, 1\}$ be a commutative PE-effect algebra, i.e. an effect algebra, in which $+$ is defined according to the following equations: $a + a + b = b + b + b = 1$.

E is then given according to the scheme on the right. Here, for instance, by the arrow which points from the element a to the element a^- and to which "$+b$" is attached, it is meant that $a \leq a^-$ and $a + b = a^-$.



Let $A, B \subseteq E$ such that $A \leq B$. We may check that if A and B are order-close, then A and B are addition-close. But the case $A = \{a, b\}$, $B = \{a^-\}$ shows that the converse does not hold.

Example 3.3. Consider now the PE-algebra corresponding to the Chang MV-algebra; cf. [DvPu, Example 5.2.39]. Let $E = \{0, 1, 2, 3, \ldots, \hat{3}, \hat{2}, \hat{1}, \hat{0}\}$. Define the addition as follows: For $m, n = 0, 1, 2, \ldots$, let $m + n$ be the usual sum of natural numbers, and let $m + \hat{n} = \hat{n} + m = \widehat{n-m}$ for $m \leq n$. Let 0 be the constant zero and $\hat{0}$ be the constant one.

Let $A, B \subseteq E$ such that $A \leq B$. If A and B are addition-close, then for some $g \in E$ we have $A \subseteq \{a\colon\ a \leq g\}$, $B \subseteq \{a\colon\ a \geq g\}$ and $g \in A, B$; it follows that these sets are order-close. But the case $A = \{0, 1, 2, \ldots\}$ and $B = \{\ldots, \hat{2}, \hat{1}, \hat{0}\}$ shows that the converse does not hold.

We note that we may construct in a similar way an example of a non-commutative PE-algebra in which addition-closeness implies order-closeness, but not vice versa. Let $(G; \leq, +)$ be a non-abelian totally ordered group, let $E = \{g\colon\ g \in G^+\} \cup \{\hat{g}\colon\ g \in G^+\}$, i.e. the disjoint union of G^+ and a copy of G^+, and define the addition as follows: For $a, b \in G^+$, let $a + b$ be the group sum, and let $a + \hat{b} = \widehat{b - a}$, $\hat{b} + a = \widehat{-a + b}$ for $a \leq b$.

Our aim is to show that (SCP) is exactly what we need to embed a PE-algebra into a complete one. First of all, this condition should hold in those algebras that are already complete.

Proposition 3.4. *A complete PE-algebra $(E; +, 0, 1)$ fulfills (SCP).*

Proof. Let $A, B \subseteq E$ such that $A \leq B$. Then we have $\bigvee A \leq \bigwedge B$, and we easily see that $\bigvee A = \bigwedge B$ iff A and B are order-close iff A and B are addition-close. □

Next, we describe the MacNeille completion of a PE-algebra. The construction works in any case and leads to an ordered partial algebra, which is complete with respect to this order. But certainly, this algebra is in general not again a PE-algebra.

Definition 3.5. Let $(E; +, 0, 1)$ be a PE-algebra.

For any $A \subseteq E$, set
$$A^c \stackrel{\text{def}}{=} L(U(A)),$$
and let $\mathbf{E} \stackrel{\text{def}}{=} \{A \subseteq E\colon\ A = A^c\}$. Define a partial order $\boldsymbol{\leq}$ on $\mathbf{E}$ by set inclusion. Furthermore, define
$$\iota\colon\ E \to \mathbf{E},\quad a \mapsto \{a\}^c.$$

Moreover, let $\boldsymbol{+}$ be the partial binary operation on $\mathbf{E}$ defined for those pairs $A, B \in \mathbf{E}$ such that $a + b$ exists for all $a \in A$ and $b \in B$, in which case we set $A \boldsymbol{+} B \stackrel{\text{def}}{=} (A + B)^c$. Define $\mathbf{0} \stackrel{\text{def}}{=} \{0\}$ and $\mathbf{1} \stackrel{\text{def}}{=} E$.

If now $(\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1})$ is again a PE-algebra, we call the pair $((\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1}), \iota)$ the *PE-MacNeille completion* of E and we say that E *possesses* a PE-MacNeille completion.

In such a case, the (bold printed) operations $\boldsymbol{^-}, \boldsymbol{^\sim}, \boldsymbol{\backslash}, \boldsymbol{/}$ will denote the additional operations for this PE-algebra as defined by (1) and (2).

Remark 3.6. Let E be a PE-algebra. Then the poset $(\mathbf{E}; \boldsymbol{\leq})$ together with the function $\iota\colon\ E \to \mathbf{E}$ is the well-known MacNeille completion of E as a poset. We have the following facts; see for instance [Fu, Chapter V.10].

(i) The operator ^{c} has the following closure properties: $\{0\}^c = \{0\}$; $A \subseteq B$ implies $A^c \subseteq B^c$; $A \subseteq A^c$; and $(A^c)^c = A^c$, where $A, B \subseteq E$.

(ii) For any $a \in E$, we have $\iota a = \{x \in E\colon\ x \leq a\}$. In particular, the function $\iota\colon\ E \to \mathbf{E}$ is injective.

(iii) $\mathbf{E}$ is a complete poset; we have $\bigwedge A_\iota = \bigcap A_\iota$ and $\bigvee A_\iota = (\bigcup A_\iota)^c$ for $A_\iota \in \mathbf{E}$, $\iota \in I$.

$\iota\colon\ E \to \mathbf{E}$ preserves the order and any existing infimum and supremum.

(iv) $\iota(E)$ is supremum- and infimum-dense in $\mathbf{E}$: For any $A \in \mathbf{E}$ there are $A_l, A_u \subseteq E$ such that $A = \bigvee \iota(A_l) = \bigwedge \iota(A_u)$.

The meaning of a PE-MacNeille completion lies in the following.

Proposition 3.7. Let the PE-algebra $(E; +, 0, 1)$ possess a PE-MacNeille completion. Then ι: $E \to \mathbf{E}$ embeds $(E; +, 0, 1)$ densely isomorphically into the complete PE-algebra $(\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1})$, preserving all existing infima and suprema.

Proof. Let us first see that $\boldsymbol{\leq} = \subseteq$ is the order of $\mathbf{E}$ also as a PE-algebra. Let $A, B \in \mathbf{E}$. If, for some $C \in \mathbf{E}$, $A \boldsymbol{+} C = B$, then, since $0 \in C$, we have $A \subseteq B$. Conversely, let $A \subseteq B$; then we have from $B^{\boldsymbol{-}} \boldsymbol{+} B = (B^{\boldsymbol{-}} + B)^c = \mathbf{1}$ that $(B^{\boldsymbol{-}} + A)^c = B^{\boldsymbol{-}} \boldsymbol{+} A$ exists, whence we may define $C = (B^{\boldsymbol{-}} \boldsymbol{+} A)^{\boldsymbol{\sim}}$. Now, since $B^{\boldsymbol{-}} \boldsymbol{+} A \boldsymbol{+} C = \mathbf{1} = B^{\boldsymbol{-}} \boldsymbol{+} B$, it follows that $A \boldsymbol{+} C = B$.

From Remark 3.6 it now follows that ι is injective, that $\iota(E)$ is supremum- and infimum-dense in $\mathbf{E}$, that $\mathbf{E}$ is complete, and that ι preserves existing infima and suprema.

Suppose now that for $a, b \in E$, $a + b$ is defined. Then we have $\iota(a + b) = \{z\colon z \leq a + b\} = L(\{z\colon\ z \geq a + b\} = L(U\{x + y\colon\ x \leq a, y \leq b\}) = L(U(\iota(a) + \iota(b))) = \iota(a) \boldsymbol{+} \iota(b)$. Suppose that $a + b$ is not defined. Then also $\iota(a) \boldsymbol{+} \iota(b)$ is undefined, since $\iota(a)$ contains a and $\iota(b)$ contains b.

We finally have that $\iota(0) = \{0\} = \mathbf{0}$ and $\iota(1) = E = \mathbf{1}$. □

What we will show is that a PE-algebra possesses a PE-MacNeille completion if and only if it fulfils (SCP). The latter condition involves by its definition two implications; we will see that each of the following two conditions is related to one of these two implications.

Definition 3.8. Let $(E; +, 0, 1)$ be a PE-algebra.

(i) We say that E is *archimedean* if for $a \in E$, whenever $n\,a$ is defined for every $n \in \mathbb{N}$, we have $a = 0$. Here, $n\,a \stackrel{\text{def}}{=} \underbrace{n + \cdots + n}_{n \text{ times}}$.

(ii) We say that E is *supremum-homogeneous* if for any $a, b_\iota \in E$, $\iota \in I$, such that either $\bigvee_\iota (a + b_\iota)$ or $\bigvee_\iota (b_\iota + a)$ exists, also $\bigvee_\iota b_\iota$ exists.

One sufficient condition for archimedeaness is exhibited by the following immediate improvement of [DvVe2, Proposition 6.11].

Proposition 3.9. *A monotone σ-complete PE-algebra $(E; +, 0, 1)$ is archimedean.*

Concerning supremum-homogenicity, we first note that this property allows to improve [DvVe1, Lemma 1.7(i), Remark 1.8] in the following way.

Proposition 3.10. *Let $(E; +, 0, 1)$ be a supremum-homogeneous PE-algebra, and let $a, b_\iota \in E$, $\iota \in I$. Then the term $\bigvee_\iota (a + b_\iota)$ exists if and only if the term $a + \bigvee_\iota b_\iota$ exists, in which case both are equal.*

Similarly, the term $\bigvee_\iota (b_\iota + a)$ exists if and only if the term $(\bigvee_\iota b_\iota) + a$ exists, in which case both are equal.

Next we give a sufficient condition also for supremum-homogenicity; we shall see that the latter property holds in PE-algebras that fulfil (RDP_1) and that are lattice-ordered. We note that these are exactly those PE-algebras we may identify with pseudo-MV algebras.

Proposition 3.11. *A lattice-ordered PE-algebra fulfilling (RDP$_1$) is supremum-homogeneous.*

Proof. Let $(E; +, 0, 1)$ be a lattice-ordered PE-algebra fulfilling (RDP$_1$), and let $\oplus$ be the addition of the corresponding pseudo-MV algebra (see the end of Section 2). Let $a, b_\iota \in E$, $\iota \in I$, such that, let's say, $\bigvee_\iota (a + b_\iota)$ exists. Choose c in a way that $\bigvee_\iota (a + b_\iota) = a + c$; we then have to show that $c = \bigvee_\iota b_\iota$.

From $a + b_\iota \leq a + c$ we have $b_\iota \leq c$ for all ι. Let $b_\iota \leq d$ for all ι; then by the monotonicity of $\oplus$ [GeIo, Proposition 1.9], it follows $a + b_\iota = a \oplus b_\iota \leq a \oplus d = a + (a^\sim \wedge d)$, so $a + c = \bigvee_\iota (a + b_\iota) \leq a + (a^\sim \wedge d)$, and we conclude $c \leq a^\sim \wedge d \leq d$. $\square$

It is not possible to drop the condition of lattice order in Proposition 3.11; (RDP$_1$) alone does not imply the supremum-homogenicity of a PE-algebra. Furthermore, given (RDP$_1$), Proposition 3.11 says that lattice order implies supremum-homogenicity; but the converse is not true: Being lattice-ordered is a strictly stronger condition than being supremum-homogeneous.

These statements are shown by the following examples. They will serve to judge about the strength of our main result of Section 4.

Example 3.12. Let G be the additive abelian group generated freely by countably many elements $g_0, g_1, \ldots$; let v: $(G; +) \to (\mathbb{R}, +)$ be the homomorphism determined by the conditions $v(g_i) = (\frac{1}{2})^i$, $i = 0, \ldots$. Define a partial order in G by setting $G^+ \stackrel{\text{def}}{=} \{x \in G\colon\ x = 0 \text{ or } v(x) > 0\}$; this means that we have for $a, b \in G$

$$a \leq b \text{ iff } a = b \text{ or } v(a) < v(b).$$

Then $\Gamma(G, g_0)$ is a commutative, archimedean PE-algebra.

$\Gamma(G, g_0)$ fulfils (RDP), so by the commutativity also (RDP$_1$). Indeed, let a_1, a_2, $b_1, b_2 \in \Gamma(G, g_0)$ satisfy $a_1 + a_2 = b_1 + b_2$; we may suppose $a_1, b_1, b_2 > 0$. We have to show that, for some $k \in G$, all elements in the scheme

$$\begin{array}{ccccc} & a_1 - k & k & \to & a_1 \\ (4) & k - a_1 + b_1 & -k + b_2 & \to & a_2 \\ & \downarrow & \downarrow & & \\ & b_1 & b_2 & & \end{array}$$

are in $\Gamma(G, g_0)$. If $v(a_1) < v(b_1)$, we put $k = 0$; if $v(a_1) > v(b_1)$, we put $k = -b_1 + a_1$. Let now $v(a_1) = v(b_1)$ and, since the case $a_1 = b_1$ is trivial, let $a_1 \neq b_1$. Then put $k = g_i$, choosing i large enough to make every element in (4) strictly positive.

Furthermore, $\Gamma(G, g_0)$ is not supremum-homogeneous. Indeed, we have on the one hand

$$\bigvee \{g_1 + a\colon\ 0 < v(a) < \tfrac{1}{2}\} = g_0,$$

since g_0 is the only element in $\Gamma(G, g_0)$ such that $v(g_0) = 1$. On the other hand, the supremum of $\{a\colon\ 0 < v(a) < \frac{1}{2}\}$ does not exist since any $b \in \Gamma(G, g_0)$ such that $v(b) = \frac{1}{2}$ is a minimal upper bound of this set, but two different elements of that kind are never comparable.

Example 3.13. Let E be the set of differentiable functions $f: [0, 1] \to [0, 1]$. Define $+$ to be the pointwise addition of two such functions whenever this leads to a result within E; let 0 and 1 the constant functions with value 0 and 1, respectively. Then $(E; +, 0, 1)$ is obviously a PE-algebra, which is commutative and archimedean.

E is obviously not lattice-ordered.

But (RDP_0) holds in E, which by [DvVe1, Proposition 3.3(iii)] implies (RDP_1). Indeed, let $f, g_1, g_2 \in E$ satisfy $f \leq g_1 + g_2$. We have $f - (f \wedge g_2) \leq f \wedge g_1$, where $\wedge$ denotes the pointwise calculated infimum and $\leq$ means the pointwise order. We must show that there lies a differentiable function d in between; for then we have $0 \leq d \leq g_1$ and $0 \leq f - d \leq g_2$ as required. Now the support of $f - (f \wedge g_2)$ is the union of closed intervals, in the interior of which this function is differentiable. Furthermore, at an edge points of any of these intervals, either $f - (f \wedge g_2)$ is also differentiable, or else $f \wedge g_1$ is strictly positive; in the latter case this applies then also to some neighbourhood. It is now not difficult to see that there is a differentiable d such that $f - (f \wedge g_2) \leq d \leq f \wedge g_1$.

We shall see that E is also supremum-homogeneous. To this end, we first show how suprema are calculated in E. Let $f_\iota \in E$, $i \in I$, such that $\bigvee_\iota f_\iota$ exists; we claim that in this case

$$(\bigvee_\iota f_\iota)(x) = \bigwedge_{\varepsilon > 0} \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota f_\iota(y) \text{ for } x \in [0,1]; \tag{5}$$

where $U_\varepsilon(x)$ is the open ε-neighbourhood of x in $[0,1]$. Indeed, since $\bigvee_\iota f_\iota$ is continuous, we have $(\bigvee_\iota f_\iota)(x) = \bigwedge_{\varepsilon>0} \bigvee_{y \in U_\varepsilon(x)} (\bigvee_\iota f_\iota)(y) \geq \bigwedge_{\varepsilon>0} \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota f_\iota(y)$ for any x. For the converse inequality, we must show that $(\bigvee_\iota f_\iota)(x) \leq \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota f_\iota(y)$ for any $\varepsilon > 0$ and $x \in [0,1]$. Suppose on the contrary that there is an $\varepsilon > 0$ and an x such that $(\bigvee_\iota f_\iota)(x) \geq \delta + \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota f_\iota(y)$ for some $\delta > 0$. It then follows $(\bigvee_\iota f_\iota)(x) \geq \delta + f_\iota(y)$ for all $y \in U_\varepsilon(x)$ and all ι, so $\bigvee_\iota f_\iota$ is not the smallest among the differentiable functions majorizing all f_ι, in contradiction to our assumption.

Now let $a, b_\iota \in E$, $\iota \in I$, be given such that $\bigvee_\iota (a + b_\iota)$ exists. Let $x \in [0,1]$. For any given $\delta > 0$, we may choose by the continuity of a some ε' such that $|a(y) - a(x)| \leq \delta$ for any $y \in U_{\varepsilon'}(x)$. Then from (5) we have $(\bigvee_\iota (a + b_\iota))(x) = \bigwedge_{0 \leq \varepsilon \leq \varepsilon'} \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota (a(y) + b_\iota(y)) = \bigwedge_{0 \leq \varepsilon \leq \varepsilon'} \bigvee_{y \in U_\varepsilon(x)} (a(y) + \bigvee_\iota b_\iota(y)) \leq \bigwedge_{0 \leq \varepsilon \leq \varepsilon'} \bigvee_{y \in U_\varepsilon(x)} (a(x) + \delta + \bigvee_\iota b_\iota(y)) = a(x) + \delta + \bigwedge_{0 \leq \varepsilon \leq \varepsilon'} \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota b_\iota(y)$. So $(\bigvee_\iota (a + b_\iota))(x) \leq a(x) + \bigwedge_{\varepsilon \geq 0} \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota b_\iota(y)$ for all x.

It follows that $\bigvee_\iota (a + b_\iota) - a = \bigvee_\iota b_\iota$. Indeed, we have $\bigvee_\iota (a + b_\iota) - a \geq b_\iota$ for any ι. Now suppose for some differentiable $c \in E$ that $c \geq b_\iota$ for all ι; it follows $c(x) = \bigwedge_{\varepsilon \geq 0} \bigvee_{y \in U_\varepsilon(x)} c(y) \geq \bigwedge_{\varepsilon \geq 0} \bigvee_{y \in U_\varepsilon(x)} \bigvee_\iota b_\iota(y) \geq (\bigvee_\iota (a + b_\iota))(x) - a(x)$ for all x, that is, $c \geq \bigvee_\iota (a + b_\iota) - a$. So we have proved that E is supremum-homogeneous.

Let us now analyse the property (SCP) in detail.

Theorem 3.14. *Let $(E; +, 0, 1)$ be a PE-algebra.*

(i) *The following statements are pairwise equivalent.*
   (α) *For any $A, B \subseteq E$ such that $A \leq B$, if A and B are order-close, then A and B are addition-close.*
   (β) *In $(\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1})$, every element possesses a left and a right complement.*
   (γ) *E is archimedean.*

(ii) *The following statements are equivalent.*
   (α) *For any $A, B \subseteq E$ such that $A \leq B$, if A and B are addition-close, then A and B are order-close.*
   (β) *In $(\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1})$, every element possesses at most one left and at most one right complement.*

   Moreover, (α) or (β) implies the following statement.
   (γ) *E is supremum-homogeneous.*

If E fulfils (RDP), (α), (β), (γ) are pairwise equivalent.

(iii) *The following statements are equivalent.*

(α) *E fulfils (SCP).*

(β) *In $(\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1})$, every element possesses exactly one left and exactly one right complement.*

Moreover, (α) or (β) implies the following statement.

(γ) *E is archimedean and supremum-homogeneous.*

If E fulfils (RDP), (α), (β), (γ) are pairwise equivalent.

Proof. (i) Assume (α). To show (β), let $A \in \mathbf{E}$; we will see that $L(A^-)$ is a left complement of A in $\mathbf{E}$; the existence of a right complement follows analogously. Clearly $L(A^-) \in \mathbf{E}$ and $L(A^-) \boldsymbol{+} A$ exists, and we have to show that $\bigvee(L(A^-) + A) = 1$, which then means that $L(A^-) \boldsymbol{+} A = E = \mathbf{1}$. Since we have $L(A^-) \leq A^-$ and $L(A^-) \leq U(L(A^-))$, it follows from (α) that $L(A^-)$ and A^- are addition-close. If now, for any $y \in E$, we have $L(A^-) + A \leq y$, it follows $y^- + L(A^-) \leq A^-$ and thus $y^- = 0$ or $y = 1$.

Assume (β). To show (γ), i.e. the archimedeaness of E, suppose that for some $x \in E$, $n\,x$ is defined for any $n \in \mathbb{N}$. Let $A = \{kx\colon\ k \in \mathbb{N}\}^c$, and let B be a left complement of A, which means that $B + A$ exists and $\bigvee(B + A) = 1$. But from $a \in A$ it follows $a + x \in A$; indeed, we have then $a \leq c$ for all $c \geq x, 2x, 3x, \ldots$, so $a \leq c \,/\, x$ for all $c \geq 2x, 3x, \ldots$, hence $a + x \leq c$ for all $c \geq x, 2x, \ldots$, which means $a + x \in A$. It follows that for all $b \in B$ and $a \in A$, $b + a + x$ exists, or $b + a \leq x^-$, and it follows $x = 0$.

Assume (γ). To show (α), let $A, B \subseteq E$ satisfy $A \leq B$ and $L(B) \leq U(A)$; we will see that then A and B are addition-close. Indeed, let $x \in E$ and suppose $A + x \leq B$. Then $A + x \subseteq L(B) \leq U(A)$, so $A + x \subseteq L(U(A)) = A$. Since no set in E is empty, we conclude that $n\,x$ exists for all $n \in \mathbb{N}$, that is, $x = 0$ by assumption. From $x + A \leq B$, we may show in an analogous manner that $x = 0$.

(ii) Assume (α). To show (β), let $A \in \mathbf{E}$ and suppose that B is a left complement of A in $\mathbf{E}$; this means $\bigvee(B + A) = 1$. So $B \leq A^-$, and if there is any $x \in E$ such that $x + B \leq A^-$, then $x + B + A$ is defined and $B + A \leq x^\sim$, and so $x = 0$. So B and A^- are addition-close, and by assumption they are order-close, hence $L(A^-) \leq U(B)$ or $L(A^-) \subseteq L(U(B)) = B^c = B$. On the other hand, $B \subseteq L(A^-)$, and so $B = L(A^-)$, which is thus proved to be the only left complement of A in $\mathbf{E}$. Similarly, we see that A has at most one right complement in $\mathbf{E}$.

Assume (β). To show (α), let $A, B \subseteq E$ satisfy $A \leq B$ and suppose that A and B are addition-close. It follows that B^- is a left complement of A, which is the same as saying that $\bigvee(B^- + A) = 1$. Indeed, $B^- + A$ exists; and if, for some $y \in E$, $B^- + A \leq y$, then $B^- + A + y^\sim$ exists, so $A + y^\sim \leq B$ and $y = 1$. B^- is by assumption the only left complement of A; now if there is a left complement, we saw above that also $l(a^-)$ is one; it follows $B^- = L(A^-)$, and since $L(A^-) = U(A)^-$, we conclude $B = U(A)$. So $L(B) \leq U(A)$, that is, A and B are order-close.

Assume again (α). To show (γ), let $a, b_\iota \in E$, $\iota \in I$, and assume that $\bigvee_\iota(a + b_\iota)$ exists; we have to show that then also $\bigvee_\iota b_\iota$ exists. Let $c = [\bigvee_\iota(a + b_\iota)] \setminus a$; we show that $c = \bigvee_\iota b_\iota$. Clearly c is an upper bound of $\{b_\iota\colon\ \iota \in I\}$. Let $A = \{b_\iota\colon\ \iota \in I\}$ and $B = \{c\}$. Then A and B are addition-close; indeed, if $x \in E$ such that $x + A \leq B$, we have $x + b_\kappa \leq c$ or $a + x + b_\kappa \leq \bigvee_\iota(a + b_\iota)$ for all κ, from which we easily conclude $x = 0$; and also $A + x \leq B$ implies $x = 0$. So A and B are by assumption order-close; this means that c is below any upper bound of $\{b_\iota\colon\ \iota \in I\}$, that is, c is the supremum of that set. We proceed similarly in the case that $\bigvee_\iota (b_\iota + a)$ exists.

Assume now (RDP) to hold, and (γ). To show (β), let $A \in \mathbf{E}$ and suppose, for some $B \in \mathbf{E}$, that $B \boldsymbol{+} A = E$. We will show $U(B)^\sim \subseteq A$. For this means $U(B)^\sim \leq U(A)$ or $U(A)^- \leq U(B)$, so $L(A^-) = U(A)^- \subseteq L(U(B)) = B$. Since anyway $B \subseteq L(A^-)$, we then have $B = L(A^-)$, that is, A has at most one left complement. In a similar manner, it is seen that A has at most one right complement.

So suppose $c \geq B$; we show $c^\sim \in A$. Given a pair $a \in A, b \in B$, we conclude by (RDP) from $a + (b+a)^\sim = (c \setminus b) + c^\sim$ that there are four elements in E according to the following scheme (here, rows and columns sum up to the elements the arrows point to [DvVe1]):

$$\begin{array}{cccc} d_1^{a,b} & d_2^{a,b} & \to & a \\ d_3^{a,b} & d_4^{a,b} & \to & (b+a)^\sim \\ \downarrow & \downarrow & & \\ c \setminus b & c^\sim. & & \end{array}$$

We conclude $1 = \bigvee_{a,b}(b+a) = \bigvee_{a,b}(b + d_1^{a,b} + d_2^{a,b}) = \bigvee_{a,b}(b + d_1^{a,b} + d_3^{a,b} + d_2^{a,b}) = \bigvee_{a,b}(c + d_2^{a,b}) = c + \bigvee_{a,b} d_2^{a,b}$ by the supremum-homogenicity and Proposition 3.10. So $c^\sim = \bigvee_{a,b} d_2^{a,b} \leq U(A)$, that is $c^\sim \in A^c = A$.

(iii) This follows easily from (i) and (ii). □

After all these long preparations, we finally arrive at our first main result.

Theorem 3.15. *Let $(E; +, 0, 1)$ be a PE-algebra fulfilling (SCP). Then E possesses a PE-MacNeille completion; that is, $(\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1})$ is a complete PE-algebra, into which ι densely isomorphically embeds E, preserving all existing infima and suprema.*

Proof. We have to show that $(\mathbf{E}; \boldsymbol{+}, \{0\}, E)$ is a PE-algebra; it is then clear by Proposition 3.7, that ι is a homomorphism of the indicated kind. So we check the axioms (E1) to (E4).

To prove associativity, i.e. (E1), we first note that

$$(X^c + Y)^c = (X + Y)^c \quad \text{for } X \subseteq E,\ Y \in \mathbf{E} \text{ such that } X + Y \text{ exists.} \tag{6}$$

Indeed, the existence of $X + Y$ implies $Y \subseteq L(X^\sim) = L(U(L(X^\sim))) = L(L(U(X))^\sim) = L((X^c)^\sim)$ and hence the existence of $X^c + Y$. We conclude, using Remark 3.6, $(X^c + Y)^c = (\bigcup_{y \in Y}(X^c + y))^c = \bigvee_{y \in Y}(X^c + y)^c = \bigvee_{y \in Y}(X + y)^c \subseteq (X + Y)^c \subseteq (X^c + Y)^c$.

Suppose now that, for $A, B, C \in \mathbf{E}$, the expression $(A \boldsymbol{+} B) \boldsymbol{+} C$ exists. Using (6), we see that $(A \boldsymbol{+} B) \boldsymbol{+} C = [(A + B) + C]^c = (A + B + C)^c$. Analogously, we may show that also $A \boldsymbol{+} (B \boldsymbol{+} C) = (A + B + C)^c$.

(E2) is the statement that every element possesses exactly one left and one right complement, and thus follows from (SCP) by Theorem 3.14(iii).

So we may introduce the unary operators $\boldsymbol{\sim}, \boldsymbol{-}$ on $\mathbf{E}$ by (1); we have $A^{\boldsymbol{\sim}} = L(A^\sim)$ and $A^{\boldsymbol{-}} = L(A^-)$. We next show that $\boldsymbol{\leq} = \subseteq$ is the order of $\mathbf{E}$ as defined for a PE-algebra, proceeding similarly as in the proof of Proposition 3.7. For $A, B, C \in \mathbf{E}$, $A \boldsymbol{+} C = B$ clearly implies $A \subseteq B$. If for $A, B \in \mathbf{E}$, on the other hand, $A \subseteq B$, then it is possible to define $C = (B^{\boldsymbol{-}} \boldsymbol{+} A)^{\boldsymbol{\sim}}$, and from $(B^{\boldsymbol{-}} \boldsymbol{+} A) \boldsymbol{+} C = E = B^{\boldsymbol{-}} \boldsymbol{+} B$, it follows by the associativity and the uniqueness of the complementation just proved that $A \boldsymbol{+} C = B$. Now one half of (E3) follows easily; the second is seen analogously.

To prove (E4), suppose for some $A \in \mathbf{E}$ that $A \boldsymbol{+} E$ or $E \boldsymbol{+} A$ is defined; it clearly follows $A = \{0\}$. □

A converse of Theorem 3.15 is the following.

Theorem 3.16. *Let $(E; +, 0, 1)$ be a PE-algebra. If E is densely isomorphically embeddable into a complete PE-algebra, then E fulfils (SCP).*

Proof. Let ι: $E \to F$ be the embedding, and let $A, B \subseteq E$ satisfy $A \leq B$. Certainly, we have then $\iota(A) \leq \iota(B)$.

Let A and B be order-close, so $L(B) \leq U(A)$. Then also $L(\iota(B)) \leq U(\iota(A))$. Indeed, let $c \in L(\iota(B))$, that is, $c \leq \iota(B)$. Since E is supremum-dense in $\mathbf{E}$, we have $c = \bigvee_\kappa \iota(c_\kappa)$ for some $c_\kappa \in E$. From $\iota(c_\kappa) \leq \iota(B)$ for any κ, we conclude $b_\kappa \leq B$ or $b_\kappa \in L(B)$. Furthermore, we see in an analogous manner that for any $d \in U(\iota(A))$ there are $d_\lambda \in E$, $\lambda \in K$, such that $d = \bigwedge \iota(d_\lambda)$ and $d_\lambda \in U(A)$. Then $c_\kappa \leq d_\lambda$ for any κ and λ, which implies $c \leq d$. So we have shown that also $\iota(A)$ and $\iota(B)$ are order-close, and they are addition-close by the completeness of F and Proposition 3.4. It follows that A and B are addition-close.

Conversely, assume that A and B are addition-close. By the supremum-density of $\iota(E)$ in F we conclude that also $\iota(A)$ and $\iota(B)$ are addition-close, so they are order-close by the completeness of F. This means $L(\iota(B)) \leq U(\iota(A))$, and so $L(B) \leq U(A)$. So A and B are order-close. □

We combine our results to a single statement.

Theorem 3.17. *Let $(E; +, 0, 1)$ be a PE-algebra. Then the following statements are pairwise equivalent.*

(α) *E is densely isomorphically embeddable into a complete PE-algebra.*
(β) *E possesses a PE-MacNeille completion.*
(γ) *E fulfils (SCP).*

Proof. This is clear from Proposition 3.7 and Theorems 3.15 and 3.16. □

Our next step will be to see that the Riesz property (RDP_1) is preserved from a PE-algebra E in its MacNeille completion $\mathbf{E}$, if the latter exists.

Theorem 3.18. *Let $(E; +, 0, 1)$ be a PE-algebra fulfilling (RDP_1). Then E is supremum-homogeneous and archimedean if and only if E possesses a PE-MacNeille completion. In this case, also $\mathbf{E}$ fulfils (RDP_1).*

Proof. The first part follows from Theorems 3.14(iii) and 3.17.

It remains to be proved that (RDP_1) holds in $\mathbf{E}$. We will actually show (RDP_0); since $\mathbf{E}$ is complete, (RDP_1) then follows by [DvVe1, Proposition 3.3]. For practical reasons, let us set for this proof $\bar{x} \overset{\mathrm{def}}{=} \iota(x)$ for $x \in E$.

So let $A, B, C \in \mathbf{E}$ be given such that $A \boldsymbol{\leq} B \boldsymbol{+} C$. Note first that

$$B \boldsymbol{+} C = \bigvee_{\substack{b \in B,\\ c \in C}} (\bar{b} \boldsymbol{+} \bar{c}) = \bigwedge_{\substack{d \in E,\\ d \geq B+C}} \bar{d},$$

as may be verified by direct calculation.

Now, for any $a \in A$, $b \in B$, and $c \in C$, we have in view of [DvVe1, Remark 1.8]

$$\bar{a} \boldsymbol{\leq} B \boldsymbol{+} C = \bigwedge_{d \geq B+C} [\bar{b} \boldsymbol{+} \bar{c} \boldsymbol{+} (\bar{d} \boldsymbol{\setminus} (\bar{b} \boldsymbol{+} \bar{c}))] = \bar{b} \boldsymbol{+} \bar{c} \boldsymbol{+} \bigwedge_{d \geq B+C} (\bar{d} \boldsymbol{\setminus} (\bar{b} \boldsymbol{+} \bar{c})). \tag{7}$$

Since (RDP_0) holds in E, there are, for any $d \geq B + C$, elements $a_B^{bcd} \leq b$, $a_C^{bcd} \leq c$, and $a_D^{bcd} \leq d \setminus (b+c)$ such that $a = a_B^{bcd} + a_C^{bcd} + a_D^{bcd}$. From $\bar{a}_D^{bcd} \leq \bar{d} \setminus (\bar{b}+\bar{c})$ and (7), we further conclude

$$\begin{aligned} \bigwedge_{\substack{b\in B,\, c\in C,\\ d\geq B+C}} \bar{a}_D^{bcd} \;&\boldsymbol{\leq} \bigwedge_{\substack{b\in B,\, c\in C,\\ d\geq B+C}} (\bar{d} \boldsymbol{\setminus} (\bar{b} \boldsymbol{+} \bar{c})) \\ &= \bigwedge_{\substack{b\in B,\\ c\in C}} [(B \boldsymbol{+} C) \boldsymbol{\setminus} (\bar{b} \boldsymbol{+} \bar{c})] \\ &= (B \boldsymbol{+} C) \boldsymbol{\setminus} \bigvee_{\substack{b\in B,\\ c\in C}} (\bar{b} \boldsymbol{+} \bar{c}) = 0. \end{aligned}$$

Since we have $\bar{a} = \bar{a}_B^{bcd} \boldsymbol{+} \bar{a}_C^{bcd} \boldsymbol{+} \bar{a}_D^{bcd}$, it follows that

$$\bar{a} = \bigvee_{\substack{b\in B,\, c\in C,\\ d\geq B+C}} (\bar{a}_B^{bcd} \boldsymbol{+} \bar{a}_C^{bcd}) \;\boldsymbol{+} \bigwedge_{\substack{b\in B,\, c\in C,\\ d\geq B+C}} \bar{a}_D^{bcd} = \bigvee_{\substack{b\in B,\, c\in C,\\ d\geq B+C}} (\bar{a}_B^{bcd} \boldsymbol{+} \bar{a}_C^{bcd}),$$

and so we get $A = \bigvee_{a\in A} \bar{a} = \bigvee_{\substack{a\in A,\, b\in B,\\ c\in C,\, d\geq B+C}} (\bar{a}_B^{bcd} \boldsymbol{+} \bar{a}_C^{bcd})$.

Set $A_B = \bigvee_{\substack{a\in A,\, b\in B,\\ c\in C,\, d\geq B+C}} \bar{a}_B^{bcd}$, and choose A_C in a way that $A = A_B \boldsymbol{+} A_C$. Then clearly $A_B \boldsymbol{\leq} B$. Moreover,

$$\begin{aligned} A_B \boldsymbol{+} A_C = A \;&= \bigvee_{\substack{a\in A,\, b\in B,\\ c\in C,\, d\geq B+C}} (\bar{a}_B^{bcd} \boldsymbol{+} \bar{a}_C^{bcd}) \\ &\leq \bigvee_{\substack{a\in A,\, b\in B,\\ c\in C,\, d\geq B+C}} \bar{a}_B^{bcd} \;\boldsymbol{+} \bigvee_{\substack{a\in A,\, b\in B,\\ c\in C,\, d\geq B+C}} \bar{a}_C^{bcd} \\ &= A_B \;\boldsymbol{+} \bigvee_{\substack{a\in A,\, b\in B,\\ c\in C,\, d\geq B+C}} \bar{a}_C^{bcd} \\ &\leq A_B \;\boldsymbol{+}\; C \end{aligned}$$

and thus $A_C \boldsymbol{\leq} C$. □

Remark 3.19. The meaning of this result lies in the following. Given a PE-algebra E fulfilling (RDP_1), we know by Theorem 2.1 that E is group representable, namely, $(E; +, 0, 1) = (\Gamma(\mathcal{G}(E), 1); +, 0, 1)$. Now, by what was just proved, we have that $\mathbf{E}$ is in such a case still group representable, that is, $(\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1}) = (\Gamma(\mathcal{G}(\mathbf{E}), \mathbf{1}); \boldsymbol{+}, \mathbf{0}, \mathbf{1})$.

Following our general concept, we will see in the subsequent chapter that $\iota\colon E \to \mathbf{E}$ extends to an isomorphic embedding of $(\mathcal{G}(E), 1)$ into $(\mathcal{G}(\mathbf{E}), \mathbf{1})$, which will allow us to derive properties of $\mathcal{G}(E)$ and consequently of E, like archimedeaness and commutativity, from those of $\mathcal{G}(\mathbf{E})$.

We finally formulate the results of this chapter for pseudo-MV algebras. Archimedeaness and supremum-homogenicity is assumed to hold for a pseudo-MV algebra if it holds for the corresponding PE-algebra. So for them, Propositions 3.9 and 3.11 read as follows.

Proposition 3.20. *Let E be a pseudo-MV algebra.*

(i) *If E is monotone σ-complete, E is archimedean.*
(ii) *E is supremum-homogeneous.*

Now we see that for pseudo-MV algebras, completability is the same as archimedeaness. This is a generalization of a result of [Höh].

Theorem 3.21. *Let E be a pseudo-MV algebra. Then the following statements are pairwise equivalent.*

(α) *E is densely isomorphically embeddable into a complete pseudo-MV algebra.*
(β) *$(E; +, 0, 1)$, the corresponding PE-algebra, possesses a PE-MacNeille completion.*
(γ) *E is archimedean.*
(δ) *E fulfils (SCP).*

In this case, let $((\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1}), \iota)$ be the PE-MacNeille completion of the corresponding PE-algebra. Then $\mathbf{E}$, being lattice-ordered and fulfilling (RDP$_1$), may be identified with a complete pseudo-MV algebra; and $\iota\colon\ E \to \mathbf{E}$ is the isomorphic embedding of pseudo-MV algebras in question.

Proof. By Proposition 3.20(ii), E is supremum-homogeneous, so (γ) and (δ) are equivalent by Theorem 3.14(iii).

Let $(E; +, 0, 1)$ be the corresponding PE-algebra. By Theorem 3.15, it follows from (δ) that E possesses the PE-MacNeille completion $((\mathbf{E}; \boldsymbol{+}, \mathbf{0}, \mathbf{1}), \iota)$. So ($\beta$) holds.

In this case, $\mathbf{E}$ is lattice-ordered and fulfils by Theorem 3.18 (RDP$_1$); and moreover, ι preserves existing infima and suprema. It follows that ι is an isomorphic embedding also of pseudo-MV algebras.

From (β), (α) follows by Proposition 3.7 and Theorem 3.18.

Let (α) hold. Since any isomorphic embedding of pseudo-MV algebras is also an isomorphic embedding of PE-algebras, (δ) follows by Theorem 3.16. □

4. Archimedeaness for *po*-groups representing PE-algebras

After the detailed discussion of the MacNeille completion in the previous section, we are now ready to relate archimedeaness of PE-algebras to archimedeaness of their representing *po*-groups.

We proceed according to what was outlined at the beginning of Section 3. An archimedean PE-algebra fulfilling (RDP$_1$) which, in addition, is supremum-homogeneous is completable, and the completed algebra is still representable by some *po*-group; the latter is archimedean and abelian, hence the same applies to the group representing the original algebra.

We shall use the following definition of archimedeaness for *po*-groups.

Definition 4.1. A *po*-group $(G; +, \leq)$ is said to be *archimedean* if, for every $a, b \in G$, $n\,a \leq b$ for every $n \geq 1$ implies $a \leq 0$.

We have the following generalization of [Goo, Theorem 16.10] to the non-commutative case.

Proposition 4.2. *Let $(G; +, \leq)$ be a po-group that fulfils (RIP) and that is monotone σ-complete. Then G is archimedean.*

Proof. Let $a, b \in G$ be such that $n\,a \leq b$ for any $n \in \mathbb{N}$. From [DvVe5, Section 4] we know that from (RIP) and the monotone σ-completeness, it follows (σ-RIP). So there is a sequence $c_1, c_2, \ldots \in G$ such that

$$c_1 = b, \quad n\,a \leq c_2 \leq c_1 \,/\, a, \quad n\,a \leq c_3 \leq c_2 \,/\, a, \quad \ldots \quad \text{for any } n.$$

We then have $a \leq \cdots \leq c_2 \leq c_1$, and so by the monotone σ-completeness the infimum $c = \bigwedge_i c_i$ exists. From $c + a \leq c_{i+1} + a \leq c_i$ for all i it follows $c + a \leq c$, and so $a \leq 0$. □

We note the following immediate consequence of Proposition 4.2.

Proposition 4.3. *Let $(E;+,0,1)$ be a monotone σ-complete PE-algebra fulfilling (RDP$_1$). Then E is commutative, and its representing unital po-group is abelian.*

Proof. Let $(\mathcal{G}(E),1)$ be the extension of E to its representing unital *po*-group. From Theorem 2.4 we know that $\mathcal{G}(E)$ is also monotone σ-complete and thus by Proposition 4.2 archimedean. Furthermore, $\mathcal{G}(E)$ is directed. It follows (see e.g. [Fu, Chapter V, Corollary 20]) that $\mathcal{G}(E)$ is abelian, hence E is commutative. □

We arrive at the main theorem of this section.

Theorem 4.4. *Let $(E;+,0,1)$ be a supremum-homogeneous PE-algebra fulfilling (RDP$_1$), and let $(\mathcal{G}(E),1)$ be the extension of E to its representing unital po-group. Then E is archimedean if and only if so is $\mathcal{G}(E)$. In this case, E is commutative and $\mathcal{G}(E)$ is abelian.*

Proof. If $\mathcal{G}(E)$ is archimedean, then so is E by [DvVe2, Proposition 6.13].

Conversely, assume that E is archimedean. By Theorem 3.18, there is an isomorphical embedding $\iota\colon\ E \to \mathbf{E}$ of E into a complete PE-algebra $\mathbf{E}$ fulfilling (RDP$_1$). In particular, $\mathbf{E}$ is monotone σ-complete, and by Theorem 2.4, so is its representing group $\mathcal{G}(\mathbf{E})$. By Proposition 4.2, $\mathcal{G}(\mathbf{E})$ is archimedean.

Since $\mathcal{G}(\mathbf{E})$ is directed and archimedean, it is abelian. It follows that $\mathbf{E}$ and E are commutative. By [DvVe2, Proposition 6.7], $\mathcal{G}(E)$ is abelian.

Due to the universal property of $\mathcal{G}(E)$, holding by Theorem 2.3, there is a homomorphism $h_\iota\colon\ \mathcal{G}(E) \to \mathcal{G}(\mathbf{E})$ of unital *po*-groups such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\iota} & \mathbf{E} \\ {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\ \mathcal{G}(E) & \xrightarrow{h_\iota} & \mathcal{G}(\mathbf{E}). \end{array}$$

We shall show that h_ι is an isomorphic embedding, i.e. that h_ι is injective.

An element of $\mathcal{G}(E)$ may be assumed to be of the form $a-b$, where $a = a_1 + \cdots + a_m$ and $b = b_1 + \cdots + b_n$ for some $a_1, \ldots, b_n \in E$, $m,n \geq 1$. We have to show that $h_\iota(a-b) = 0$ implies $a - b = 0$, or in other words, that from

$$\iota(a_1) + \cdots + \iota(a_m) = \iota(b_1) + \cdots + \iota(b_n), \tag{8}$$

it follows $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. So assume (8) to hold. Since (RDP$_1$) holds in $\mathcal{G}(\mathbf{E})$, we know by [DvVe2, Lemma 5.3] that there are $\bar{d}_{11}, \ldots, \bar{d}_{mn} \in \mathbf{E}$ such that the following scheme holds:

$$\begin{array}{ccccc} \bar{d}_{11} & \ldots & \bar{d}_{1n} & \rightarrow & \iota(a_1) \\ \vdots & & \vdots & & \vdots \\ \bar{d}_{m1} & \ldots & \bar{d}_{mn} & \rightarrow & \iota(a_m) \\ \downarrow & & \downarrow & & \\ \iota(b_1) & \ldots & \iota(b_n) & & \end{array} \tag{9}$$

Since E is supremum-dense in $\mathbf{E}$, we have, for all pairs i,j, $\bar{d}_{ij} = \bigvee_\kappa \iota(d_{ij}^\kappa)$ for some $d_{ij}^\kappa \in E$, $\kappa \in K$. It follows

$$\bigvee_{\substack{\kappa_{i1},\ldots,\\ \kappa_{in} \in K}} (\iota(d_{i1}^{\kappa_{i1}}) + \cdots + \iota(d_{in}^{\kappa_{in}})) = \iota(a_i), \quad \text{for } i = 1, \ldots, m,$$

and consequently, a similar equation holds in E, namely

$$\bigvee_{\substack{\kappa_{i1},\dots,\\ \kappa_{in}\in K}} (d_{i1}^{\kappa_{i1}} + \cdots + d_{in}^{\kappa_{in}}) = a_i \quad \text{for } i = 1, \dots, m.$$

This means that we have in $\mathcal{G}(E)$

$$a_1 + \cdots + a_m = \bigvee_{\substack{\kappa_{11},\dots,\\ \kappa_{mn}\in K}} (d_{11}^{\kappa_{11}} + \cdots + d_{mn}^{\kappa_{mn}}).$$

In a similar way, also the sum $b_1 + \cdots + b_n$ may be expressed; only the order of the summands over which the supremum is taken will be different. But $\mathcal{G}(E)$ is abelian; so we have $a_1 + \cdots + a_m = b_1 + \cdots + b_n$.

From the injectivity of h_ι, we further conclude that, for any $g \in \mathcal{G}(E)$, $h_\iota(g) \geq 0$ implies $g \geq 0$. Indeed, $h_\iota(\mathcal{G}(E)^+)$ is infimum-dense in $\mathcal{G}(\mathbf{E})^+$, whence for some $g_\lambda \in \mathcal{G}(E)^+$, $\lambda \in L$, we have $h_\iota(g) = \bigwedge_\lambda h_\iota(g_\lambda) = h_\iota(\bigwedge_\lambda g_\lambda)$ and so $g = \bigwedge_\lambda g_\lambda \geq 0$.

Now it easily follows that $\mathcal{G}(E)$ is archimedean and thus abelian. □

So we have proved that an archimedean PE-algebra fulfilling (RDP_1) has an archimedean representing group – under the assumption of supremum-homogenicity. One may wonder about the strength of this statement; in particular, the question seems natural if the latter condition cannot be weakened. But, as we saw in the last section, it cannot be weakened so easily. For it is implied by (SCP), which, by Theorem 3.17, is in turn equivalent to completability, the central property needed for our proof. Furthermore, Example 3.12 shows that supremum-homogenicity is not implied by the further conditions of Theorem 4.4, not even by (RDP_1) and archimedeaness together.

We may then, on the other hand, ask if the requirement of supremum-homogenicity does not restrict the class of PE-algebras to which Theorem 4.4 applies only to those that correspond to pseudo-MV algebras, that is, to lattice-ordered ones. But Example 3.13 shows that also this is not the case.

We conclude that, our method of proof, the MacNeille completion, given, we cannot go further than we did; any stronger statement about archimedean PE-algebras fulfilling (RDP_1) would require completely new ideas. On the other hand, our results do go beyond those which have already been established earlier, in particular beyond those for pseudo-MV algebras.

So let us finally see how the results of this section "properly" specialize to the latter kind of algebras. From Proposition 4.3 we have the following.

Proposition 4.5. *A monotone σ-complete pseudo-MV algebra is commutative, i.e. an MV-algebra.*

Our main result, Theorem 4.4, about the preservation of archimedeaness in a representing *po*-group, translates as follows to pseudo-MV algebras; this is due to [Dv].

Theorem 4.6. *A pseudo-MV algebra E is archimedean if and only if the extension $(\mathcal{G}(E), 1)$ of E to its representing unital ℓ-group is archimedean. In this case, E is commutative, i.e. an MV-algebra; and $\mathcal{G}(E)$ is abelian.*

Proof. This follows from Proposition 3.20(ii) and Theorem 4.4. □

References

[Dv] A. Dvurečenskij, *Pseudo-MV algebras are intervals in ℓ-groups*, J. Austr. Math. Soc., to appear.

[DvPu] A. Dvurečenskij and S. Pulmannová, *New Trends in Quantum Structures*, Kluwer Academic Publ., Dordrecht, Ister Science, Bratislava, 2000.

[DvVe1] A. Dvurečenskij and T. Vetterlein, *Pseudoeffect algebras. I. Basic properties*, Inter. J. Theor. Phys. *40* (2001), 685–701.

[DvVe2] A. Dvurečenskij and T. Vetterlein, *Pseudeffect algebras. II. Group representations*, Inter. J. Theor. Phys. *40* (2001), 703–726.

[DvVe3] A. Dvurečenskij and T. Vetterlein, *Congruences and states on pseudoeffect algebras*, Found. Phys. Letters *14* (2001), 425–446.

[DvVe4] A. Dvurečenskij and T. Vetterlein, *On pseudo-effect algebras which can be covered by pseudo-MV-algebras*, Found. Phys. Letters, submitted.

[DvVe5] A. Dvurečenskij and T. Vetterlein, *Infinitary lattice and Riesz properties for pseudoeffect algebras and po-groups*, submitted.

[FoBe] D. J. Foulis and M. K. Bennett, *Effect algebras and unsharp quantum logics*, Found. Phys. *24* (1994), 1325–1346.

[Fu] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963.

[GeIo] G. Georgescu and A. Iorgulescu, *Pseudo-MV algebras*, Multi. Val. Logic *6* (2001), 95–135.

[Goo] K. R. Goodearl, *Partially Ordered Abelian Groups with Interpolation*, Mathematical Surveys and Monographs 20, American Mathematical Society, Providence, 1986.

[Höh] U. Höhle, *Commutative, residuated l-monoids*, in: U. Höhle, E. P. Klement (eds.), *Non-Classical Logic and Their Representation to Fuzzy Subsets*, Theory Decis. Libr., Ser. B, 32, Kluwer Academic Publ., Dordrecht, 1995, pp. 53–106.

[Rie] Z. Riečanová, *MacNeille completions of D-posets and effect algebras*, Inter. J. Theor. Phys. *39* (2000), 859–869.

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