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Compatibility and decompositions of effects

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Compatibility and decompositions of effects *

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Abstract. Compatibility relations in effect algebras and their connections with refinements of the orthogonal partitions of unity are studied. Properties of blocks as maximal sets of compatible elements are discussed. Some special kinds of effect algebras are characterized using properties of compatibility. Using refinements, an additional structure on the effect test spaces is introduced and used to a characterization of different types of effect algebras.

1 Introduction

Effect algebras [14], alternatively D-posets [24] have been introduced as an algebraic generalization of the set of Hilbert space effects, i.e., self-adjoint operators between 0 and I on a Hilbert space. These effects play an important role in the theory of unsharp quantum measurements and positive operator valued (POV) measures [3].

Besides the Hilbert space effects, effect algebras include the most important structures used in quantum and classical logic: orthomodular lattices, orthomodular posets, Boolean algebras, and also MV-algebras, an algebraic base of multivalued logic [5]. For basic properties of effect algebras and D-posets see [11].

In the present paper we study compatibility relations on effect algebras and their connections with refinements of the orthogonal partitions of unity. We use properties of compatibility to a characterization of some types of effect algebras and their blocks.

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Using refinements, we introduce additional structures to effect test spaces and characterize test spaces of different types of effect algebras.

2 Basic definitions and properties of effect algebras

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations $0, 1$ satisfying the following conditions.

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.

(E4) If $a \oplus 1$ is defined then $a = 0$.

Effect algebras in this setting have been introduced in [14]. Essentially equivalent structures, called D-posets, were introduced in [24]. Another equivalent structures called weak orthoalgebras were introduced in [18], and so called quasilogics were introduced in [2].

For brevity we denote the effect algebra $(E; \oplus, 0, 1)$ by E. We write $a \leq b$ if there is a $c \in E$ with $a \oplus c = b$. It is easy to check that each effect algebra is cancellative, i.e., $a \oplus b = a \oplus c$ implies $b = c$, and positive, i.e., $a \oplus b = 0$ implies $a = b = 0$. We can derive from this that $\leq$ is a partial order with 0 as the least and 1 as the greatest element. Moreover, it is possible to introduce a new partial operation $\ominus$ such that $b \ominus a$ is defined iff $a \leq b$ and then $a \oplus (b \ominus a) = b$. We can also prove that $a \oplus b$ is defined iff $a \leq b'$, equivalently, iff $b \leq a'$. For convenience, we will write $a \perp b$ if $a \oplus b$ is defined, and say that a and b are *orthogonal*.

Let E_0 be a subset of E such that $a \in E_0$ implies $a' \in E_0$ and $a, b \in E_0$, $a \perp b$ implies $a \oplus b \in E_0$. Let $a, b \in E_0$ such that $a \leq b$. Then $b' \in E_0$ and $a \perp b'$ implies that $b \ominus a = (b' \oplus a)' \in E_0$, so that E_0 is closed under $\ominus$. Then E_0 is called the *sub-effect algebra* of E. Another possibility to construct a substructure of E is to consider an element $a \in E$ and the interval $[0, a]$ of E, and to restrict the $\oplus$ operation letting a act as the unit element. We denote such effect algebra by $[0, a]_E$. Observe that for x, y in $[0, a]_E$, $x \perp_a y$ iff $x \oplus y$ exists in E and $x \oplus y \leq a$.

An effect algebra satisfying $a \perp a \;\Rightarrow\; a = 0$ is called an *orthoalgebra*. An effect algebra E is an *orthomodular poset* (OMP, for short) if for $a, b, c \in E$, $a \perp b$, $b \perp c$, $c \perp a$ implies that $a \oplus b \oplus c$ exists [14]. An orthoalgebra is an *orthomodular lattice* (OML, for short) if it is lattice ordered. An effect algebra E is said to have the *Riesz decomposition property* (RDP) [32] if for any $a_1, a_2, b_1, b_2 \in E$ with $a_1 \oplus a_2 = b_1 \oplus b_2$ there are elements $(w_{ij})_{i,j=1}^{2}$ such that $w_{11} \oplus w_{12} \oplus w_{21} \oplus w_{22}$ is defined and $a_i = \oplus_{j=1}^{2} w_{ij}$, $b_j = \oplus_{i=1}^{2} w_{ij}$. Alternatively, E has the Riesz decomposition property if $a \leq b \oplus c$ implies $a = b_1 \oplus c_1$, $b_1 \leq b$, $c_1 \leq c$. If E is lattice ordered and has the RDP, then E can be organized into an MV-algebra (cf. e.g., [11]). An interesting weakening of the RDP was recently introduced in [22]: an effect algebra E is called *homogeneous* if $a \leq b \oplus c$, and $a \leq (b \oplus c)'$ implies $a = b_1 \oplus c_1$, $b_1 \leq b$, $c_1 \leq c$. Finally, an MV-algebra which is an orthoalgebra is a Boolean algebra.

3 Compatibility in effect algebras

Let E be an effect algebra. Let $C = (c_1, \ldots, c_n)$ be a finite sequence of elements of E. We say that C is *jointly orthogonal* (or simply *orthogonal*) if the sum $c_1 \oplus \cdots \oplus c_n$ is defined. We then write $\oplus C = c_1 \oplus \cdots \oplus c_n$. For $n = 0$ we define $\oplus C = 0$.

Let $\sigma := (P_1, \ldots, P_k)$ be an ordered partition of the set $\{1, \ldots, n\}$. We put $||\sigma|| := k$. Then $\sigma(C) = (\oplus_{i \in P_1} c_i, \ldots, \oplus_{i \in P_k} c_i)$ is an orthogonal sequence and $\oplus \sigma(C) = \oplus_{l=1}^{k} \oplus_{i \in P_l} c_i = \oplus C$. We will say that C is a *refinement* of $\sigma(C)$. To every ordered partition σ of $\{1, \ldots, n\}$ we can define a function $f_\sigma : \{1, \ldots, n\} \to \{1, \ldots, k\}$ by $f_\sigma(i) = j$ iff $i \in P_j$. Conversely, every such surjection f defines a partition of $\{1, \ldots, n\}$. We will call f_σ the *refinement mapping*.

Observe that if $C = (c_1, \ldots, c_n)$ is an orthogonal sequence, then there may exist different partitions of $\{1, \ldots, n\}$ with different refinement mappings leading to the same orthogonal sequence $D = \{d_1, \ldots, d_k\}$ of which C is a refinement. This may happen only if there is $c \in C$ such that $c \perp c$.

Let E be an effect algebra. A finite family M of elements of E is said to be *jointly compatible*, (or simply *compatible*), if there is an orthogonal sequence $C = (c_1, \ldots, c_n)$ such that for every $a \in M$ there is a set $I_a \subset \{1, \ldots, n\}$ such that $a = \oplus_{i \in I_a} c_i$. We say that C *covers* M, and call C a *cover* of M. If M is any subset of E, we say that M is compatible if every finite subset of M

is compatible.

An ordered pair (a, b) of elements of M is said to be *Mackey compatible* if there is an orthogonal sequence (a_1, b_1, c) such that $a = a_1 \oplus c$, $b = b_1 \oplus c$. The sequence (a_1, b_1, c) is called a *Mackey decomposition* of (a, b). Clearly, if (a, b) is Mackey compatible then (b, a) is Mackey compatible with the decomposition (b_1, a_1, c). We will write $a \leftrightarrow b$ if (a, b) are Mackey compatible.

Let F be a subset of E. We say that events $A = (a_1, \ldots, a_n)$ are *compatible in* F if A is compatible with a cover $C \subset F$. A subset M of E is called *compatible in* F if any finite subset of M is compatible with covers in F. We say that M is *internally compatible* if M is compatible in M. We say that elements (a, b) are *Mackey compatible in* F if there is a Mackey decomposition (a_1, b_1, c) of (a, b) with $a_1, b_1, c \in F$, and we write a $\leftrightarrow^F b$

In the following propositions we collect some basic properties of compatibility [23], [11].

Proposition 1 *Let a, b be elements of an effect algebra E.*

(i) $a \leftrightarrow a$, $0 \leftrightarrow a$, $1 \leftrightarrow a$, $\forall a \in E$.

(ii) $a \leq b \;\;\Rightarrow\;\; a \leftrightarrow b$, $a \perp b \;\;\Rightarrow\;\; a \leftrightarrow b$.

(iv) $a \leftrightarrow b \;\;\Rightarrow\;\; a' \leftrightarrow b$.

(v) *If elements $a_1, \ldots, a_n$ are jointly compatible, they are pairwise Mackey compatible. If $n \geq 3$, the converse need not hold even in OMP* ([28]).

Proof of the above statement is straightforward.

Proposition 2 *Let (a, b) be an ordered pair of elements in E. The following statements are equivalent.*

(C1) $a \leftrightarrow b$.

(C2) *a, b are jointly compatible.*

(C3) $a = x \oplus y$ *with* $x \leq b$, $y \leq b'$.

(C4) *There are elements u, v such that $u \leq a, b \leq v$ and $a \ominus u = v \ominus b$.*

Proof. (C1)$\Rightarrow$(C2). Let (a_1, b_1, c) be a Mackey decomposition of (a, b). The orthogonal sequence (a_1, b_1, c) covers (a, b), so a, b are jointly compatible.

(C2)$\Rightarrow$(C3): Let $(c_1, \dots, c_n)$ be a cover of a, b. Then $a = \oplus_{i\in I_a} c_i$, $b = \oplus_{i\in I_b} c_i$, where I_a, I_b are subsets of $\{1, \dots, n\}$. Putting $P_1 = I_a \cap I_b$, $P_2 = I_a \cap I_b^c$, $P_3 = I_a^c \cap I_b$, $P_4 = I_a^c \cap I_b^c$, where I_a^c, I_b^c are complements of I_a, I_b in $\{1, \dots, n\}$, respectively, we obtain a partition of $\{1, \dots, n\}$. Clearly, $I_a = P_1 \cup P_2$, and $I_b = P_1 \cup P_3$. Then $a_1 = \oplus_{i\in P_2} c_i$, $b_1 = \oplus_{i\in P_3} c_i$, $c = \oplus_{i\in P_1} c_i$ is a Mackey decomposition of (a, b). Put $d = (c_1 \oplus \dots \oplus c_n)'$. Then $b \oplus b' = \oplus_{i=1}^n c_i \oplus d = 1$, and by cancellativity, $b' = \oplus_{i\in I_b^c} c_i \oplus d$. Therefore $a_1 = \oplus_{i\in I_a\cap I_b^c} c_i \leq b'$, and $c \leq b$. Putting $x = c$, $y = a_1$ concludes the proof.

(C3)$\Rightarrow$(C4). Let $a = x \oplus y$, $x \leq b$, $y \leq b'$. Then $x \leq a$, $x \leq b$ and $a \leq b \oplus y$, $b \leq b \oplus y$. In addition, $a \ominus x = y = (b \oplus y) \ominus b$. Putting $u = x, v = b \oplus y$ yields the desired statement.

(C4)$\Rightarrow$(C1). Let $u \leq a, b \leq v$ and $a \ominus u = v \ominus b$. Then $a = u \oplus (a \ominus u)$, $b = u \oplus (b \ominus u)$, v=$b \oplus (v \ominus b) = u \oplus (b \ominus u) \oplus (a \ominus u)$, which shows that $(a \ominus u, b \ominus u, u)$ is a Mackey decomposition of (a, b). □

We will say that elements a, b in E are *strongly compatible* if there is a Mackey decomposition (a_1, b_1, c) of the ordered pair (a, b) such that $a_1 \wedge b_1 = 0$ (in the sense that $x \leq a_1$, $x \leq b_1$ implies $x = 0$). We will write $a \leftrightarrow^c b$. More generally, we say that a finite set $A = (a_1, \dots, a_n)$ of elements are *jointly strongly compatible* if there is a cover $C = (c_1, \dots, c_m)$ of A such that for every a_i, a_j in A,

$$\oplus_{k\in I_{a_i}\cap I_{a_j^c}} c_k \wedge \oplus_{k\in I_{a_i^c}\cap I_{a_j}} c_k = 0.$$

We then say that C *strongly covers* A (or that C is a strong cover of A).

Lemma 1 *Let (a_1, b_1, c) be a Mackey decomposition of (a, b). Let $c \leq \bar{c} \leq a, b$. Then $(a \ominus \bar{c}, b \ominus \bar{c}, \bar{c})$ is a Mackey decomposition of (a, b).*

Proof. We can write $a = \bar{c} \oplus (a \ominus \bar{c})$, $b = \bar{c} \oplus (b \ominus \bar{c})$. Then $a \oplus (b \ominus \bar{c}) \leq a \oplus (b \ominus c) = a_1 \oplus b_1 \oplus c$. It follows that $(a \ominus \bar{c}, b \ominus \bar{c}, \bar{c})$ is a Mackey decomposition of (a, b). □

Proposition 3 *Let a, b be elements of an effect algebra E. Let (a_1, b_1, c) be a Mackey decomposition of (a, b). The following statements are equivalent.*

(i) *(a_1, b_1, c) is a strong Mackey decomposition of (a, b);*

(ii) *c is a maximal lower bound of a, b;*

(iii) $d := a_1 \oplus b_1 \oplus c$ *is a minimal upper bound of* a, b.

Proof. (i) $\Rightarrow$ (ii): Let (a_1, b_1, c) be a strong Mackey decomposition of (a, b). Let $\bar{c} \in E$ be such that $c \le \bar{c} \le a, b$. There in an $x \in E$ such that $\bar{c} = c \oplus x$. We have

$$\begin{aligned} a &= \bar{c} \oplus (a \ominus \bar{c}) && (1) \\ &= c \oplus x \oplus (a \ominus \bar{c}) = c \oplus a_1, && (2) \end{aligned}$$

hence $a_1 = x \oplus (a \ominus c)$,

$$\begin{aligned} b &= \bar{c} \oplus (b \ominus \bar{c}) && (3) \\ &= c \oplus x \oplus (b \ominus \bar{c}) = c \oplus b_1, && (4) \end{aligned}$$

hence $b_1 = x \oplus (b \ominus \bar{c})$. It follows that $x \le a_1, b_1$, so $x = 0$. Therefore c is a maximal lower bound of a, b.

(ii) $\Rightarrow$ (i): Let c be a maximal lower bound of a, b and let (a_1, b_1, c) be a Mackey decomposition of (a, b). Assume that $x \le a_1, b_1$. Then $a_1 = x \oplus \bar{a}_1$, $b_1 = x \oplus \bar{b}_1$, so that $a = x \oplus c \oplus \bar{a}_1$, $b = x \oplus c \oplus \bar{b}_1$. Hence $c \le x \oplus c \le a, b$. As c is a maximal lower bound of a, b then $x = 0$, and (a_1, b_1, c) is a strong Mackey decomposition of (a, b).

(ii) $\Leftrightarrow$ (iii): Assume that (a_1, b_1, c) is a Mackey decomposition of (a, b). Put $d := a_1 \oplus b_1 \oplus c$. Then $a \ominus c = d \ominus b$. Let there be a $\bar{d}$ with $a, b \le \bar{d} \le d$. There is $y \in E$ such that $d = \bar{d} \oplus y$. So we obtain $a \ominus c = (\bar{d} \oplus y) \ominus b = (\bar{d} \ominus b) \oplus y$, and hence $(a \ominus c) \ominus y = \bar{d} \ominus b$, which implies that $a \ominus (c \oplus y) = \bar{d} \ominus b$. Hence $c \oplus y \le a$, and similarly we prove that $c \oplus y \le b$. So if d is not a minimal upper bound of a, b, then c is not a maximal lower bound of a, b. By reversing the implications we obtain the converse. It follows that c is a maximal lower bound of a, b iff d is a minimal upper bound of a, b. $\square$

Remark 1. Let E be an effect algebra.

(a) E is an orthoalgebra iff every Mackey decomposition is strong. Indeed, $(a, a', 0)$ is a Mackey decomposition of (a, a') for every $a \in E$. If it is strong, then $a \wedge a' = 0$. The converse is clear.

(b) E is an OMP iff for every Mackey decomposition (a_1, b_1, c) of (a, b), $a_1 \oplus b_1 \oplus c = a \vee b$. That is, every Mackey decomposition is strong and unique. The converse need not hold, there are orthoalgebras with unique Mackey decompositions which are not OMPs.

(c) E is an MV-algebra iff E is lattice ordered and $a \leftrightarrow b$ for every a, b in E [6], [29], [11]. Indeed, according to [6], a lattice ordered effect algebra can be organized into an MV-algebra iff $a \ominus (a \wedge b) = (a \vee b) \ominus b$ holds for every $a, b \in E$. It is easy to see that $a \ominus (a \wedge b), b \ominus (a \wedge b), a \wedge b$ is a Mackey decomposition of (a, b). If E is a lattice and $a \leftrightarrow b$, then it can be derived from Lemma 1 and Proposition 3, that $(a \ominus (a \wedge b), b \ominus (a \wedge b), a \wedge b)$ is a strong Mackey decomposition of (a, b). It follows that $((a \ominus (a \wedge b)) \oplus b$ is a minimal upper bound of a, b, which is $a \vee b$.

Recall that a partially ordered set P satisfies the *Riesz interpolation property* (RIP) if whenever $x, y \leq a, b$ there is an element c in P such that $x, y \leq c \leq a, b$. In partially ordered abelian groups the Riesz interpolation property and the Riesz decomposition property are equivalent. In an effect algebra, the Riesz decomposition property implies the Riesz interpolation property, but the converse need not hold [32].

Proposition 4 *Let E be an effect algebra with RIP. If the elements a, b in E are strongly compatible, there is a unique strong Mackey decomposition (a_1, b_1, c) of (a, b) and $c = a \wedge b$, $d = a_1 \oplus b_1 \oplus c = a \vee b$.*

Proof. Let (a_1, b_1, c) be a strong Mackey decomposition of (a, b). Then $c \leq a, b$ and let $x \in E$ be any lower bound of a, b. By RIP, there is $y \in E$ such that $x, c \leq y \leq a, b$. Since c is maximal, we have $y = c$. Therefore $c = a \wedge b$. Similarly we prove that $d = a \vee b$. □

Corollary 1 *An effect algebra E with RIP is an MV-algebra iff any two elements are strongly compatible.*

Proof. If any two elements are strongly compatible, it follows by Proposition 4 that E is a lattice. Since any two elements are compatible, E is an MV-algebra. The converse statement is well known (see e.g. [30]). □

Recall that for $u \in E$, $[0, u]_E$ denotes the interval $0 \leq x \leq u$ in E with the operation $\oplus$ defined with respect u as the unit element. Two elements $a, b \in [0, u]_E$ are said to be compatible in $[0, u]_E$ if there is a Mackey decomposition (a_1, b_1, c) of (a, b) such that $a_1 \oplus b_1 \oplus c \in [0, u]_E$. In [4], the following characterization of different types of effect algebras has been given

Theorem 1 *Let E be an effect algebra.*

1. *E satisfies RDP if and only if every pair (a, b) of elements of E is compatible in every interval $[0, x]_E$ which contains a, b.*

2. *E is an MV-algebra if and only if for every elements (a, b) there is a Mackey decomposition (a_1, b_1, c) such that $a_1 \oplus b_1 \oplus c \in [0, x]_E$ whenever $a, b \leq x$.*

The following characterization of effect algebras with the RDP has been proved in [22, Theorem 3.5]. It shows what we need to add to the joint compatibility of all finite subfamilies of elements of E to obtain the RDP.

Proposition 5 *Let E be an effect algebra. The following statements are equivalent.*

(a) *E satisfies the RDP.*

(b) *Every finite family of elements in E is jointly compatible and the following weakening of the RDP holds for any $x, a, b \in E$:*

(*) If $x \leq a \oplus b$, and $x \leq (a \oplus b)'$, then there are $a_1 \leq a$. $b_1 \leq b$ such that $x = a_1 \oplus b_1$.

Proof. (a)$\Rightarrow$(b): Let $a_1, \ldots, a_n$ be elements of E. We will proceed by induction on n. If $n = 1$, there is nothing to prove. Now assume that an orthogonal cover exists for $n - 1$, and consider elements $(a_1, \ldots, a_n)$. By induction hypothesis, there is an orthogonal cover of $(a_1, \ldots, a_{n-1})$ say $(c_1, \ldots, c_k)$. Put $c = \oplus_{i=1}^k c_i$, then (c, a_n) have an orthogonal cover - Mackey decomposition (y_1, y_2, z). So $a_n = z \oplus y_2$, where $z \leq c$, $y_2 \leq c'$. Since $z \leq c = \oplus_{i=1}^k c_i$, there are elements $d_i \leq c_i$, $i = 1, \ldots, k$ such that $z = \oplus_{i=1}^k d_i$, and then $(d_1, \ldots, d_k, c_1 \ominus d_1, \ldots, c_k \ominus d_k, y_2)$ is an orthogonal cover of $(a_1, \ldots, a_n)$. It is immediate that (*) is satisfied.

(b)$\Rightarrow$(a): Let a, b, x be elements of E such that $a, b \leq x$. We wish to prove that a, b are compatible in the interval $[0, x]_E$. First we prove that for any cover C of (a, x) with $a \leq x$ there exists a refinement D such that a cover of a in D is contained in a cover of x in D (see [22, Prop. 3.2]). So let $C = (c_1, \ldots, c_n)$ be an orthogonal cover of (a, x). Then there are subsets $A \subset \{1, \ldots, n\}$, $X \subset \{1, \ldots, n\}$ such that $a = \oplus_A c_i$, $x = \oplus_X c_i$. Without any loss of generality, we my assume that $A \setminus X = \{1, \ldots, r\}$, $A \cap X = \{r+1, \ldots, s\}$, and $X \setminus A = \{s+1, \ldots, t\}$ with $r+s+t = n$. Define $a_1 = \oplus_{i=1}^r c_i$, $d = \oplus_{i=r+1}^s c_i$, $x_1 = \oplus_{i=s+1}^t c_i$. Then $a = a_1 \oplus d$, $x = x_1 \oplus d$. From $a \leq x$ and cancellativity we get $a_1 \leq x_1$, and since C is an orthogonal family, we also have $a_1 \leq x_1'$. Now for any $c_i, i = 1, \ldots, r$, we have $c_i \leq x_1$ and $c_i \leq x_1'$. Using (*) we obtain that $c_i = \oplus_{j=s+1}^t d_j^i$, where $d_j^i \leq c_j$, $j = s+1, \ldots, t$. It is then easy to check that $(d^i_j)_{i=1,\dots,r;j=s+1,\dots,t} \cup (c_i)_{i=r+1,\dots,s} \cup (c_i \ominus \oplus_{j=1}^r d^j_i)_{i=s+1,\dots,t})$ is refinement of C with the desired property.

Now we can find a cover $W = (w_1, \dots, w_n)$ of a, b, x and subsets A, B, X of $\{1, \dots, n\}$ with $A \subset X$, $B \subset X$ with $a = \oplus_A w_i$, $B = \oplus_B w_i$, $X = \oplus_X w_i$. Putting $a_1 = \oplus_{A\setminus B} w_i$, $b_1 = \oplus_{B\setminus A} w_i$, $c = \oplus_{A\cap B} w_i$ we obtain a Mackey decomposition of a, b in $[0, x]_E$. $\square$

The following proposition collects some characterizations of MV-algebras among effect algebras with the RDP.

Proposition 6 *Let E be an effect algebra with the RDP. The following statements are equivalent.*

(i) *E is an MV-algebra.*

(ii) *Every two elements in E are strongly compatible.*

(iii) *E is lattice ordered.*

(iv) *The partial operation $\oplus$ can be extended uniquely to a total operation $+$ which is commutative, associative, monotone and such that*

$$a' = \wedge\{x : a + x = 1\}.$$

Proof. (i)$\Rightarrow$(ii): See Corollary 1.

(ii)$\Rightarrow$(iii): Follows by Proposition 4.

(iii)$\Rightarrow$(iv): Recall that the RDP implies that for any elements a, b,

$$b \leq a \oplus a' \;\Rightarrow\; b = c \oplus b_1, c \leq a \text{ and } b_1 \leq a'.$$

Therefore any two elements a, b have a Mackey decomposition. Since E is lattice ordered, by Lemma 1, (a, b) have also a strong Mackey decomposition (a_1, b_1, c) with $c = a \wedge b$, $a_1 \oplus b_1 \oplus c = a \vee b$. It follows that $a \ominus (a \wedge b) = (a \vee b) \ominus b$. Define

$$a + b := a \oplus (a' \wedge b).$$

Clearly, $+$ coincides with $\oplus$ if $a \perp b$, and is monotone in b. By the interplay between $\oplus$ and $\ominus$, we obtain

$$\begin{aligned} (a+b)' &= a' \ominus (a' \wedge b) = (a' \vee b) \ominus b && (5)\\ &= (a \wedge b' \oplus b)' = (b+a)'. && (6) \end{aligned}$$

Therefore $+$ is commutative. Further, using $(a \ominus a \wedge b) = (a \vee b) \ominus b$, and the fact that $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$ if $a \perp b, b \perp c$, we have

$$\begin{aligned}
(a+b)+c &= (a \oplus a' \wedge b) \oplus ((a \oplus a' \wedge b))' \wedge c \\
&= (a \oplus a' \wedge b) \oplus (a' \ominus a' \wedge b) \wedge a' \wedge c \\
&= (a \oplus (a' \ominus a' \wedge b) \wedge a' \wedge c) \oplus a' \wedge b \\
&= (a \oplus (a' \ominus a' \wedge b) \wedge (a \oplus a' \wedge c)) \oplus a' \wedge b \\
&= (a' \ominus (a' \ominus a' \wedge b) \vee (a' \ominus a' \wedge c))' \oplus a' \wedge b \\
&= (a' \wedge b \vee (a' \ominus a' \wedge c))' \oplus a' \wedge b \\
&= ((a' \wedge b \vee (a' \ominus a' \wedge c)) \ominus a' \wedge b)' \\
&= ((a' \ominus a' \wedge c) \ominus (a' \ominus a' \wedge c) \wedge a' \wedge b)' \\
&= (a \oplus a' \wedge c) \oplus (a \oplus a' \wedge c)' \wedge b \\
&= (a+c)+b.
\end{aligned}$$

This proves associativity of $+$.

If $a+x=1$, then $0=(a+x)' = (a \oplus a' \wedge x)' = a' \ominus a' \wedge x$, hence $a' = a' \wedge x$, so that $a' \leq x$. Since $a + a' = a \oplus a' = 1$, the operation $+$ has the desired property.

(iv)$\Rightarrow$(i): Let there exist an operation $+$ with the desired properties. First we prove that $(p'+q)'+q$ is an upper bound of both p and q in E. We have $1 = (p'+q)+(p'+q)' = p' + (q + (p'+q)')$ assuming commutativity and associativity of $+$. Hence $p \leq (p'+q)'+q$ by the last property of $+$. By monotony of $+$, $q \leq (p'+q)'+q$. Put $s = (p'+q)'+q$. We have just proved that s is an upper bound of p, q. Suppose that there is x in E with $x \geq p$, $x \geq q$. We want to prove that $x \geq s$. We may write

$$x = (x \ominus q) \oplus q \tag{7}$$
$$= (x' \oplus q)' \oplus q = (x'+q)'+q, \tag{8}$$

since $+$ is an extension of $\oplus$. From $x \geq p$ we obtain $x' \leq p'$, and by monotony of $+$, we get $x'+q \leq p'+q$. Since $'$ is order reversing, by repeated application of the monotony of $+$ we obtain

$$(p'+q)'+q \leq (x'+q)'+q,$$

and so

$$x' = ((x'+q)'+q)' \leq ((p'+q)'+q)' = s;$$

i.e., $s \leq x$, as required. This proves that $p \vee q = (p' + q)' + q$.

It remains to prove the uniqueness of $+$. Let E be an effect algebra with the RDP. Assume that there are two total extensions of $\oplus$ with the desired properties and denote them by $+_1$ and $+_2$. We want to prove that $+_1 = +_2$. Assume that there are $q, r \in E$ such that $q +_1 r \neq q +_2 r$. Putting $p = r'$, we get $q +_1 p' \neq q +_2 p'$ and hence $(p' +_1 q)' \neq (p +_2 q)'$. Put $z := (p' +_1 q)'$. Then $q \leq q +_1 p' = z'$. Hence $q \perp z$, so $q \oplus z = q +_1 z$. By the last part of the above proof, we have $p \vee q = (p +_1 q)' +_1 q = z +_1 q = z \oplus q$. Similarly, letting $y := (p' +_2 q)'$, we prove that $y \perp q$, and we obtain $p \vee q = y \oplus q = y +_2 q$. We conclude $y = z$, in contradiction with our suppositions. $\square$

Remark 2. The equivalence (i)$\Leftrightarrow$(iv) was proved in [26] in relations with the K_0 theory for AF C*-algebras. Our proof based purely on the algebraic properties of E. Combining it with the result of [13], which proves that the Elliott's invariant is an interval in a partially ordered group with the RDP and so it has a RDP itself, the result of [26] follows.

Remark 3. Let (a_1, b_1, c) be a strong Mackey decomposition of (a, b). Then (b_1, a_1, d') where $d = a_1 \oplus b_1 \oplus c$ is a strong Mackey decomposition of (a', b'). Indeed, by Proposition 3 c is a maximal lower bound of a, b and d is a minimal upper bound of a, b. Then d' is a maximal lower bound of a', b', and (b_1, a_1, d') is a Mackey decomposition of (a', b'). By Proposition 3 (ii), (b_1, a_1, d') is a strong Mackey decomposition of (a', b'). Strong compatibility of (a, b'), resp. (a', b) can be derived iff a_1 is a maximal lower bound of a, b', resp. b_1 is a maximal lower bound of a', b. In particular, if c is any lower bound of a, a', then $(a \ominus c, a' \ominus c, c)$ is a Mackey decomposition of a, a', but a, a' are strongly compatible iff they have a maximal lower bound.

Evidently, strong Mackey decompositions need not exist even for compatible elements, and if they exist, they may be not unique. In the following theorem we show that in any Hilbert space effect algebra $\mathcal{E}(H)$, there exist strong Mackey decompositions (not necessarily unique) for any two compatible elements. This follows from the existence of maximal lower bounds [25, Theorem 4.5].

Theorem 2 *In $\mathcal{E}(H)$, any two Mackey compatible effects have also a strong Mackey decomposition.*

Proof. If $a, b \in \mathcal{E}(H)$ are Mackey compatible, there are $c, d \in \mathcal{E}(H)$ such that $c \leq a, b \leq d$ and $a \ominus c = d \ominus b$. Put $S := \{x \in \mathcal{E}(H) : c \leq x \leq a, b\}$. Let $K \subset S$ be a chain. S is partially ordered and nonempty, since $c \in S$.

K can be considered as an increasing net in $\mathcal{E}(H)$ and by [35, Lemma 1], $\lim_K < x\phi, \phi > = < x_0\phi, \phi >$ for some $x_0 \in \mathcal{E}(H)$ and every $\phi \in H$. It follows that $x_0 \in S$ and $c \leq x \leq x_0$ for every $x \in K$. By Zorn's lemma, S has a maximal element c_1, say, such that $c \leq c_1$. By Lemma 1, $(a \ominus c_1, b \ominus c_1, c_1)$ is a Mackey decomposition of (a, b), and by Proposition 3, it is a strong Mackey decomposition. □

4 Blocks in effect algebras

Recall than an *observable* on an effect algebra E is a morphism $\alpha : B \to E$ from a Boolean algebra B to E. Let $\text{ran}(\alpha) = \{\alpha(b) : b \in B\}$ denote the range of α. We say that two observables α and β are *coexistent* if there is an observable γ such that $\text{ran}(\alpha) \cup \text{ran}(\beta) \subseteq \text{ran}(\gamma)$. An observable α is finite if there is a finite set $\{b_1, \ldots, b_k\}$ in B such that $\alpha(b_1) \oplus \cdots \oplus \alpha(b_k) = 1$. Then $(\alpha(b_i)_{i=1}^k$ is a finite partition of unity in E and $\text{ran}(\alpha)$ consists of $\oplus$-sums over all subsets of it. Clearly, $\text{ran}(\alpha)$ is a compatible set. A system $A = (\alpha_i)_{i \in I}$ will be called *coexistent* (in A) if every two observables α, β in A are coexistent with $\gamma \in A$, where $\text{ran}(\alpha) \cup \text{ran}(\beta) \subseteq \text{ran}(\gamma)$. Clearly, the system A is coexistent iff $\cup_{i \in I} \text{ran}(\alpha_i)$ is compatible with covers in itself.

By a *block* of E we mean a maximal subsystem A of E such that every finite subset of elements of A is compatible in A. Alternatively, a block is union of the ranges of a maximal coexistent system of finite observables on E. Using Zorn lemma, we can prove that every effect algebra E can be expressed as the union of its blocks. The structure of blocks in effect algebras is not known, in general. In fact, we can only say that the blocks bear a structure of a compatible quasi-effect algebra (see [4]). A *quasi-effect algebra* is a structure $(A; \leq, \oplus, ', 1, 0)$ where $(A; \leq, ', 1, 0)$ is a bounded involutive poset, and $\oplus$ is a partially defined binary operation such that

1. $\exists(a \oplus b)$ implies $\exists(b \oplus a)$ and $a \oplus b = b \oplus a$.
2. $a \oplus a' = 1$.
3. $\exists(a \oplus 1)$ implies $a = 0$.
4. $a \leq b$ and $\exists(a \oplus c)$ and $\exists(b \oplus c)$ implies $a \oplus c \leq b \oplus c$.

Let α be an observable on E. Then $A = \text{ran}(\alpha)$ can be organized into a quasi-effect algebra $(A; \leq_A, \oplus_A, '_A, 1_A, 0_A)$ where $\leq_A$ is the restriction of $\leq$ in

E to A, $'_A = '$, $1_A = 1$, $0_A = 0$, and $\exists(a \oplus_A b)$ iff $\exists(a \oplus b)$ and $a \oplus b \in A$, and then $a \oplus_A b = a \oplus b$ ([4, Lemma 1]).

We say that a system $(\alpha_i)_{i\in I}$ of observables is directed if to every $i, j \in I$ there is $k \in I$ such that ran$(\alpha_i)\cup$ ran$(\alpha_j) \in$ ran(α_k). Let $(\alpha_i)_{i\in I}$ be a directed system of observables. Put $A = \cup_{i\in I}$ran(α_i). With partial order $\leq$ and orthosupplementation $'$ and $0, 1$ inherited from E, A is an involutive, bounded poset. Define $\oplus_A$ as follows: $a \oplus_A b$ is defined if there is $i \in I$ with $a, b, a \oplus b \in$ ran(α_i) and then put $a \oplus_A b = a \oplus b$. Clearly, $(A; \leq_A, \oplus_A, '_A, 1, 0)$ is a quasi-effect algebra. We may conclude that in general, a block of an effect algebra is a compatible quasi effect algebra.

Theorem 3 *A block A of an effect algebra E is a sub-effect algebra of E if and only if the following condition is satisfied: $\{a_1, a_2 \ldots, a_n\}$ is a compatible set and $a_1 \perp a_2$ implies that $\{a_1, a_2, \ldots, a_n, a_1 \oplus a_2\}$ is a compatible set.*

Proof. If a block A is a subeffect algebra of E, then $a_1, a_2 \in A$ and $a_1 \perp a_2$ implies $a_1 \oplus a_2 \in A$. Since every finite subset of A is compatible, we have, for any finites set $\{b_1, \ldots, b_n\}$ in A that $\{a_1, a_2, b_1, \ldots, b_n, a_1 \oplus a_2\}$ is a compatible set.

Conversely, if $a_1, a_2 \in A$, $a_1 \perp a_2$ and for any $b_1, \ldots, b_n$ in A $a_1, a_2, b_1, \ldots, b_n, a_1 \oplus a_2$ is compatible, then the maximality of A implies that $a_1 \oplus a_2 \in A$. □

Observe that in the above theorem, the condition $a_1, a_2, \ldots, a_n$ compatible and $a_1 \perp a_2$ implies $a_1, a_2, \ldots, a_n, a_1 \oplus a_2$ compatible can be replaced by the condition $a_1, a_2, \ldots, a_n$ compatible and $a_1 \leq a_2$ implies $a_1, a_2, \ldots, a_n, a_2 \ominus a_1$ compatible.

In analogy with orthomodular posets, we say that an effect algebra E is *regular* if any pairwise compatible finite set is jointly compatible. In a regular effect algebra, a block is a maximal system of pairwise compatible elements.

The following special cases are known.

Case 1. Let E be an orthoalgebra. It is well known that the range of every observable α is a subeffect algebra of E which, endowed with "local" lattice operations $\vee_\alpha, \wedge_\alpha$ taken with respect to ran(α), is a Boolean algebra. Moreover, the mapping $\alpha : B \to$ ran(α) is a Boolean algebra homomorphism. The corresponding "global" suprema and infima in E need not exist, and even if they exist, they need not belong to ran(α).

Let $(\alpha_i)_{i\in I}$ be a maximal coexistent system of finite observables, so that $\cup_{i\in I}$ ran(α_i) is a block. Then $(\text{ran}(\alpha_i))_{i\in I}$ form a directed system of Boolean algebras. Indeed, if ran$(\alpha_i) \subseteq$ ran(α_j), then by well known theorems by

Sikorski and Varadarajan [36], there is a function $f : \Omega_j \to \Omega_i$ where Ω_i, Ω_j are finite sets such that $\alpha_i(\Omega_i) = 1$, $\alpha_j(\Omega_j) = 1$, such that for every $X \in 2^{\Omega_i}$, $\alpha_i(X) = \alpha_j(f^{-1}(X))$. Let $a, b \in \mathrm{ran}(\alpha_i)$, $a = \alpha_i(X)$, $b = \alpha_i(Y)$. Then

$$\begin{aligned} a \wedge_i b &= \alpha_i(X \cap Y) = \alpha_j(f^{-1}(X \cap Y)) \\ &= \alpha_j(f^{-1}(X) \cap f^{-1}(Y)) \\ &= \alpha_j(f^{-1}(X)) \wedge_j \alpha(f^{-1}(Y)) = a \wedge_j b. \end{aligned}$$

Let $A = \cup_{i\in I}\mathrm{ran}(\alpha_i)$ be a block. It is clear that A is a bounded orthocomplemented poset. For every $a, b, c \in A$ there is a $k \in I$ such that $a, b, c \in\mathrm{ran}(\alpha_k)$. It follows that A is a subeffect algebra of E and with suprema and infima taken in A, it is a distributive lattice, hence a Boolean algebra. The lattice operations in a block coincide with those in E iff E is an orthomodular poset.

Case 2. Let E be a lattice ordered effect algebra. It has been proved in [32], that the maximal pairwise compatible subsets are subeffect algebras of E which are closed under the lattice operations and form MV-algebras. It follows that lattice ordered effect algebras are regular and blocks are MV-algebras with the lattice operations inherited from E.

Case 3. Homogeneous effect algebras introduced in [22] have blocks that form subeffect algebras of E satisfying the Riesz decomposition property. Homogeneous effect algebras include orthoalgebras and lattice ordered effect algebras as special cases. In both these subcases, blocks form distributive lattices, but the nature of lattice operations may be different.

Case 4. According to [8], if an effect algebra satisfies the Riesz interpolation property (RIP) and so called difference meet property (DMP), that is if $a, b, c \in E$, $a \le b$ are such that $a \wedge c, b \wedge c$ exist in E, then $(b \ominus a) \wedge c$ exists in E, then E can be covered by MV-algebras, which are maximal sets of pairwise strongly compatible subsets of E. The lattice operations in the MV-algebras coincide with the lattice operations in E. Since every lattice ordered effect algebra satisfies RIP and DMP, and in a lattice ordered effect algebras any two compatible elements are also strongly compatible, this case generalizes the case of lattice ordered effect algebras. It is not known whether the maximal strongly compatible sets coincide with the maximal compatible sets.

Case 5. Let H be a Hilbert space (complex, separable). Consider a maximal subset A of the effect algebra $\mathcal{E}(H)$ consisting of pairwise commuting elements. Using well-known results of von Neumann [27] and Varadarajan [36], there is an observable (self-adjoint operator) u and Borel functions f_a :

$\mathbb{R} \to [0,1]$ such that every $a \in A$ is a function of u, $a = f_a(u)$. Since A is maximal, the set $\{f_a : a \in A\}$ consists of all measurable functions. The functions $f_a : \mathbb{R} \to [0,1]$ can be endowed with a structure of an MV-algebra, if we define ([4])

$$(f \oplus g)(x) = max(1, f(x) + g(x)),\ x \in \mathbb{R}.$$

Using the functional calculus, we can transfer the MV-algebra structure to the operators in A. In this way, $\mathcal{E}(H)$ can be covered by MV-algebras. By the functional calculus, we have

$$\begin{aligned} a \wedge_A b &= \int_0^1 max(f_a(\lambda), f_b(\lambda)u(d\lambda) \\ &= \int_a^b \frac{1}{2}(f_a + f_b - |f_a - f_b|)u(d\lambda) \\ &= \frac{1}{2}(a + b - |a - b|) \\ &= a \sqcap b. \end{aligned}$$

According to [20], $a \sqcap b \geq 0$ and it is a maximal lower bound of a, b. Since, again by the functional calculus, $a + (b - a \sqcap b) = \frac{1}{2}(a + b + |a - b|) = a \sqcup b \leq 1$, a, b are strongly compatible. The infimum of a, b in A is then equal to $a \sqcap b$, which is equal to $a \wedge b$ iff $a \wedge b$ exists in $\mathcal{E}(H)$. We note that $a \wedge b$ need not exist in $\mathcal{E}(H)$ even if a and b commute. Let $a \in \mathcal{E}(H)$ be a regular effect such that $0, 1$ do not belong to the point spectrum of a (recall that a is regular if a and a' are incomparable). By [1], the infimum $a \wedge a'$ in $\mathcal{E}(H)$ exists iff

$$\int_{0+}^{1-} \lambda du(\lambda)$$

and

$$\int_{0+}^{1-} (1 - \lambda)du(\lambda)$$

where $\lambda \to u(\lambda)$ is the spectral measure of a, are comparable, and then the infimum is the smaller of both. But if the point spectrum of a does not contain $0, 1$, then the above integrals coincide with a and $a' = 1 - a$, respectively, which are regular, hence incomparable. Therefore $a \wedge a'$ does not exist.

We conclude that in the MV-algebras A corresponding to maximal sets of commuting effects, the lattice operations in A need not coincide with the lattice operations in $\mathcal{E}(H)$.

It can be shown that there are noncommuting effects a, b in $\mathcal{E}(H)$ such that (a, b) have a Mackey decomposition $(a \ominus a \sqcap b, b \ominus a \sqcap b, a \sqcap b)$. Let ϕ and ψ be different unit vectors in H. Put $a = \lambda\phi$, $b = \mu\psi$, $0 < \lambda, \mu < 1$. We can choose λ and μ such that $a + b \leq I$. Then $a \wedge b = 0 = a \sqcap b$ in $\mathcal{E}(H)$, and $(a, b, 0)$ is a strong Mackey decomposition of (a, b).

We conclude that the above MV-algebras do not correspond to maximal compatible sets in $\mathcal{E}(H)$.

5 Effect test spaces

In [10], [11] and [19], close relations between effect algebras and so-called D-test spaces ([10], [11]) or effect test spaces ([19]) were shown. D-test spaces and effect test spaces are essentially equivalent and they are a generalized form of test spaces introduced by Foulis and Randall [16], [34]. Let us recall some basic facts about effect test spaces.

Let X be a nonempty set, elements of X are called *outcomes*. Let I be a nonempty set. A function $F : I \to X$ is said to be of *finite multiplicity* if for every $x \in X$, $F^{-1}(x)$ is of finite cardinality. Observe that if $I = \emptyset$ then $X^I = \{\emptyset\}$. For arbitrary sets I, J and functions $F \in X^I$, $G \in X^J$ define $F \subset G$ if there is an injection $\sigma : I \to J$ such that $F = G \circ \sigma$, i.e., $F(i) = G(\sigma(i))$ for all $i \in I$. If $F \subset G$ and $\sigma : I \to J$ is a bijection, we say that F and G are *equivalent*, in symbols $F \sim G$, sometimes we will identify equivalent functions, and write $F = G$. Let $F \in X^I$, $G \in X^J$. We can now define unambiguously a function $F \dot{\cup} G$ as follows: Let $K = I' \cup J'$, $I' \cap J' = \emptyset$, and $\phi : I \to I', \psi : J \to J'$ be bijections. Then $F \dot{\cup} G \in X^K$, and $F \dot{\cup} G(k) = F(i)$ if $k = \phi(i), i \in I$, and $F \dot{\cup} G(k) = G(j)$ if $k = \psi(j), j \in J$. Evidently, $\dot{\cup}$ is commutative and associative.

For $F \in X^I$, let $\mathcal{R}(F) := \{x \in X : x = F(i) \text{ for some } i \in I\}$, denote the range of F.

Definition 1 *Let $\mathcal{J}$ be a nonvoid family of index sets, $X \neq \emptyset$, and let $\mathcal{T} = \{T \in X^I : I \in \mathcal{J}\}$. We say that $(X, \mathcal{T})$ is an* **effect test space** *if the following conditions are satisfied:*

(i) *any $T \in \mathcal{T}$ is of finite multiplicity,*

(ii) *for every* $x \in X$ *there is a* $T \in \mathcal{T}$ *such that* $x \in \mathcal{R}(T)$*,*

(iii) *if* $S, T \in \mathcal{T}$ *and* $S \subset T$*, then* $S \sim T$*.*

An element $T \in \mathcal{T}$ *is said to be a* **test***.*

Notice that if $T \in \mathcal{T}$ and $T \in X^I$, then $I \neq \emptyset$. For otherwise by (iii), $\emptyset \subset T \;\Rightarrow\; \emptyset = T$ for all tests T, contradicting (ii). A function $G \in X^J$ is called an *event* if there is a $T \in \mathcal{T}$ such that $G \subset T$. Denote the set of all events by $\mathcal{E} = \mathcal{E}(X, \mathcal{T})$.

Definition 2 *Let* F, G *be events. We say that* F *and* G *are*

(i) **orthogonal** *(written* $F \perp G$*) if there is a* $T \in \mathcal{T}$ *such that* $F \dot{\cup} G \subset T$*,*

(ii) **local complements** *of each other (*$F \operatorname{loc} G$*) if there si a* $T \in \mathcal{T}$ *with* $F \dot{\cup} G = T$*,*

(iii) **perspective with axis** H *(*$F \approx_H G$*, or simply* $F \approx G$*) if they share a common complement* H*.*

Observe that if $\emptyset \approx G$ then $G \sim \emptyset$. Moreover, arbitrary two tests R, S are perspective with axis $\emptyset$.

Definition 3 *An effect test space* $(X, \mathcal{T})$ *is* **algebraic** *if for* $F, G, H \in \mathcal{E}$*,* $F \approx G$ *and* $F \perp H$ *implies* $G \perp H$*.*

Equivalently, $(X, \mathcal{T})$ is algebraic iff $F, G, H \in \mathcal{E}$, $F \operatorname{loc} H$ and $F \approx G$ implies $G \operatorname{loc} H$. If $(X, \mathcal{T})$ is algebraic, then the relation $\approx$ becomes an equivalence relation. Denote by $\Pi = \Pi(X, \mathcal{T})$ the set of all equivalence classes of events. For $F \in \mathcal{E}$, define $\Pi(F)' = \Pi(G)$, where G is any local complement of F. Then $\Pi(F)'$ is well defined. For $F, G \in \mathcal{E}$ define $\Pi(F) \oplus \Pi(G) := \Pi(H)$ iff $F \perp G$ and $H = F \dot{\cup} G$. Putting $0 := \Pi(\emptyset)$ and $1 := \Pi(T)$, for any $T \in \mathcal{T}$, it can be proved that $\Pi(X, \mathcal{T})$ becomes an effect algebra.

Conversely, let $(E, \oplus, 0, 1)$ be an effect algebra. A finite sequence $(a_1, a_2, \ldots, a_n)$ of nonzero elements of E is a *finite partition of unity* if $a_1 \oplus a_2 \cdots \oplus a_n$ is defined in $\mathcal{E}$ and equals to 1. Put $X := E \setminus \{0\}$ and $\mathcal{F}$ be the family of all finite partitions of unity in E. Then $(X, \mathcal{F})$ is an algebraic effect test space, where two events $A = (a_i)_{i \leq n}$, $B = (b_j)_{j \leq m}$ are perspective iff $\oplus_{i \leq n} a_i = \oplus_{j \leq m} b_j$. The logic $\Pi(X, \mathcal{F})$ is isomorphic with E. Such a test space is called the *standard test space* for the effect algebra E.

In what follows, we consider an algebraic effect test space $(X, \mathcal{T})$. Let $T \in \mathcal{T}$ be a test, we may assume that $T \in X^{I_T}$, where I_T denotes the index set of T. Let $\sigma(I_T)$ be an ordered partition of I_T. Then there is a mapping $f_\sigma : I_T \to \sigma(I_T)$, such that $f_\sigma(i) = j$ iff $i \in P_j$ where $P_j \in \sigma(I_T)$. Let $||\sigma||$ denote the cardinality of $\sigma(I_T)$. We may consider $\{j : P_j \in \sigma(I_T)\}$ as a new index set and assume that there exists a test $\sigma(T)$ in $\mathcal{T}$ such that

$$\sigma(T)(j) \approx \dot{\cup}_{i \in f_\sigma^{-1}(j)} T(i) = \dot{\cup}_{i \in P_j} T(i).$$

We will say that $\sigma(T)$ is a *coarsening* of T, or that T is a *refinement* of $\sigma(T)$.

Consider the index set $I_{\sigma(T)}$ of $\sigma(T)$. Let τ be any ordered partition of $I_{\sigma(T)}$. We can build a function $\tau(\sigma(T))$ putting

$$\begin{aligned} \tau(\sigma(T))(j) &\approx \dot{\cup}_{k \in f_\tau^{-1}(j)} \dot{\cup}_{i \in f_\sigma^{-1}(k)} T(i) \\ &= \dot{\cup}_{k \in Q_j} (\dot{\cup}_{i \in P_k} T(i)), \end{aligned}$$

where $Q_j \in \tau(\sigma(T))$.

Using the refinement mappings $f_\sigma, f_\tau, f_\kappa$ we can show associativity of compositions . If σ is an ordered partition of I_T, τ is an ordered partition of $I_{\sigma(T)}$ and κ is an ordered partition of $I_{\tau(\sigma(T))}$, then we have

$$\kappa \cdot (\tau \cdot \sigma)(T) = (\kappa \cdot \tau) \cdot \sigma(T).$$

For every $n \in \mathbb{N}$ let σ_n denote the n-ary partition of I_T, so $||\sigma_n|| = n$. Then $I_{\sigma_n(T)} = \{1, \cdots, n\}$, and we obtain an n-ary test

$$\sigma_n(T) = (\sigma_n(T)(1), \cdots, \sigma_n(T)(n)).$$

In what follows, we show examples of test spaces, where coarsenings and refinements make sense.

Example 1. Let $X = 2^Y \setminus \{\emptyset\}$ be the set of all nonempty subsets of nonempty set Y, and consider a classical test space $(X, \mathcal{T})$, i.e., for every test $T \in \mathcal{T}$ the cardinality of $T^{-1}(x)$, $x \in X$, is at most 1. For any test $T \in X^{I_T}$ and any ordered partition $\sigma = \{P_j : j \in J\}$ of I_T, the sets $\cup\{T(i) : i \in P_j\}$, $j \in J$ belong to X. Therefore it makes sense to define a function $F : J \to X$ putting $F(j) = \cup\{T(i) : i \in P_j\}$. Instead of 2^Y we can consider a subfamily of it, e.g., a σ field of sets, and consider partitions with members in that family.

Example 2. Let $(X, +, 0)$ be a positive cone. Let U be a subset of X such that $v \in X$, $u \leq v$ for some $u \in U$ implies $v \notin U$. Put $I_n = \{1, \ldots, n\}$

$n \in \mathbb{N}$, and define a function $f : I_n \to X$ to be a test if $\sum_{i=1}^{n} f(i)$ belongs to U. Let $\mathcal{T}$ denote the set of all tests. Then $(X \setminus \{0\}, \mathcal{T})$ is an effect test space, and the structure of X enables us to take into account partitions of index sets and the corresponding coarsenings, resp. refinements.. If f is an event, we denote by $\sum f := \sum_{i \in I_f} f(i)$, where I_f is the index set of f. An event h is a local complement of f if $\sum_i f(i) + \sum_j h(j) \in U$. Events f and g are perspective if they share a common complement. The test space is algebraic iff whenever two events share one local complement, they share all local complements. In this case, the quotient with respect to the perspectivity relation is an effect algebra. For example, if $U = \{u\}$ is a one-element set, then by cancellativity of the positive cone, the test space is algebraic, and we obtain an interval effect algebra. More generally, a positive cone may be replaced by a partial abelian monoid (cf. [37], [31], [30]).

Example 3. Let $(E \setminus \{0\}, \mathcal{T})$ be a standard effect test space of an effect algebra E. Tests in $\mathcal{T}$ are orthogonal subsets $(a_1, \dots, a_n)$ of nonzero elements such that $\oplus_{i=1}^{n} a_i = 1$. If $(P_1, \dots, P_s)$ is any partition of the set $\{1, \dots, n\}$, then $(b_1, \dots, b_s)$, where $b_i = \oplus_{\{j \in P_i\}} a_j$, is also a test. This example is a special case of Example 2.

Example 4. Let $(X, \mathcal{T})$ be an effect test space, $\mathcal{E}$ the set of its events. Put $Z := \mathcal{E} \setminus \{\emptyset\}$. Consider the set $\mathcal{F}$ of functions $\tilde{F} : I_n \to Z$, where I_n denotes the set $\{1, \dots, n\}$, $n \in \mathbb{N}$, such that $\dot{\cup}_{i=1}^{n} \tilde{F}(i) \in \mathcal{T}$. We prove that $(Z, \mathcal{F})$ is an effect test space. Let $E \in \mathcal{E} \setminus \{\emptyset\}$, then for any local complement E' of E, the function $\tilde{F} : \{1\} \to Z$, $\tilde{F}(1) = E$ if E is a test, $\tilde{F} : \{1, 2\} \to Z$, $\tilde{F}(1) = E$, $\tilde{F}(2) = E'$ otherwise, belongs to $\mathcal{F}$. Hence the union of ranges of functions in $\mathcal{F}$ covers Z. Assume that $\tilde{F} : I_m \to Z$, $\tilde{G} : I_n \to Z$ belong to $\mathcal{F}$, and $\tilde{F} \subset \tilde{G}$. Then there must be an injection $\sigma : I_m \to I_n$ such that $\tilde{F}(\sigma(i)) = \tilde{G}(j)$ whenever $\sigma(i) = j$. Then

$$\dot{\cup}_{\{j \in I_n : j = \sigma(i)\}} \tilde{G}(j) = \dot{\cup}_{\{i \in I_m\}} \tilde{F}(i) \in \mathcal{T}$$

and hence $\tilde{G}(k) = 0$ for $k \in I_n \setminus \sigma(I_m)$, so that $\tilde{F} = \tilde{G}$. This proves that $(Z, \mathcal{F})$ is an effect test space. Denote by $\tilde{\mathcal{E}}$ the set of events of $(Z, \mathcal{F})$. For every $\tilde{E} : I_n \to Z$ in $\tilde{\mathcal{E}}$, define $w(\tilde{E}) := \dot{\cup}_{i \in I_n} \tilde{E}(i)$. Clearly $w(\tilde{E})$ belongs to $\mathcal{E}$ and $w : \tilde{\mathcal{E}} \to \mathcal{E}$ is a surjective mapping. In addition, if $\tilde{E} \perp \tilde{F}$, then $w(E) \perp w(F)$ and $w(\tilde{E} \dot{\cup} \tilde{F}) = w(\tilde{E}) \dot{\cup} w(\tilde{F})$, and also $w(\tilde{E}) \perp w(\tilde{F})$ implies $\tilde{E} \perp \tilde{F}$.

Assume that $(X, \mathcal{T})$ is an algebraic effect test space. We show that $(Z, \mathcal{F})$ is also an algebraic effect test space. Moreover, the corresponding effect algebras $\Pi(X)$ and $\Pi(Z)$ are isomorphic. To prove it, assume that $\tilde{E}, \tilde{F}, \tilde{G} \in \tilde{\mathcal{E}}$, and $\tilde{E} \approx_{\tilde{G}} \tilde{F}$. Then $w(\tilde{E}) \dot{\cup} w(\tilde{G}) \in \mathcal{T}$, $w(\tilde{F}) \dot{\cup} w(\tilde{G}) \in \mathcal{T}$, and hence $w((\tilde{E}) \approx_{w(\tilde{G})} w(\tilde{F})$.

Conversely, let $w(\tilde{E}) \approx_{w(\tilde{G})} w(\tilde{F})$. Then $w(\tilde{E}) \dot{\cup} w(\tilde{G}) \in \mathcal{T}$, $w(\tilde{F}) \dot{\cup} w(\tilde{G}) \in \mathcal{T}$, so that $\tilde{E} \dot{\cup} \tilde{G} \in \mathcal{F}$, $\tilde{F} \dot{\cup} \tilde{G} \in \mathcal{F}$, and hence $\tilde{E} \approx_{\tilde{G}} \tilde{F}$.

Define $\nu : \Pi(Z) \to \Pi(X)$ by $\nu(\Pi(\tilde{E})) = \Pi(w(\tilde{E}))$. If $\tilde{E} \perp \tilde{F}$, then

$$\begin{aligned} & \nu(\Pi(\tilde{E}) \oplus \Pi(\tilde{F})) = \nu(\Pi(\tilde{E} \dot{\cup} \tilde{F})) \\ = \quad & \Pi(w(\tilde{E} \dot{\cup} \tilde{F})) = \Pi(w(\tilde{E}) \dot{\cup} w(\tilde{F})) \\ = \quad & \Pi(w(\tilde{E})) \oplus \Pi(w(\tilde{F})) \\ = \quad & \nu(\Pi(\tilde{E})) \oplus \nu(\Pi(\tilde{F})). \end{aligned}$$

Clearly, $\nu(\Pi(0)) = 0$, and $\nu(\Pi(\tilde{T})) = \Pi(w(\tilde{T}) = 1$ whenever $\tilde{T} \in \mathcal{F}$. Hence ν is a surjective morphism of effect algebras. Since

$$\begin{aligned} & \nu(\Pi(\tilde{E}) \perp \nu(\Pi(\tilde{F})) \Rightarrow \Pi(w(\tilde{E})) \perp \Pi(w(\tilde{F})) \\ \Rightarrow \quad & w(\tilde{E}) \perp w(\tilde{F}) \Rightarrow \tilde{E} \perp \tilde{F}, \end{aligned}$$

ν is an isomorphism.

It is easy to see that if $\tilde{E}$ is any event in $(Z, \mathcal{F})$, $\tilde{E} : I_n \to Z$, and $\kappa(I_n) = (P_1, \ldots, P_{||\kappa||})$ is any partition of I_n, then $\tilde{F} : I_{||\kappa||} \to Z$, $\tilde{F}(i) = \dot{\cup}(\tilde{E}(j) : j \in P_i)\}$ is an event in $(Z, \mathcal{F})$.

6 Test spaces with refinements

The following definition is inspired by [21].

Definition 4 *Let $(X, \mathcal{T})$ be an effect test space. We will say that $(X, \mathcal{T})$ is a* **leveled effect test space** *if for every test $T : I_T \to X$ in $\mathcal{T}$ and every finite partition $\sigma(I_T) = (P_1, \ldots, P_n)$, $||\sigma|| = n$, $n \in \mathbb{N}$, of the set I_T, there is a test $\sigma(T) : \{1, \ldots, n\} \to X$ such that $\sigma(T)(i) = \dot{\cup}_{j \in P_i} T(j)$.*

Let $(X, \mathcal{T})$ be a leveled algebraic effect test space, $\mathcal{E}$ its set of events. We say that an event F is a *refinement* of an event E if there is a mapping $\kappa : I_F \to I_E$ such that for every $i \in I_E$, $E(i) \approx \dot{\cup}\{F(j) : j \in \kappa^{-1}(i)\}$. Clearly, if F is a refinement of E, then $F \approx E$. We say that two events E and G have a *common refinement*, if there is an event F which is a refinement of both E and G. Clearly, if E and G have a common refinement, then $E \approx G$.

We say that events $E_1, \dots, E_k$ *can be built* from an event F if there are sub-events $F_1, \dots, F_k$ of F which are refinements of $E_1, \dots, E_k$, respectively.

Let $A, B : \{1, 2\} \to X$ be binary events, and let $C : I_C \to X$ be their common refinement. Then there are partitions (I_1, I_2) and (J_1, J_2) of the set I_C such that $A(1) = \dot\cup_{i \in I_1} C(i)$, $A(2) = \dot\cup_{i \in I_2} C(i)$, $B(1) = \dot\cup_{j \in J_1} C(j)$, $B(2) = \dot\cup_{j \in J_2} C(j)$. We can construct a partition of I_C consisting of sets $I_1 \cap J_1, I_1 \cap J_2, I_2 \cap J_1, I_2 \cap J_2$ and define the events $W_{ij} = \dot\cup_{k \in I_i \cap J_j} C(k)$, $i, j = 1, 2$, the four-element event $(W_{ij})_{i,j=1}^2$ is a common refinement of A and B. Any two binary events A, B which have a common refinement, have also a common refinement of the latter type. We say that the events A, B admit a *strong common refinement* if they have a common refinement $(W_{ij})_{i,j=1}^2$ such that $W_{12} \cap W_{21} = \emptyset$, that is, there is no nonempty event V under W_{12} and W_{21}.

Theorem 4 *Let $(X, \mathcal{T})$ be an algebraic leveled effect test space. Let $\mathbb{E} = \Pi(\mathcal{E})$ be the corresponding effect algebra. The following statements are equivalent.*

(i) *Elements $a_1, \dots, a_n$ in $\mathbb{E}$ are compatible.*

(ii) *There are events $A_1, \dots, A_n$ in $\mathcal{E}$, $\Pi(A_i) = a_i$, that can be built from a test T in $\mathcal{T}$.*

(iii) *There are binary tests B_i with $\Pi((B_i)(1)) = a_i$, $i = 1, \dots, n$ which have a common refinement.*

Moreover, elements a, b in E are strongly compatible if and only if there are binary events A, B, such that $\Pi(A) = a, \Pi(B) = b$ and A, B admit a strong common refinement.

Proof. (i)$\Rightarrow$(ii). By (i), there is an orthogonal set $(c_1, \dots, c_k)$ such that $a_i = \oplus_{j \in I_i} c_j$, $I_i \subset \{1, \dots, k\}$, $i = 1, \dots, n$. Hence there are orthogonal events $C_1, \dots, C_k$ with $\Pi(C_i) = c_i$, $i = 1, \dots, k$, and a test $T \in \mathcal{T}$ to which $C_1, \dots, C_k$ belong. We then have $a_i = \Pi(\dot\cup_{j \in I_i} C_j)$, $i = 1, \dots, n$, and $A_i = \dot\cup_{j \in I_i} C_j$, $i = 1, \dots, n$ can be built from T.

(ii)$\Rightarrow$(iii). Let $a_i = \Pi(A_i)$, $i = 1, \dots, n$, and $A_1, \dots, A_n$ can be built from a test T. We may assume, with no loss of generality, that $A_1, \dots, A_n$ are subevents of T. Then there are subsets $I_1, \dots, I_n$ of the index set I_T of T such that $A_k = \dot\cup_{j \in I_k} T(j)$, $k = 1, \dots, n$. Taking partitions $(I_k, I_T \setminus I_k)$ of I_T, we obtain binary tests B_k, $k = 1, \ldots, n$ with $B_k(1) \approx A_k = \dot\cup_{i \in I_k} T(j)$, which can be built from T.

(iii)$\Rightarrow$(i). Let B_i, $i = 1, \ldots, n$ be binary tests with $\Pi(B_i) = a_i$ which can be built from a test T. Then there are subsets I_i of I_T with $B_i \approx \dot\cup_{j \in I_i} T(i)$. Taking a suitable coarsening if necessary, we may assume that I_T is finite, $I_T = \{1, \ldots, k\}$. Then $c_j = \pi(T(j)$, $j = 1, \ldots, k$ is an orthogonal sequence, and $a_i = \Pi(\dot\cup_{j \in I_i} T(j)) = \oplus_{j \in I_i} c_j$.

To prove the remaining part, assume that (a, b) in E have a strong Mackey decomposition (a_1, b_1, c) and put $d = a_1 \oplus b_1 \oplus c$. Let A, B be any events such that $\Pi(A) = a$, $\Pi(B) = b$. Then there are orthogonal events A_1, B_1, C, D' belonging to a test T, such that $\Pi(A_1) = a_1, \Pi(B_1) = b_1, \Pi(C) = c$ and $\Pi(D') = d'$. We have $\Pi(A_1 \dot\cup C) = a$, $\Pi(B_1 \dot\cup C) = b$, hence $A \approx A_1 \dot\cup C$, $B \approx B_1 \dot\cup C$, and (A_1, B_1, C, D') is a common refinement of the binary tests (A, A') and (B, B'). From $a_1 \wedge b_1 = 0$, we obtain that there is no event $V \neq \emptyset$ lying under A_1 and B_1.

The proof of the opposite statement is analogous. $\square$

In the following theorem, we give a characterization of the most important types of effect algebras by properties of leveled effect test spaces. Notice that statements similar to (ii) and (iii) in terms of ordinary effect test spaces have been proved in [7].

Theorem 5 *Let $(X, \mathcal{T})$ be a leveled algebraic effect test space, $\mathcal{E}$ its set of events and $\mathbb{E} = \Pi(\mathcal{E})$ the corresponding effect algebra. Following statements hold.*

- (i) *$\mathbb{E}$ is an orthoalgebra if and only if every test is an injective function.*
- (ii) *$\mathbb{E}$ is an OMP if and only if every pairwise orthogonal events E, F, G can be built from one test T.*
- (iii) *$\mathbb{E}$ has the RDP if and only if every two perspective binary events have a common refinement.*
- (iv) *$\mathbb{E}$ is an MV-algebra if and only if every two perspective binary events have a strong common refinement.*

Proof. (i) Is well-known. (ii) Easily follows from the fact that an effect algebra E is an OMP iff for any a, b, c in $\mathbb{E}$ which are pairwise orthogonal $a \oplus b \oplus c$ exists.

(iii) Let $\mathbb{E}$ have the RDP. Let $A : \{1,2\} \to X$, $B : \{1,2\} \to X$ be perspective binary events. Put $\pi(A(1)) = a_1$, $\pi(A(2)) = a_2$, $\pi(B(1)) = b_1$, $\pi(B(2)) = b_2$. By assumption $A \approx B$, which implies $A(1)\dot{\cup}A(2) \approx B(1)\dot{\cup}B(2)$, and this in turn implies $a_1 \oplus a_2 = b_1 \oplus b_2$. By RDP, there are elements $w_{11}, w_{12}, w_{21}, w_{22}$, jointly orthogonal and such that $a_1 = w_{11} \oplus w_{12}$, $a_2 = w_{21} \oplus w_{22}$, $b_1 = w_{11} \oplus w_{21}$, $b_2 = w_{12} \oplus w_{22}$. Then there are events W_{ij}, $i, j \in \{1,2\}$, such that $w_{ij} = \Pi(W_{ij})$, which all belong to one test. The event $W = (W_{11}, W_{12}, W_{21}, W_{22})$ is a common refinement of A and B.

Conversely, assume that every two perspective binary events have a common refinement. Let $a_1 \oplus a_2 = b_1 \oplus b_2$, where a_1, a_2, b_1, b_2 are elements in E. Then there are perspective binary events A, B such that $A(i) = a_i$, $i = 1,2$ and $B(j) = b_j$, $j = 1,2$. By supposition, they have a common refinement $W = (W_{11}, W_{12}, W_{21}, W_{22})$, $A_i \approx \dot{\cup}_{j=1}^2 W_{1j}$, $i = 1,2$, $B_j = \dot{\cup}_{i=1}^2 W_{ij}$, $j = 1,2$. It follows that $a_1 = \oplus_{j=1}^2 \Pi(W_{ij})$, $i = 1,2$, $b_j = \oplus_{i=1}^2 \Pi(W_{ij})$, $j = 1,2$, which yields the RDP.

(iv) Let $\mathbb{E}$ be an MV-algebra and let A, B be perspective binary events. Put $\Pi(A(i) = a_i$ and $\Pi(B_j) = b_j$, $i, j = 1,2$. Then we have $a_1 \oplus a_2 = b_1 \oplus b_2$, and since E is an MV-algebra, there are jointly orthogonal elements $w_{ij}, i, j = 1,2$ such that $a_i = \oplus_{j=1}^2 w_{ij}$, $b_j = \oplus_{i=1}^2 w_{ij}$ and $w_{12} \wedge w_{21} = 0$ [17]. Then there are orthogonal events W_{ij}, $i, j = 1,2$ such that $\Pi(W_{ij}) = w_{ij}$, so that $W_{12} \cap W_{21} = \emptyset$. It follows that A and B have a strong common refinement. By reversing the arguments, the converse statement is proved. $\square$

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