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Martin Kochol

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EQUIVALENT VERSIONS OF THE 3-FLOW CONJECTURE

Martin Kochol

Abstract. The 3-flow conjecture of Tutte is that every bridgeless graph without a 3-edge cut has a nowhere-zero 3-flow. We present two equivalent versions of this conjecture. The first one is that it suffices to prove the 3-flow conjecture for 5-edge-connected graphs. The second variant is that every 4-edge-connected graph has a nowhere-zero $\mathbb{Z}_3$-flow with prescribed values on edges incident to a vertex of valency 4 or 5.

The graphs considered in this paper are all finite, unoriented, and loopless. Multiple edges are allowed. If G is a graph, then $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. To each edge of G there are associated two distinct *arcs* (see [4]). Arcs on distinct edges are distinct. If an arc on an edge is denoted by x the other is denoted by x^{-1}. If the ends of an edge e of G are the vertices u and v, one of the arcs on e is said to be *directed* from u to v and the other one is directed from v to u. Let $D(G)$ denote the set of arcs on G. If $v \in V(G)$, then the set of arcs of G directed out from v is denoted by $\omega_G^+(v)$.

In this paper, every Abelian group is additive. If G is a graph and A is an Abelian group, then an *A-chain* in G is a mapping $\varphi : D(G) \to A$ such that $\varphi(x^{-1}) = -\varphi(x)$ for every $x \in D(G)$. Furthermore, the mapping $\partial\varphi : V(G) \to A$ so that

$$\partial\varphi(v) \;=\; \sum_{x\in\omega_G^+(v)} \varphi(x) \quad (v \in V(G))$$

is called the *boundary* of φ. By [3, 4, 5], $\sum_{v\in V(G)} \partial\varphi(v) = 0$.

An A-chain φ in G is called *nowhere-zero* if $\varphi(x) \neq 0$ for any $x \in D(G)$. A (nowhere-zero) A-chain φ in G is called a (*nowhere-zero*) *A-flow* in G if $\partial\varphi(v) = 0$ for every vertex v of G. If k is a positive integer, then by a (*nowhere-zero*) *k-flow* φ in a graph G we mean a (nowhere-zero) $\mathbb{Z}$-flow in G such that $|\varphi(x)| < k$ for every $x \in D(G)$.

By Tutte [8], a graph has a nowhere-zero k-flow iff it has a nowhere-zero $\mathbb{Z}_k$-flow (see, e.g., [2, 7, 9]). The 3-flow (resp., 5-flow) conjecture of Tutte is that every bridgeless graph without 3-edge-cuts (resp., every bridgeless graph) has a nowhere-zero 3-flow (resp., 5-flow). It is easy to check that the 3-flow conjecture reduces to 4-edge-connected graphs. We show that it suffices to prove this conjecture for 5-edge-connected graphs.

In [5] we have proved the following statement.

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Lemma 1. *Let G and H be two distinct graphs without nowhere-zero k-flows. Suppose that e is an edge of G with ends u and v and let u', v' be two distinct vertices from H. Delete e from G and identify the pairs of vertices u, u' and v, v' to two new vertices, respectively. Then the resulting graph G' does not admit a nowhere-zero k-flow.*

Proof. Suppose that G' has a nowhere-zero k-flow and thus, by Tutte [8], also a nowhere-zero $\mathbb{Z}_k$-flow φ'. Let ψ and φ be the restrictions of φ' to $D(H)$ and $D(G-e)$ (which are subsets of $D(G')$), respectively. Then $\partial\psi(v) = \sum_{x\in\omega_H^+(v)} \psi(x) = 0$ for every $v \in V(H) \setminus \{u', v'\}$, whence, by [4, Lemma 1], $\partial\psi(u') + \partial\psi(v') = 0$. If $\partial\psi(u') = \partial\psi(v') = 0$, then ψ is a nowhere-zero $\mathbb{Z}_k$-flow in H, a contradiction. Otherwise extend φ to a mapping φ'' on $D(G)$ so that $\varphi''(x) = \partial\psi(u')$ and $\varphi''(x^{-1}) = \partial\psi(v')$ where x is the arc on e directed from u to v. Then φ'' is a nowhere-zero $\mathbb{Z}_k$-flow in G, a contradiction. This implies the statement. $\square$

We say that G is *strongly $\mathbb{Z}_k$-connected* in a vertex v of G if every mapping $\psi : \omega_G^+(v) \to \mathbb{Z}_k \setminus \{0\}$ satisfying $\sum_{x\in\omega_G^+(v)} \psi(x) = 0$ can be extended to a nowhere-zero $\mathbb{Z}_k$-flow in G.

An edge cut is called *trivial* if it is the set of edges incident to a single vertex.

Theorem 1. *The following statements are pairwise equivalent.*

(1) *Every 4-edge-connected graph is strongly $\mathbb{Z}_3$-connected in each vertex of degree 4 or 5.*
(2) *Every 4-edge-connected graph has a nowhere-zero 3-flow.*
(3) *Every 5-edge-connected graph has a nowhere-zero 3-flow.*
(4) *Every 5-edge-connected graph with no nontrivial 5-edge-cut is strongly $\mathbb{Z}_3$-connected in each vertex of degree 5.*
(5) *Every 4-edge-connected graph with each vertex of degree 4 or 5 is strongly $\mathbb{Z}_3$-connected in every vertex.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. It is well known (see, e.g., [6, Theorem 6.6]) that (2) may be reduced to 5-regular 4-edge-connected graphs. Similarly (5) $\Rightarrow$ (1).

We prove (4) $\Rightarrow$ (5). Suppose that G is a 4-edge-connected graph with each vertex of degree 4 or 5 which is not strongly $\mathbb{Z}_3$-connected in a vertex v and let $|V(G)|$ be the smallest possible. Then $|V(G)| \geq 4$ (otherwise every pair of distinct vertices of G is joined by at least two parallel edges, whence G is strongly $\mathbb{Z}_3$-connected in every vertex).

Suppose that G has a nontrivial 4- or 5-edge cut C. Consider graphs G_1 and G_2 arising from G after contracting the subgraphs which are components in $G - C$ to the new vertices w_1 and w_2, respectively, and deleting all loops. Without loss of generality we can assume $v \in V(G_1)$. Since $|V(G_1)|, |V(G_2)| < |V(G)|$, G_1 and G_2 are strongly $\mathbb{Z}_3$-connected in v and w_2, respectively. Then, clearly, G is strongly $\mathbb{Z}_3$-connected in v, a contradiction.

Thus there is no nontrivial 4- or 5-edge cut in G. Suppose that G contains a vertex v' of degree 4 and distinct from v. If v' is joined with a vertex v'' by at least two parallel edges, then the edges with exactly one end in $\{v', v''\}$ form a nontrivial edge cut of cardinality ≤ 5 (because $|V(G)| \geq 4$). Thus there are four distinct vertices $v_1, \ldots, v_4$ adjacent to v'. Adding to $G - v'$ new edges (v_1, v_2) and (v_3, v_4), we get a 4-edge-connected graph G' with vertices of degrees 4 and 5.

Since $|V(G')| < |V(G)|$, G' is strongly $\mathbb{Z}_3$-connected in v, whence G is strongly $\mathbb{Z}_3$-connected in v, a contradiction.

Thus all vertices except possibly v have degree 5. Then either G (if v has degree 5) or the graph arising from G after replacing one of the edges incident to v with two parallel edges (if v has degree 4) is a counterexample to (3). Therefore (4) $\Rightarrow$ (5).

We prove (3) $\Rightarrow$ (4). Suppose that G is a 5-edge-connected graph containing a vertex v of degree 5 such that G is not strongly $\mathbb{Z}_3$-connected in v. Then the arcs from $\omega_G^+(v)$ can be denoted as $x_1, \ldots, x_5$ so that there does not exist a nowhere-zero $\mathbb{Z}_3$-flow φ in G satisfying $\varphi(x_1) = \cdots = \varphi(x_4) = 1$ and $\varphi(x_5) = 2$. Let e_i be the edge of G corresponding to x_i for $i = 1, \ldots, 5$. Replace the vertex v with three vertices v_1, v_2, v_3 so that v_1 is incident to e_1, e_2; v_2 is incident to e_3, e_4; and v_3 is incident to e_5 and add two new edges (v_1, v_3), (v_2, v_3). We get a graph H which does not have a nowhere-zero 3-flow. H has precisely three vertices of degree 3, namely, v_1, v_2, v_3. Take the graph K_4 and let $w_1, \ldots, w_4$ be its vertices. Replace the edge (w_1, w_2) with a copy of H more precisely, delete the edge (w_1, w_2) and identify the pairs of vertices v_1, w_1 and v_2, w_2 to two new vertices of degree 5. In the same way replace the edge (w_3, w_4) with another copy of H. We get a graph H' having exactly two vertices of degree three, v_3 and v_3'. Repeat this process and replace every edge of K_4 with a copy of H' so that the copies of v_3 and v_3' are identified with the vertices of K_4. We get a 5-edge-connected graph H''. It is known (see [2, 9]) that K_4 does not admit a nowhere-zero 3-flow. Hence, by Lemma 1, H' and H'' do not admit nowhere-zero 3-flows. Thus (3) $\Rightarrow$ (4). $\square$

Let v be a vertex of K_4 and e_1, e_2, e_3 be the edges incident to v. Suppose that H is the graph arising from K_4 after replacing e_i with n_i (≥ 2) parallel edges $e_{i,1}, \ldots, e_{i,n_i}$ for $i = 1, 2, 3$. Let x_i ($x_{i,j}$) be the arc on e_i ($e_{i,j}$) directed out from v for $i = 1, 2, 3$ and $j = 1, \ldots, n_i$. Then H does not admit a nowhere-zero $\mathbb{Z}_3$-flow ψ satisfying $\sum_{j=1}^{n_i} \psi(x_{i,j}) = 1$ for $i = 1, 2, 3$, because otherwise we get a nowhere-zero $\mathbb{Z}_3$-flow φ in K_4 (setting $\varphi(x) = \psi(x)$ for $x \in D(K_4 - v)$ and $\varphi(x_i) = -\varphi(x_i^{-1}) = 1$ for $i = 1, 2, 3$). Thus, for every $n \geq 6$ there exists a 4-edge-connected graph H which is not strongly $\mathbb{Z}_3$-connected in a vertex v of degree n.

Note that, by [6], every 5-edge-connected graph has a nowhere-zero 3-flow if and only if every 5-regular 5-connected simple hamiltonian graph has a nowhere-zero 3-flow.

A *circular k-flow* φ in a graph G is a nowhere-zero $\mathbb{Z}_k$-flow in G such that $\varphi(x) \in \{\pm 1\}$ for every $x \in D(G)$. The circular flow conjecture of Jaeger [1] is that every $4t$-edge-connected graph has a circular $(2t+1)$-flow. For $t = 1$ this is equivalent to the 3-flow conjecture. For $t = 2$ Jaeger [1] proved that the 5-flow conjecture holds true if every 9-edge-connected graph has a circular 5-flow (see [1, 6, 9] for more details). Thus the 3- and 5-flow conjectures are implied by the following one.

Conjecture 1. Every $(4t+1)$-edge-connected graph has a circular $(2t+1)$-flow.

Since Conjecture 1 is equivalent to the circular flow conjecture for $t = 1$, it would be of some interest to study whether these two conjectures are equivalent for every $t \geq 2$.

A graph G is called *A-connected* if for every mapping $b : V(G) \to A$ satisfying $\sum_{v \in V(G)} b(v) = 0$ there exists a nowhere-zero A-chain φ in G such that $\partial\varphi(v) =$

$b(v)$ for every vertex v of G (see [3, 9]). Jaeger, Linial, Payan and Tarsi [3] constructed 4-edge-connected graphs which are not $\mathbb{Z}_3$-connected and conjectured that every 5-edge-connected graph is $\mathbb{Z}_3$-connected. This conjecture, if true, would imply the 3-flow conjecture by Theorem 1.

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MÚ SAV, Štefánikova 49, 814 73 Bratislava 1, Slovakia
E-mail address: `kochol@savba.sk`

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