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Perfect Effect Algebras are Categorically Equivalent with Abelian Interpolation po-Groups

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Abstract

We introduce perfect effect algebras and we show that every perfect algebra is an interval in the lexicographical product of the group of all integers with an Abelian directed interpolation po-group. To show that we introduce prime ideals of effect algebras with the Riesz decomposition property, (RDP). We show that the category of perfect effect algebras is categorically equivalent with the category of Abelian directed interpolation po-groups. Moreover, we prove that any perfect effect algebra is a subdirect product of antilattice effect algebras with the (RDP).

1 Introduction

Effect algebras are partial algebras with a partially defined addition $+$. Such algebras entered mathematics in 1994 by Foulis and Bennett [FoBe] as a common basis for the system $\mathcal{E}(H)$ of all Hermitian operators A on on a Hilbert space H such that $O \le A \le I$, where O and I are the zero and identity

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operators on a Hilbert space H. Such operators are of great importance for modelling events of unsharp nature in quantum mechanics. More about effect algebras or on equivalent structures, D-posets introduced in [KoCh], is in [DvPu].

One of a very important class of effect algebras is the class of effect algebras with the Riesz decomposition property, i.e., when roughly speaking we can do a refinement of decompositions of unity 1. Such algebras are always intervals in unital interpolation po-groups, see [Rav]. In addition, every MV-algebra (introduced by Chang [Cha] in 1957), which models Boolean algebras as well as many-valued reasoning, can be understood as a lattice effect algebra with the Riesz decomposition property. Such algebras are always intervals in unital ℓ-groups.

Motivating by MV-algebras, we introduce perfect effect algebras. In such algebras every element is either in its radical or in its logical complement. Such algebras are always intervals in the lexicographical product of the group of all integers with an Abelian directed interpolation po-group. Moreover, we show that we have a categorical equivalence of the category of perfect effect algebras and the category od Abelian interpolation directed po-groups. This generalizes an analogical result of Di Nola and Lettieri [DiLe] for perfect effect algebras.

The paper is organized as follows. Effect algebras are presented in Section 2 with the main accent to the Riesz decomposition property. Ideals of effect algebras (maximal ideals, prime and values) are studied in Section 3. In Section 4, we introduce infinitesimal elements of effect algebras and the radical, the most important part of any effect algebra. We show that in contrast to MV-algebras, not every radical consists of all its infinitesimal. Therefore, we introduce effect algebras with the Rad-property. Perfect effect algebras are introduced in Section 5, where we show the categorical equivalence of the category of perfect effect algebras with the category of Abelian directed interpolation po-groups. In Section 6, we study quotient effect algebras, and in Section 7 we prove that any perfect effect algebra is a subdirect product of antilattice effect algebras with the (RDP).

2 Effect Algebras with the Riesz Decomposition Property

An *effect algebra* is a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

(i) $a+b$ is defined in E iff $b+a$ is defined, and in such the case $a+b = b+a$;

(ii) $a + b, (a + b) + c$ are defined iff $b + c$ and $a + (b + c)$ are defined, and in such the case $(a + b) + c = a + (b + c)$;

(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a+a' = 1$;

(iv) if $a + 1$ is defined in E, then $a = 0$.

If we define $a \leq b$ iff there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering, and we write $c := b - a$.

For example, if (G, u) is an Abelian unital po-group with a strong unit u, [2] and if $\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$ is endowed with the restriction of the group addition $+$, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra.

Let E and F be two effect algebras. A mapping $h : E \to F$ is said to be a *homomorphism* if (i) $h(a+b) = h(a)+h(b)$ whenever $a+b$ is defined in E, and (ii) $h(1) = 1$. A bijective homomorphism h such that h^{-1} is homomorphism is said to be an *isomorphism* of E and F.

We say that an effect algebra E satisfies (i) the *Riesz interpolation property*, (RIP) for short, if, for all x_1, x_2, y_1, y_2 in E, $x_i \leq y_j$ for all i, j implies there exists an element $z \in E$ such that $x_i \leq z \leq y_j$ for all i, j; (ii) the *Riesz decomposition property*, (RDP) for short, if $x \leq y_1 + y_2$ implies that there exist two elements $x_1, x_2 \in$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 + x_2$.

We recall that (1) if E is a lattice, then E has trivially the (RIP); the converse is not true as wee see below. (2) E has the (RDP) iff, [DvPu, Lem 1.7.5], $x_1 + x_2 = y_1 + y_2$ implies there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$, and $y_2 = c_{12} + c_{22}$. (3) the (RDP) implies the (RIP), but the converse is not true (e.g. if $E = L(H)$, the system of all closed subspaces of a Hilbert space H, then E is a complete lattice but without the (RDP)). On the other hand, every finite poset with the (RIP) is a lattice.

[2] An element $u \in G^+$ is said to be a *strong unit* for a po-group G, if given an element $g \in G$, there is an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

We recall that a poset $(E; \leq)$ is an *antilattice* if only comparable elements of E have a supremum (infimum). It is clear that any linearly ordered poset is an antilattice and every finite effect algebra with the (RIP) is a lattice.

There exists an effect algebra with (RIP) which is not a lattice:

Example 2.1 *Let G be the additive group $\mathbb{R}^2$ with the positive cone of all (x, y) such that either $x = y = 0$ or $x > 0$ and $y > 0$. Then $u = (1,1)$ is a strong unit for G. The effect algebra $E = \Gamma(G, u)$ is an antilattice having the* (RIP) *and the* (RDP) *but E is not a lattice.*

A partially ordered Abelian group $(G; +, 0)$ is said to satisfy the *Riesz decomposition property* provided, given x, y_1, y_2 in G^+ such that $x \leq y_1 + y_2$, there exist x_1, x_2 in G^+ such that $x = x_1 + x_2$ and $x_j \leq y_j$ for each j. This condition is equivalent by [Goo, Prop. 2.1] with the following two equivalent conditions:

(a) Given x_1, x_2, y_1, y_2 in G such that $x_i \leq y_j$ for all i, j, there exists z in G such that $x_i \leq z \leq y_j$ for all i, j.

(b) Given x_1, x_2, y_1, y_2 in G^+ such that $x_1 + x_2 = y_1 + y_2$, there exist $z_{11}, z_{12}, z_{21}, z_{22}$ in G^+ such that $x_i = z_{i1} + z_{i2}$ for each i and $y_j = z_{1j} + z_{2j}$ for each j.

According to [Goo], a group G with the Riesz decomposition property is said to be the *interpolation group*.

It is clear that if (G, u) is a unital group interpolation group, then $E = \Gamma(G, u)$ is with the (RDP).

We recall that by a *universal group* for an effect algebra E we mean a pair (G, γ) consisting of an additive Abelian group G and a G-valued measure $\gamma : E \to G$ (i.e., $\gamma(a + b) = \gamma(a) + \gamma(b)$ whenever $a + b$ is defined in E) such that the following conditions hold: (i) $\gamma(E)$ generates G. (ii) If H is an additive Abelian group and $\phi : E \to H$ is an H-valued measure, then there is a group homomorphism $\phi^* : G \to H$ such that $\phi = \phi^* \circ \gamma$. Due to [FoBe], every effect algebra possesses a universal group.

Ravindran [Rav] ([DvPu, Thm 1.17.17]) proved the following important result.

Theorem 2.2 *Let E be an effect algebra with the Riesz decomposition property. Then there exists a unital interpolation group (G, u) with a strong unit u such that $\Gamma(G, u)$ is isomorphic with E, and there is a G-valued injective measure γ such that (G, γ) is a universal group for E.*

Let G be a directed Abelian po-group and define the lexicographical product

$$G(\mathbb{Z}) := \mathbb{Z} \times_{lex} G, \tag{2.1}$$

where $\mathbb{Z}$ is the group of all integers. Then the element $(1,0)$ is a strong unit in the po-group $G(\mathbb{Z})$ and

$$E(G) := \Gamma(G(\mathbb{Z}),(1,0)), \tag{2.2}$$

is an effect algebra. Every element $a \in E(G)$ is of the form either $a = (1,-g)$ or $a = (0,g)$, where $g \in G^+$. In addition, if G is a directed interpolation group, then $G(\mathbb{Z})$ is an interpolation group, [Goo, Cor. 2.12], and $E(G)$ satisfies the Riesz interpolation property.

In what follows, we introduce perfect effect algebras, and we show that every perfect effect algebra is isomorphic with $E(G)$ for some interpolation Abelian po-group G.

3 Ideals of Effect Algebras

In the present Section, we introduce ideals, prime and maximal ideals and values.

Let a be any element of an effect algebra E and n an integer ($n \ge 0$). We define recurrently

$$0\,a := 0, \quad 1\,a = a, \quad (n+1)\,a = n\,a + a, \ n \ge 1,$$

supposing that $n\,a$ and $n\,a + a$ are defined in E. By the *isotropic index* of an element $a \in E$, $i(a)$, we mean

$$i(a) := \sup\{n \ge 1 : \ n\,a \in E\}.$$

An *ideal* of an effect algebra E is a non-empty subset I of E such that (i) $x \in E$, $y \in I$, $x \le y$ imply $x \in I$, and (ii) if $x, y \in I$ and $x + y$ is defined in E, then $x + y \in I$. An ideal I is said to be the *Riesz ideal* if $x \in I$, $a, b \in E$ and $x \le a + b$, there exist $a_1, b_1 \in I$ such that $x \le a_1 + b_1$ and $a_1 \le a$ and $b_1 \le b$.

For example, if E is with the (RDP), then any ideal of E is Riesz.

We denote by $\mathcal{I}(E)$ the set of all ideals of E; then $\{0\}, E \in \mathcal{I}(E)$. We recall that if E is linearly ordered, then $\mathcal{I}(E)$ is linearly ordered with respect to the set-theoretical inclusion. Indeed, let I_1, I_2 be two ideals of E and assume $a \in I_1 \setminus I_2$ and $b \in I_2 \setminus I_1$. We can assume that, e.g., $a \le b$ which gives $a \in I_2$, a contradiction.

Proposition 3.1 *Let A be a subset of an effect algebra E satisfying the* (RDP)*. Then the ideal $I_0(A)$ of E generated by A is the set*

$$\begin{aligned} I_0(A) = \{x \in E : \ & \exists\, n \in \mathbb{N},\ \exists\, a_1^0, \ldots, a_n^0 \in E, \exists\, a_1, \ldots, a_n \in A, \\ & a_i^0 \le a_i,\ i = 1, \ldots, n, \quad x = a_1^0 + \cdots + a_n^0\}. \end{aligned}$$

Proof. If $A = \emptyset$, then $I_0(A) = \{0\}$. Let $A \neq \emptyset$. Then $A \subseteq I_0(A)$. Assume $x \in E$, $y \in I_0(A)$ and $x \le y$. Then $y = a_1^0 + \cdots + a_n^0$, where $a_i^0 \in E$, $a_i^0 \le a_i$ for some $a_1, \ldots a_n \in A$. The (RDP) implies $x \in I_0(A)$.

Suppose $a, b \in I_0(A)$ and $a+b \in E$. Then $a = a_1^0 + \cdots + a_n^0$, $b = b_1^0 + \cdots + b_m^0$, where $E \ni a_i^0 \le a_i \in A$, $E \ni b_j^0 \le b_j \in A$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$ which gives $a + b \in I_0(A)$.

Since any ideal containing A must contains $I_0(A)$, we have the assertion in question. □

Proposition 3.2 *Let J be an ideal of an effect algebra E with the* (RDP) *and $a \in E$. Then the ideal $I_0(J, a)$ of E generated by J and a is given by*

$$\begin{aligned} I_0(J, a) = \{x \in E : \ & \exists\, y \in J,\ \exists\, k \in \mathbb{N},\ \exists\, a_1^0, \ldots, a_k^0 \in E, \\ & a_i^0 \le a,\ x = y + a_1^0 + \cdots + a_k^0\}. \end{aligned}$$

Proof. It follows from Proposition 3.1 □

In particular, the ideal $I_0(a)$ of E generated by an element a is by Proposition 3.2

$$I_0(a) = \{x \in E : \ \exists\, k \in \mathbb{N},\ \exists\, a_1^0, \ldots, a_k^0 \in E,\ a_i^0 \le a,\ x = a_1^0 + \cdots + a_k^0\}. \tag{3.1}$$

Let a be a non-zero element of E. By Zorn's lemma, there exists an ideal V not containing a and which is maximal with respect to this property. Such an ideal is said to be a *value* of a in E. We denote by $\Gamma(a)$ the set of all values of the element a. We say that an ideal J *covers* I if $I \subset J$ and, for any ideal K of E, $I \subseteq K \subseteq J$ implies either $I = K$ or $J = K$; and we say that J is a *cover* of I.

Proposition 3.3 *Every value in an effect algebra E has a cover.*

Proof. Let $a > 0$ and V be a value of a in E. Let $V(a)$ be the intersection of all ideals of E that contain V and a. Then $V(a)$ covers V: For it if J is an ideal of E and $J \supset V$, then $a \in J$ by the maximality of V, hence $V(a) \subseteq J$. $\square$

Proposition 3.4 *For any effect algebra E satisfying the* (RDP), *$\mathcal{I}(E)$ is a distributive lattice. Moreover, for all $I, I_s \in \mathcal{I}(E)$ $(s \in S)$, $I \cap \bigvee_s I_s = \bigvee_s (I \cap I_s)$. In addition, $I \vee (I_1 \cap I_2) = (I \vee I_1) \cap (I \vee I_2)$.*

Proof. Let $\{I_s\}_{s\in S}$ be a family of ideals of E and let I_0 be the ideal of E generated by $\bigcup_s I_s$. Clearly, $\bigvee_s (I \cap I_s) \subseteq I \cap \bigvee_s I_s$. Let $x \in I \cap \bigvee_s I_s$, then by Proposition 3.1, $x = x_1 + \cdots + x_n$, with $x_i \in I_{s_i}$, $i = 1 \ldots, n$, and while simultaneously $x \in I$, we have $x_i \in I$ for $i = 1, \ldots, n$ which gives $x \in \bigvee_s (I \cap I_s)$.

For the rest, it is clear that $I \vee (I_1 \cap I_2) \subseteq (I \vee I_1) \cap (I \vee I_2)$. Assume $x \in (I \vee I_1) \cap (I \vee I_2)$. Then by Proposition 3.2, $x = x_1 + x_2 = y_1 + y_2$, where $x_1, y_1 \in I$ and $x_2 \in I_1$, $y_2 \in I_2$. By the (RDP), there exists four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$ and $y_2 = c_{12} + c_{22}$. Hence, $c_{11}, c_{12}, c_{21} \in I$ and $c_{22} \in I_1 \cap I_2$. Hence $x = (c_{11} + c_{12} + c_{21}) + c_{22} \in I \vee (I_1 \cap I_2)$. $\square$

We say that an ideal P of an effect algebra E with the (RDP) is *prime* if, for all ideals I and J of E, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. We denote by $\mathcal{P}(E)$ the set of all prime ideals of E.

Proposition 3.5 *If an effect algebra E satisfies the* (RDP), *then every value is a prime ideal of E.*

Proof. Let V be a value of an element $a \in E \setminus \{0\}$. Assume $I \cap J \subseteq V$. By Proposition 3.4, $V = V \vee (I \cap J) = (V \vee I) \cap (V \vee J)$. Since $a \notin V$, we have that either $a \notin V \vee I$ or $a \notin V \vee J$. The maximality of V gives $V = V \vee I$ or $V = V \vee J$. $\square$

Proposition 3.6 *Let E be an effect algebra with the* (RDP). (1) *The following statements are equivalent*

(i) *P is a prime ideal.*

(ii) $I \cap J = P$ *implies* $P = I$ *or* $P = J$.

(iii) $I_0(a) \cap I_0(b) \subseteq P$, $a, b \in E$, *imply* $a \in P$ *or* $b \in P$.

(2) *If*

(iv) *The set* $\{I \in \mathcal{I}(E) : I \supseteq P\}$ *is linearly ordered with respect to the set-theoretical inclusion,*

then (iv) *implies any of* (i)–(iii).

Proof. (i) $\Rightarrow$ (ii). Let $I \cap J = P$. By (i), $I \subseteq P$ or $J \subseteq P$. Assume $I \subseteq P$, then $I \subseteq P = I \cap J \subseteq J$ which proves $I = P$.

(ii) $\Rightarrow$ (i). Let $I \cap J \subseteq P$. By Proposition 3.4, $P = P \vee (I \cap J) = (P \vee J) \cap (P \vee J)$ which gives $P = P \vee I$ or $P = P \vee J$, i.e., $P \supseteq I$ or $P \supseteq J$.

(i) $\Rightarrow$ (iii). Let $I_0(a) \cap I_0(b) \subseteq P$. Then $I_0(a) \subseteq P$ or $I_0(b) \subseteq P$, i.e., $a \in P$ or $b \in P$.

(iii) $\Rightarrow$ (i). Let $I \cap J \subseteq P$. Then $I = \bigvee_{a \in I} I_0(a)$ and $J = \bigvee_{b \in J} I_0(b)$. By Proposition 3.4, $I \cap J = \bigvee_{a,b}(I_0(a) \cap I_0(b)) \subseteq P$. If we would found some $a \in I \setminus P$ and some $b \in J \setminus P$, then $I_0(a) \cap I_0(b) \subseteq P$ which gives $a \in P$ or $b \in P$, a contradiction.

(iv) $\Rightarrow$ (ii). Let $P = I \cap J$. Then $P \subseteq I$ and $P \subseteq J$, so that $I \subseteq J$ or $J \subseteq I$ which proves $P = I$ or $P = J$. $\square$

It would be interesting to know whether is (iv) in Proposition 3.6 equivalent with (i).

We present now a criterion when an effect algebra is a lattice with the (RDP) using the notion of prime ideals.

Proposition 3.7 *Let an effect algebra* E *satisfy the condition: For any* $a_1, a_2, b_1, b_2 \in E$ *with* $a_1 + a_2 = b_1 + b_2$ *there exists four elements* $c_{11}, c_{12}, c_{21}, c_{22} \in E$ *such that* $a_1 = c_{11}+c_{12}$, $a_2 = c_{21}+c_{22}$, $b_1 = c_{11}+c_{21}$, *and* $b_2 = c_{12}+c_{22}$, *and such that, for any prime ideal* P *of* E, $c_{12} \in P$ *or* $c_{21} \in P$. *Then* $c_{12} \wedge c_{21} = 0$, *and* E *is a lattice with the* (RDP).

Proof. It evident that E satisfies the (RDP). Assume now $x \leq c_{12}, c_{21}$ and let $x \neq 0$. Take a value P of x. By Proposition 3.5, P is a prime ideal. Hence, $c_{12} \in P$ or $c_{21} \in P$, which gives $x \in P$, a contradiction. Hence $x = 0$.

Claim. If E is an effect algebra with the (RDP) and $a \wedge b = 0$, then $a+b, a \vee b$ are defined in E, and $a \vee b = a + b$.

Indeed, if x is an upper bound of a and b, then $x = c + a$ for some $c \in E$. Hence, $b = b_1 + b_2$, where $b_1 \leq c$ and $b_2 \leq a$, which yields $b_2 = 0$ and therefore, $b = b_1$ and $b + a$ is defined in E. It is clear now that $a + b = a \vee b$.

To finish the proof, we see that $a + a' = 1 = b' + b$ which gives four elements $x_{11}, x_{12}, x_{21}, x_{22} \in E$ such that $a = x_{11} + x_{12}$, $a' = x_{21} + x_{22}$, $b = x_{11} + x_{21}$ and $b' = x_{21} + x_{22}$. We can assume that $x_{12} \wedge x_{21} = 0$. Hence we have $x_{11} + (x_{12} \vee x_{21}) = x_{11} + x_{12} + x_{21} \in E$ by above, so that $x_{11} + (x_{12} \vee x_{21}) = (x_{11} + x_{12}) \vee (x_{11} + x_{21}) = a \vee b \in E$. $\square$

Example 3.8 *Let* (G, u) *be a unital po-group, with* (RDP) *from Example* 2.1*, then* $E = \Gamma(G, u)$ *has only one maximal ideal* I*, namely* $I = \{(0,0)\}$*. In addition,* E *is an antilattice, and* $\mathcal{I}(E) = \{\{0\}, E\}$*.*

Example 3.9 *Let* (G, u) *be a unital po-group, where* $G = \mathbb{R}^2$ *a* $u = (1,1)$*, and* $G^+ := \{(0,0)\} \cup \{(x,y) \in \mathbb{R}^2 : \ x > 0, \ y \geq 0\}$*. Then* (G, u) *is with the* (RDP)*, and* $I := \{(x, 0) \in \mathbb{R}^2 : \ 0 \leq x < 1\}$ *is a unique maximal ideal of* $E = \Gamma(G, u)$*. In addition,* E *is an antilattice, and* $\mathcal{I}(E) = \{\{0\}, I, E\}$*.*

From [Gla, Lem 3.3.3] we have that if G is an ℓ-group, then an ideal P of G is prime iff, for all $f, g \in G$, there is a $c \in P$ such that $f \leq c + g$ or $g \leq c + f$. We recall that an analogical assertion is not valid for all effect algebras. Indeed, take E from Example 3.9, and set $f = (0.4, 0.5)$ and $g = (0.4, 0.6)$. Then there is no $c = (x, 0) \in I$ such that $f \leq c + g$ or $g \leq c + f$.

4 Infinitesimal Elements of Effect Algebras

In the present Section, we define infinitesimal elements of an effect algebra E, and we show that it is an element of the radical of E. Radicals, as intersection of all maximal ideals, are important parts of effect algebras, containing many information on the effect algebras. But in contrast to lattice effect algebras with the (RDP), it can happen that not every element of the radical is an infinitesimal.

It is possible to prove that if $a \wedge b$ is defined in an effect algebra E, then

$$(a - (a \wedge b)) \wedge (b - (a \wedge b)) = 0. \tag{4.1}$$

An element a is said to be *infinitesimal* if $i(a) = \infty$, and denote by $\mathrm{Infinit}(E)$ the set of all infinitesimals of E. Then (i) $0 \in \mathrm{Infinit}(E)$, (ii) if $b \in E$, $a \in \mathrm{Infinit}(E)$ and $b \leq a$, then $b \in \mathrm{Infinit}(E)$, and (iii) $1 \notin \mathrm{Infinit}(E)$.

We say that an effect algebra E is *Archimedean* if $\mathrm{Infinit}(E)=\{0\}$.

A proper ideal I of an effect algebra E is said to be *maximal* if is not a proper subset of another proper ideal of E (or equivalently, I is a value of 1). By Zorn's lemma E possesses at least one maximal ideal, and let $\mathcal{M}(E)$ be the set of all maximal ideals of E. We define the *radical* of E, $\mathrm{Rad}(E)$, via

$$\mathrm{Rad}(E)=\bigcap\{I:\ I\in\mathcal{M}(E)\}.$$

Proposition 4.1 *Let E be an effect algebra satisfying the* (RDP). *Then*

$$\mathrm{Infinit}(E)\subseteq\mathrm{Rad}(E). \tag{4.2}$$

Proof. Suppose that $a\notin\mathrm{Rad}(E)$. Then there exists a maximal ideal I of E such that $a\notin I$. Hence, the ideal $I_0(I,a)$ generated by I and a must coincide with E. Therefore, there exist elements $z\in I$, $a_1^0,\ldots,a_k^0\in E$ with $a_i^0\le a$ $(i=1,\ldots,k)$ such that $1=z+a_1^0+\cdots+a_k^0$. If $a\in\mathrm{Infinit}(E)$, then $ka\le a'$ and one would have $z+a_1^0+\cdots+a_k^0=a+a'$, so that $z=a+(a'-(a_1^0+\cdots+a_k^0))\ge a$, whence $a\in I$, a contradiction. Hence, $a\notin\mathrm{Infinit}(E)$ and $\mathrm{Infinit}(E)\subseteq\mathrm{Rad}(E)$. $\square$

Proposition 4.2 *Let E be a lattice effect algebra with the* (RDP). *If $a_1+a_2=b_1+b_2$ for some $a_1,a_2,b_1,b_2\in E$, then*

$$\begin{aligned} a_1-(a_1\wedge b_1) &= b_2-(a_2\wedge b_2),\\ a_2-(a_2\wedge b_2) &= b_1-(a_1\wedge b_1).\end{aligned}$$

Proof. Due to the (RDP), there are four elements $c_{11},c_{12},c_{21},c_{22}\in E$ such that $a_1=c_{11}+c_{12}$, $a_2=c_{21}+c_{22}$, $b_1=c_{11}+c_{21}$, and $b_2=c_{12}+c_{22}$. Due to [DvPu, Sec.1.10], we can assume that $c_{12}\wedge c_{21}=0$. Then $c_{11}=c_{11}+0=(c_{11}+c_{12})\wedge(c_{11}+c_{21})=a_1\wedge b_1$, and in a similar way $c_{22}=a_2\wedge b_2$, which proves the assertion in question. $\square$

It is valuable to recall that it can happen that in (4.2) we have no equality. For example, in Example 3.9, $\mathrm{Infinit}(E)=\{(0,0)\}$ and $\mathrm{Rad}(E)=\{(x,0):0\le x<1\}$, while in Example 3.8, $\mathrm{Infinit}(E)=\{(0,0)\}=\mathrm{Rad}(E)$. Inspiring that, we are saying that an effect algebra E has the *Rad-property* if

$$\mathrm{Infinit}(E)=\mathrm{Rad}(E). \tag{4.3}$$

An effect algebra E is said to be *simple* if it has only trivial ideals. For example, Example 3.8 gives a simple effect algebra.

Theorem 4.3 *Assume that E is an effect algebra with the* (RDP) *of one of the following kinds:*

(1) E is a lattice.

(2) E is a simple effect algebra.

(3) $E = E(G) := \Gamma(G(\mathbb{Z}),(1,0))$, where $G(\mathbb{Z})$ is the lexicographical product of $\mathbb{Z}$ with a directed Abelian po-group G with interpolation.

(4) For any $x \in E$, $x \le 1 - x$ or $1 - x \le x$.

Then E has the Rad-property. In addition, in (3), $\mathrm{Infinit}(E)$ is a unique maximal ideal of E.

Proof. (1) Let $a > 0$ be not infinitesimal. Then there exists a natural number $m \ge 1$ such that $ma \in E$ and $ma \not\le a'$. This yields by Proposition 4.2 $c_1 := a - ((ma)' \wedge a) = (ma) - ((ma) \wedge a')$, and $c_1 > 0$. Let P be a value of c_1; by Proposition 3.5, P is a prime ideal of E. Assume that M is any maximal ideal of E containing P. Since $((ma) - ((ma) \wedge a')) \wedge (a' - ((ma) \wedge a')) = 0$ by (4.1), we have by Proposition 3.6, $a' - ((ma) \wedge a') \in P \subseteq M$. The maximality of M entails $(ma) \wedge a' + a = (a' - ((ma) \wedge a'))' \notin M$. Hence $a \notin M$ which proves $\mathrm{Rad}(E) \subseteq \mathrm{Infinit}(E)$.

(2) It is evident.

(3) It is clear that $\mathrm{Infinit}(E(G)) = \{(0,g) :\ g \in G^+\}$, and $\mathrm{Infinit}(E(G))$ is an ideal of $E(G)$. We assert that it is a (unique) maximal ideal of $E(G)$. Indeed, take $(1,-g)$, where $g \in G^+$. Then $(1,-g)' = (0,g)$, so that $(1,0)$ is an element of the ideal generated by $\mathrm{Infinit}(E(G))$, and the element $(1,-g)$.

(4) Assume $x \in \mathrm{Rad}(E)$. Then only $x \le 1 - x$, otherwise $1 \in \mathrm{Rad}(E)$, which is impossible. Hence, $x + x$ is defined in E. Since $\mathrm{Rad}(E)$ is an ideal of E, we have $2x \in \mathrm{Rad}(E)$. Repeating this process with $2x$, we see that $2x + 2x \in \mathrm{Rad}(E)$, etc. Therefore, nx is defined in E for any integer $n \ge 1$, i.e., $x \in \mathrm{Infinit}(E)$. $\square$

5 Perfect Effect Algebras and Abelian po-Groups with Interpolation

In this Section, we give the main result on perfect effect algebras showing that every perfect algebra E is of the form (2.2) for some interpolation directed po-group G. In addition, we show that the category of perfect effect algebras is categorically isomorphic with the category of Abelian directed interpolation po-groups, which generalizes an analogical result of Di Nola and Lettieri [DiLe] who proved that for MV-algebras (= lattice effect algebras with the (RDP)).

Proposition 5.1 *Let E be an effect algebra with the* (RDP) *satisfying the Rad-property. Then* $\mathrm{Rad}(E)$ *is a commutative semigroup with respect to* $+$ *such that*

(i) 0 *is a neutral element,*

(ii) *the cancellation law holds in* $\mathrm{Rad}(E)$*,*

(iii) *if* $a+b=0$ *for* $a,b\in \mathrm{Rad}(E)$*, then* $a=b=0$.

(iv) $\mathrm{Rad}(E)\cap \mathrm{Rad}'(E)=\emptyset$*,*[3]

(v) *if* $a,b\in \mathrm{Rad}'(E)$*, then* $a+b$ *is not defined in* E*,*

(vi) *if* $a\in \mathrm{Rad}(E)$ *and* $b\in \mathrm{Rad}'(E)$*, then* $a\le b$.

Proof. Assume $a,b\in \mathrm{Rad}(E)$. Then $a=a_1+c$ and $b=b_1+c$, where $a_1,b_1,c\in E$ and $a_1+b_1+c\in E$. Then $a_1,b_1,c\in \mathrm{Rad}(E)$. Due to the Rad-property, we have $a_1+b_1+c\in \mathrm{Infinit}(E)=\mathrm{Rad}(E)$, therefore, $(a_1+b_1+c)+(a_1+b_1+c)\in E$ which gives $a+b$ is defined in E, consequently, $a+b\in \mathrm{Rad}(E)$. The conditions (i)–(iii) are now evident.

(iv) Assume $a\in \mathrm{Rad}(E)\cap \mathrm{Rad}'(E)$. Then $a\in \mathrm{Rad}'(E)$, so that $a'\in \mathrm{Rad}(E)$ which gives $1=a+a'\in \mathrm{Rad}(E)$, a contradiction.

(v) Assume $a+b\in E$. Then $a\le b'\in \mathrm{Rad}(E)$, which implies $a\in \mathrm{Rad}(E)$, a contradiction.

(vi) If $b\in \mathrm{Rad}'(E)$, then $b'\in \mathrm{Rad}(E)$, which implies $a+b'$ is defined in $\mathrm{Rad}(E)$. Hence $a\le (b')'=b$. $\square$

We say that an effect algebra E with the (RDP) and the Rad-property is *perfect* if, for any element $a\in E$ either $a\in \mathrm{Rad}(E)$ or $a'\in \mathrm{Rad}(E)$. For example, $E(G)$, where $E(G)$ is defined by (2.2) for a directed po-group G with interpolation is a perfect algebra. In what follows, we show that every perfect effect algebra is of this nature.

Let $\mathcal{PEA}$ be the category of all perfect effect algebras where the objects are perfect effect algebras and morphisms are homomorphisms of effect algebras. Let $\mathcal{GI}$ be the category of Abelian po-groups, where the objects are

[3] If $A\subseteq E$, then $A':=\{a' :\ a\in A\}$.

directed Abelian po-groups with interpolation and morphisms are homomorphisms of po-groups, i.e., positive homomorphisms of po-groups.

We define a mapping $\mathcal{E}: \mathcal{GI} \to \mathcal{PEA}$ by

$$\mathcal{E}(G) := E(G) = \Gamma(\mathbb{Z} \times_{lex} G, (1,0)), \tag{5.1}$$

and if h is a morphism of Abelian po-groups, we set

$$\mathcal{E}(h)(x) = \begin{cases} (0, h(g)) & \text{if } x = (0,g), \\ (0, -h(g)) & \text{if } x = (1,-g), \end{cases} \tag{5.2}$$

where $g \in G^+$.

Then $\mathcal{E}$ is a functor.

Proposition 5.2 *$\mathcal{E}$ is a faithful and full functor from the category $\mathcal{GI}$ of directed Abelian po-groups with interpolation into the category $\mathcal{PEA}$ of perfect effect algebras.*

Proof. Let h_1 and h_2 be two morphisms from G into G' such that $\mathcal{E}(h_1) = \mathcal{E}(h_2)$. Then $(0, h_1(g)) = (0, h_2(g))$ for any $g \in G^+$, consequently $h_1 = h_2$.

To prove that $\mathcal{E}$ is a full functor, suppose that $f: E(G) \to E(G')$ is a morphism of effect algebras. Then $f(0,g) = (0,g')$ for a unique $g' \in G'^+$. Define a mapping $h: G^+ \to G'^+$ by $h(g) = g'$ iff $f(0,g) = (0,g')$. Then $h(g_1+g_2) = h(g_1)+h(g_2)$ if $g_1, g_2 \in G^+$. Assume now that $g \in G$ is arbitrary. Then $g = g_1 - g_2 = g_1' - g_2'$, where $g_1, g_2, g_1', g_2' \in G^+$, which gives $g_1 + g_2' = g_1' + g_2$, i.e., $h(g) = h(g_1) - h(g_2)$ is a well-defined extension of h from G^+ onto G. In addition, h is a homomorphism of po-groups. Therefore, $\mathcal{E}(h) = f$ as desired. $\square$

Proposition 5.3 *Let E be a perfect effect algebra. Then there is a unique (up to isomorphism) directed Abelian po-group G with interpolation such that $E \cong E(G)$.*

Proof. Let E be a perfect effect algebra. By Proposition 5.1, $\mathrm{Rad}(E)$ is a cancellative semigroup satisfying conditions of Birkhoff [Fuc, Thm II.4] which guarantee that $\mathrm{Rad}(E)$ is a positive cone of a unique (up to isomorphism) po-group G. Since E is with the (RDP), G is a directed Abelian po-group with interpolation.

Without loss of generality, we can assume that any element $g \in G$ can be written in the form $g = g_1 - g_2$ with $g_1, g_2 \in \mathrm{Rad}(E)$. The mapping $\gamma : \mathrm{Rad}(E) \to G$ given by

$$\gamma(a) := a, \;\; a \in \mathrm{Rad}(E), \tag{5.3}$$

is injective and preserves $+$ in $\mathrm{Rad}(E)$.

Define $E(G)$ by (2.2). By Proposition 5.1, $E(G)$ is a perfect effect algebra, and define a mapping $f : E \to E(G)$ by $f(a) = (0, a)$ and $f(a') = (1, -a)$ if $a \in \mathrm{Rad}(E)$. Then f is an isomorphism of the effect algebras E and $E(G)$. $\square$

Extend the mapping $\gamma : \; \mathrm{Rad}(E) \to G$ defined by (5.3) to a mapping $\hat{\gamma} : E \to G$ via

$$\hat{\gamma}(a) = \begin{cases} \gamma(a) & \text{if } a \in \mathrm{Rad}(E), \\ -\gamma(a') & \text{if } a \notin \mathrm{Rad}(E). \end{cases} \tag{5.4}$$

Theorem 5.4 *Let E be a perfect effect algebra. Then the pair $(G, \hat{\gamma})$, where G is from Proposition* 5.3 *and $\hat{\gamma}$ is defined in* (5.4)*, is a universal group for E.*

Proof. Due to the Rad-property, $\hat{\gamma}$ is a mapping from E into G that is injective, preserves $+$, and $\hat{\gamma}(E)$ generates G. Assume that H is an additive Abelian group and $\phi : \; E \to H$ is an H-valued measure. Define a mapping $\phi^* : \; G^+ \to H$ via $\phi^*(g) = \phi(\hat{(}\gamma(a))$, where a is a unique element in $\mathrm{Rad}(E)$ such that $\hat{\gamma}(a) = g$. Then ϕ^* preserves addition in G^+, so that ϕ^* can be uniquely extended to a group isomorphism (denoted again as) ϕ^* via $\phi^*(g) = \phi^*(g_1) - \phi^*(g_2)$ whenever $g = g_1 - g_2$ $(g_1, g_2 \in G^+)$. Hence we have $\phi(a) = \phi^*(\hat{\gamma}(a))$ for any $a \in E$ which proves that $(G, \hat{\gamma})$ is a universal group for E. $\square$

Proposition 5.5 *The functor $\mathcal{E}$ from the category $\mathcal{GI}$ into the category $\mathcal{PEA}$ is a right-adjoint.*

Proof. We show that given a perfect effect algebra E there is a universal arrow (G, f), i.e., f is a homomorphism from E into $\mathcal{E}(G)$ such that if G' is an object from $\mathcal{GI}$ and f' is a homomorphism from E into $\mathcal{E}(G)$, then there is a unique homomorphism $f^* : G \to G'$ such that $\mathcal{E}(f^*) \circ f = f'$.

Let $(G, \hat{\gamma})$ be a universal group for E guaranteed by Theorem 5.4. Then it is straightforward to verify that $(G, \hat{\gamma})$ is a universal arrow for E in question. $\square$

Define a morphism $\mathcal{P} : \mathcal{PEA} \to \mathcal{GI}$ via $\mathcal{P}(E) := G$ whenever $(G, \hat{\gamma})$ is a universal group for E. It is clear that if f is a morphism from E into F, then f can be uniquely extended to a homomorphism $\mathcal{P}(f)$ from G into G_1, where $(G_1, \hat{\gamma}_1)$ is a universal group for the perfect effect algebra F.

Proposition 5.6 *$\mathcal{P}$ is a functor from the category $\mathcal{PEA}$ into the category $\mathcal{GI}$ which is left-invariant to $\mathcal{E}$.*

Proof. It follows from the construction of universal groups. $\square$

We present now the main result on categorical equivalence of the category of perfect effect algebras and the category of directed Abelian po-groups with interpolation.

Theorem 5.7 *$\mathcal{E}$ is a categorical equivalence of the category $\mathcal{GI}$ of directed Abelian po-groups with interpolation and the category $\mathcal{PEA}$ of perfect effect algebras.*

Proof. According to [MaL, Thm IV.4.1], it is necessary to show that, for a perfect effect algebra E, there is an object G in $\mathcal{GI}$ such that $\mathcal{E}(G)$ is isomorphic to E. To show that it is sufficient to take G, where $(G, \hat{\gamma})$ is a universal group for E from Theorem 5.4. Then $\mathcal{E}(G)$ and E are isomorphic. $\square$

To complete our categorical equivalences of appropriate categories of effect algebras and interpolation po-groups, let $\mathcal{EA}_{\mathcal{RDP}}$ be the category of effect algebras with the (RDP) whose objects are effect algebras and morphisms are homomorphisms of effect algebras, and let $\mathcal{UIG}$ be the category of unital interpolation po-groups (G, u) with a fixed strong unit u and morphisms are homomorphisms of unital po-groups, i.e., positive homomorphisms of unital po-groups which preserve fixed strong units. Using Theorem 2.2 and methods used for the categories of perfect effect algebras, we can prove the following important result.

Theorem 5.8 *The mapping $\Gamma : \mathcal{UIG} \to \mathcal{EA}_{\mathcal{RDP}}$ defines the categorical equivalence of the category $\mathcal{UIG}$ of unital interpolation po-groups and the category of effect algebras with the* (RDP).

In addition, suppose that $h: \Gamma(G,u) \to \Gamma(H,v)$ is a homomorphism of effect algebras with the (RDP)*, then there is a unique homomorphism $f: (G,u) \to (H,v)$ of unital po-groups such that $h = \Gamma(f)$, and*

(i) *if h is surjective, so is f;*

(ii) *if h is injective, so is f.*

6 Ideals and Quotients of Effect Algebras

In the present Section, we show a one-to-one bijection among the set of all ideals of an effect algebra with the (RDP) and ideals of the corresponding representation unital po-groups. In addition, we study quotient effect algebras and antilattice effect algebras.

Let P be a (proper) ideal of an effect algebra E with the (RDP). We define a relation $\sim_P$ on E via $a \sim_P b$ iff $a-e = b-f$ for some $e, f \in P$. According to [DvPu, Sec. 3.1.2], we have that $\sim_P$ is an equivalence such that (i) $a+b \in E$, $a_1+b_1 \in E$, $a \sim_P a_1$, $b \sim_P b_1$ imply $(a+b) \sim_P (a_1+b_1)$, (ii) $a \sim_P b$ implies $a' \sim_P b'$, (iii) $a+b \in E$, $c \sim_P a$ imply there exists an element $d \in E$ such that $d \sim_P b$ and $d+c \in E$, (iv) $a+b, a_1+b_1 \in E$, $a_1 \sim_P a$, $(a_1+b_1) \sim_P (a+b)$ imply $b_1 \sim_P b$. If we define $a/P := [a] := [a]_P := \{b \in E: \ b \sim_P a\}$, then $E/P := \{[a]_P : \ a \in E\}$ is an effect algebra, where $[a]+[b] = [c]$ iff there exist $a_1 \in [a], b_1 \in [b], c_1 \in [c]$ such that $a_1 + b_1 = c_1$. As the constant elements in E/P we take $[0]$ and $[1]$. We recall that

$$[a]_P \le [b]_P \text{ in } E/P \quad \Leftrightarrow \quad \text{there exists } a_1 \in [a]_P \text{ such that } a_1 \le b. \tag{6.1}$$

Indeed, if $[a]_P \le [b]_P$, there exists $c \in E$ such that $[a]_P + [c]_P = [b]_P$. Then there are elements $u, d, e, f \in P$ and $a_0 \in [a]_P, b_1 \in [b]_P$ such that $a_0 + c = b_1$, $a_0 - u = a - v$, $b_1 - e = b - f$. Since $a_0 = (a-v)+u$, $b_1 = (b-f)+e$, we have $(a-v)+u+c = (b-f)+e$. Due to the (RDP), there are $e_1, e_2, e_3 \in P$ with $e_1 \le a-v$, $e_2 \le u$, $e_3 \le c$ and $e = e_1+e_2+e_3$, which yields

$$(a-(v_1+e_1)) + (u-e_2) + (c-e_3) = b-f,$$
$$(a-(v_1+e_1)) + (u-e_2) + (c-e_3) + f = b.$$

Set $a_1 = a-(v+e_1)$, then $a_1 \le b$.

Proposition 6.1 *Let P be a proper ideal of an effect algebra E with the* (RDP)*. Then E/P is an effect algebra with the* (RDP).

Proof. Assume that $[a_1]+[a_2]=[b_1]+[b_2]$. Without loss of generality we can assume that a_1+a_2, b_1+b_2 are defined in E. There are $e,f\in P$ such that $(a_1+a_2)-e=(b_1+b_2)-f$, so that $a_1+a_2=((b_1+b_2)-f)+e$. By the (RDP) holding in E, there are $c_{11},c_{12},c_{21},c_{22}$ in E such that

$$\begin{aligned} a_1 &= c_{11}+c_{12}, & (b_1+b_2)-f &= c_{11}+c_{21},\\ a_2 &= c_{21}+c_{22}, & e &= c_{12}+c_{22}. \end{aligned}$$

This gives $c_{11}+c_{21}+f=b_1+b_2$. Again due to the (RDP), there are $d_{11},d_{12},d_{21},\ d_{22},d_{31},d_{31}\in E$ such that

$$\begin{aligned} c_{11} &= d_{11}+d_{12}, & b_1 &= d_{11}+d_{21}+d_{31},\\ c_{21} &= d_{21}+d_{22}, & b_2 &= d_{12}+d_{22}+d_{32},\\ f &= d_{31}+d_{32}. \end{aligned}$$

It is clear that $c_{12},c_{22},d_{31},d_{32}\in P$, which gives $[a_1]=[c_{11}]=[d_{11}]+[d_{12}]$, $[a_2]=[c_{21}]=[d_{21}]+[d_{22}]$, $[b_1]=[d_{11}]+[d_{21}]$, $[b_2]=[d_{12}]+[d_{22}]$, which proves that E/P is with the (RDP). $\square$

We say that an effect algebra E is *finitely subdirectly irreducible* if, for any two ideals I and J of E with $I\cap J=\{0\}$, we have $I=\{0\}$ or $J=\{0\}$.

For $\emptyset\neq A\subseteq E$, we set $A^\perp:=\{x\in E:\ x\wedge a=0 \text{ for any } a\in A\}$. We define $a^\perp:=\{a\}^\perp$ for $a\in E$. Then

$$a^\perp\cap a^{\perp\perp}=\{0\},\quad a\in E. \tag{6.2}$$

Proposition 6.2 *Let E be an effect algebra with the* (RDP)*. If $\emptyset\neq A\in E$, then $A^\perp$ is an ideal of E.*

Proof. $0\in A^\perp$. If $x,y\in E$ and $x\le y\in A^\perp$, then $x\in A^\perp$. Assume now $x,y\in A^\perp$ and let $x+y\in E$. Fix $a\in A$. If $z\le x+y$ and $z\le a$, then $z=x_1+y_1$, where $x_1\le x$, $y_1\le y$, and $x_1,y_1\in a^\perp$. While $x_1,y_1\le a$, we have $x_1=x_1\wedge a=0=y_1\wedge a=y_1$ which proves $z=0$. $\square$

Proposition 6.3 *An effect algebra E with the* (RDP) *is finitely subdirectly irreducible if and only if E is an antilattice.*

Proof. If an effect algebra E with the (RDP) is not finitely subdirectly irreducible, then there exist two non-zero ideals I and J such that $I \cap J = \{0\}$. Hence, if $a \in I$ and $b \in J$ are non-zero elements, then $a \wedge b = 0$, whence E cannot be an antilattice.

Conversely, assume that E is finitely subdirectly irreducible and let there be $a, b \in E \setminus \{0\}$ with $a \wedge b = 0$. Then $a \in b^\perp$ and $b \in a^\perp$. In view of Claim from the proof of Proposition 3.7, $0 \neq a + b = a \vee b \in E$, so that $a^\perp \cap b^\perp = (a+b)^\perp$. While $(a+b)^\perp \cap (a+b)^{\perp\perp} = \{0\}$ and $a + b \in (a+b)^{\perp\perp}$, the irreducibility implies $(a+b)^\perp = \{0\}$, i.e., $a^\perp \cap b^\perp = \{0\}$, which gives $b \in a^\perp = \{0\}$ or $a \in b^\perp = \{0\}$, i.e., $b = 0$ or $a = 0$, a contradiction. $\square$

If $A \subseteq E$ and P is an ideal of E, then we set $A/P := \{a/P : \ a \in A\}$.

Proposition 6.4 *Let E be an effect algebra with the* (RDP) *and let P be a proper ideal of E.*

(i) *If I is an ideal of E, so is I/P in E/P. Moreover, if I is a proper ideal of E containing P, then I/P is a proper ideal of E/P.*

(ii) *If M is an ideal of E/P, so is*

$$\kappa(M) := \{x \in E : \ x/P \in M\}.$$

If M is a proper ideal of E so is $\kappa(M)$ in E/P.

Proof. (i) $0/P \in I/P$. Let $x/P \leq y/P$, where $y \in I$. There exists $x_1 \in [x]_P$ such that $x_1 \leq y$, which gives $x_1 \in I$, and $x_1/P = x/P \leq y/P$. Assume $x/P + y/P$ is defined in E/P for some $x, y \in I$. There are $x_1 \in [x]_P$, $y_1 \in [y]_P$ and $e, f, u, v \in P$ such that $x_1 - e = x - f \in I$, $y_1 - u = y - v \in I$, $x_1 + y_1 \in E$. Then $x/P + y/P = x_1/P + y_1/P = (x_1 + y_1)/P = ((x - f) + e + (y - v) + u)/P = ((x - f) + (y - v))/P$ and $(x - f) + (y - v) \in I$.

Let now $I \supseteq P$ and $1/P = x/P$, where $x \in I$. There are $e, f \in P$ such that $1 - e = x - f$, i.e., $1 - x = e - f \in P \subseteq I$ which gives a contradiction.

(ii) We have $\kappa(M) \supseteq P$. If $x \leq y \in \kappa(M)$, then $x/P \leq y/P \in M$, so that $x/P \in \kappa(M)$. Let now $x, y \in \kappa(M)$ and $x + y \in E$. Then $(x+y)/P = x/P + y/P \in \kappa(M)$.

Finally, assume M is a proper ideal of E/P. Then $1/P \notin M$, hence, $1 \notin \kappa(M)$. $\square$

It is worth recalling that it can happen that $M/P = E/P$ even for a maximal ideal M. Indeed, take a Boolean algebra E as an effect algebra, let

M be a maximal ideal and a let be an element of E such that $a \notin M$, $0 \neq a \neq 1$, then for the ideal P generated by a we have $1/P = a'/P \in M/P = E/P$.

Proposition 6.5 *A proper ideal P of E is prime if and only if E/P is an antilattice.*

Proof. Let P be a prime ideal of E and I and J be two ideals of E/P with $I \cap J = \{0\}$. We set $I_0 = \{x \in E : \ x/P \in I\}$ and $J_0 = \{x \in E : \ y/P \in J\}$. Then I_0 and J_0 are ideals of E such that $I_0 \cap J_0 = P$, which by (ii) of Proposition 3.6 gives $I_0 = P$ or $J_0 = P$, i.e., $I = \{0\}$ or $J = \{0\}$. Applying Proposition 6.3, we have E/P is an antilattice.

Conversely, let P/I be an antilattice and let I and J be two ideals of E such that $I \cap J = P$. Then $I/P \cap J/P = \{0\}$, which by Proposition 6.3 yields $I/P = \{0\}$ of $J/P = \{0\}$, that is $I = P$ or $J = P$, and therefore, by Proposition 3.6, P is a prime ideal of E. $\square$

Corollary 6.6 *An effect algebra with the* (RDP) *is an antilattice if and only if $\{0\}$ is a prime ideal of E.*

Proof. Since $E \cong E/\{0\}$, the assertion follows from Proposition 6.5. $\square$

Proposition 6.7 *An ideal M of an effect algebra E with the* (RDP) *is maximal if and only if E/M is a simple effect algebra.*

Proof. Let M be a maximal ideal of E, and let J be a proper ideal of E/M. Define $J_0 := \{x \in E : \ x/M \in J\}$. Then J_0 is an ideal of E containing M. The maximality of M entails $J = \{0/M\}$.

Conversely, let E/M be simple and let I_1 be a proper ideal of E containing M. The set $\{a/M : \ a \in I_1\}$ is an ideal of E/I. Therefore by Proposition 6.4, this set coincides with the zero-ideal $\{0/M\}$ of E/M, i.e., $I_1 = M$, which proves that M is maximal. $\square$

Corollary 6.8 *Every simple effect algebra with the* (RDP) *is an antilattice.*

Proof. Let M be a maximal ideal of E. Then $M = \{0\}$ and M is a prime ideal. While $E \cong E/M$, by Proposition 6.5, E is an antilattice. $\square$

Proposition 6.9 *Let E be an effect algebra with the* (RDP). *Then $a \wedge b \in E$ if and only if $a \vee b \in E$. In addition, if I is an ideal of E, $a, b \in I$ and $a \vee b$ is defined in E, then $a \vee b \in I$.*

Proof. Let $a \wedge b \in E$. Due to the (RDP), there are $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c \in E$. Since $c \le a \wedge b$, then $a_1 \ge a - (a \wedge b)$, $b_1 \ge b - (a \wedge b)$, which yield $x_0 = a \wedge b + (a - (a \wedge b)) + (b - (a \wedge b)) \in E$. According to (4.1) and Claim from the proof of Proposition 3.7, we have $a_0 + b_0 = a_0 \vee b_0$, where $a_0 = a - (a \wedge b)$ and $b_0 = b - (a \wedge b)$. We state $x_0 = a \vee b$. Assume $x \ge a, b$; then $x \ge a_0 + (a \wedge b)$ and $x \ge b_0 + (a \wedge b)$, which gives $x - (a \wedge b) \ge a_0 \vee b_0$, i.e., $x \ge x_0 = a_0 \vee b_0 + (a \wedge b) = a_0 + b_0 + (a \wedge b) = a \vee b$.

Assume now $a \vee b \in E$. Then $(a \vee b)' = a' \wedge b' \in E$ which gives by the above $a' \vee b' \in E$. Hence, $a \wedge b = (a' \vee b')' \in E$.

Let I be an ideal of E. Then by the first part of the present proof, $a \wedge b \in E$ and $a \vee b = a + (b - (a \wedge b))$ which proves $a \vee b \in I$. $\square$

Proposition 6.10 *Let E be an effect algebra with the* (RDP) *and let P be a proper ideal of E. If $a \wedge b \in E$, then $[a \wedge b]_P = [a]_P \wedge [b]_P$ and $[a \vee b]_P = [a]_P \vee [b]_P$. In addition, if E is a lattice, so is E/P.*

Proof. It is clear that $[a \wedge b]_P \le [a]_P, [b]_P$. Assume $[x]_P \le [a]_P$ and $[x]_P \le [b]_P$. According to (6.1), there are $x_1, x_2 \in [x]_P$ such that $x_1 \le a$ and $x_2 \le b$. Since $x_1 \sim_P x_2$, there are $e, f \in P$ with $x_1 - e = x_2 - f$. Hence, for $x_0 = x_1 - e$, we have $x_0 \in [x]_P$, and $x_0 \le x_1$, $x_0 \le x_2$. Consequently, $x_0 \le a, b$ which yields $x_0 \le a \wedge b$, i.e., $[x]_P = [x_0]_P \le [a \wedge b]_P$.

The second equality follows from the first part of the present proof and of Proposition 6.9. $\square$

We recall that an *ideal* of a po-group G is any directed convex subgroup of G. An ideal I of a po-group G is said to be (i) *maximal* if it is a proper subset of G and it is not contained in any proper ideal of G, (ii) *prime* if, for all ideal P and Q of G with $P \cap Q \subseteq I$, we have $P \subseteq I$ or $J \subseteq I$, and (iii) a *value* of a non-zero element g if $g \notin I$ and it is maximal with respect to this property. Let (G, u) be a unital Abelian po-group; by $\mathcal{I}(G, u)$, $\mathcal{M}(G, u)$ and $\mathcal{P}(G, u)$ we denote the set of all ideals, maximal ideals and prime ideals, respectively, of (G, u).

Theorem 6.11 *Let (G, u) be a unital interpolation po-group and let $E = \Gamma(G, u)$. For any ideal I of E, we assign*

$$\phi(I) = \{x \in G : \ \exists\ x_i, y_j \in I, \quad x = x_1 + \cdots + x_n - y_1 - \cdots - y_m\}. \quad (6.3)$$

Then $\phi(I)$ is an ideal of (G,u). The mapping ϕ defines a one-to-one mapping preserving the set-theoretical inclusion. The inverse mapping ψ is given by

$$\psi(K) := K \cap [0,u], \ \ K \in \mathcal{I}(G,u). \tag{6.4}$$

The restriction of ϕ to $\mathcal{P}(E)$ or $\mathcal{P}(E)$ gives a bijection between $\mathcal{M}(E)$ and $\mathcal{M}(G,u)$, and $\mathcal{P}(E)$ and $\mathcal{P}(G,u)$, respectively, which preserves the set-theoretical inclusion.

Proof. Let I be an ideal of E. Then $I \subseteq \phi(I)$. Let $x,y \in \phi(I)$, then $x+y \in \phi(I)$ and $-x \in \phi(I)$. Assume $0 \le x \le y \in \phi(I)$, for $x \in G$. Then $y = y_1 + \cdots + y_n$ with $y_i \in I$ for $i = 1,\ldots,n$. The interpolation in G implies $x = x_1 + \cdots + x_n$ for $x_i \le y_i \in I$, which proves $x \in \phi(I)$. It is clear that $\phi(I)$ is directed.

Assume that I and J are two different ideals of E. Then there exists $x \in I \setminus J$ or $x \in J \setminus I$. In the first case we assert $x \in \phi(I)\setminus\phi(J)$. Indeed, it is clear that $x \in \phi(I)$. If also $x \in \phi(J)$, then $x = x_1 + \cdots + x_n - y_1 - \cdots - y_m$, where $x_i, y_j \in J$. Hence, $x + y_1 + \cdots + y_m = x_1 + \cdots + x_n$ and $0 \le x \le x_1 + \cdots + x_n$. The interpolation in (G,u) gives $x = x_1^0 + \cdots + x_n^0$ with $x_i^0 \le x_i \in J$, which entails $x \in J$, a contradiction.

Assume now K is an ideal of (G,u). Then $\psi(K) \subseteq E$, and $0 \in \psi(K)$. If $x \le y$ for $x \in E$ and $y \in \psi(K)$, then $x \in K$, i.e., $x \in \psi(K)$. If now $a,b \in \psi(K)$ and if $a+b$ is defined in E, then $a+b \in K$, that is $a+b \in \psi(K)$.

It is clear that $\phi(\psi(K)) = K$ for any $K \in \mathcal{I}(G,u)$ and $\psi(\phi(I)) = I$ for any $I \in \mathcal{I}(E)$.

Since ϕ is bijective, so is its restriction to $\mathcal{M}(E)$ and $\mathcal{P}(E)$, respectively.

Assume now P is a prime ideal of E and let $K_1 \cap K_2 \subseteq \phi(P)$, where K_1, K_2 are ideals of (G,u). From (6.4) we have

$$\psi(K_1) \cap \psi(K_2) = \psi(K_1 \cap K_2),$$

that is $\psi(K_1) \cap \psi(K_2) \subseteq P$ which gives $\psi(K_1) \subseteq P$ or $\psi(K_2) \subseteq P$. Hence, $K_1 \subseteq \phi(P)$ or $K_2 \subseteq \phi(K_2)$.

In a similar manner we prove that if P is a prime ideal of (G,u), so is $\psi(P)$ in E. $\square$

We can prove that if a is a non-zero element of E, then an ideal V is a value of a iff $\phi(V)$ is a value of a in (G,u).

Let I be an ideal of (G,u). For $f,g \in G$, we define $f \sim_I g$ iff $f-g, g-f \in I$. Then $\sim_I$ is a congruence on G, and $(G/I, u/I)$ is an interpolation group, see [Goo, Prop 2.3].

Theorem 6.12 *Let I be a proper ideal of an effect algebra E with the* (RDP) *and let $E \cong \Gamma(G,u)$. Then $E/I \cong \Gamma(G/\phi(I), u/\phi(I))$.*

Proof. Let $f: G \to G/\phi(I)$ be the canonical mapping defined via $f(g) := g/\phi(I)$, $g \in G$, and $h: E \to E/I$ given by $h(a) := [a]_I$, $a \in E$.

We can prove that, for $a,b \in E$, $[a]_I = [b]_I$ iff $a/\phi(I) = b/\phi(I)$. Indeed, if $[a]_I = [b]_I$, then there are $e,f \in I$ such that $a - e = b - f$ which proves $a/\phi(I) = b/\phi(I)$.

Conversely, if $a/\phi(I) = b/\phi(I)$, then there is $x \in \phi(I)$ such that $a - b = x$. Since $x = x_1 + \cdots + x_n - y_1 - \cdots - y_m$, where $x_i, y_j \in I$, we have $a + y_1 + \cdots + y_m = b + x_1 + \cdots + x_n$. Due to the (RDP) holding in G, we find elements $c_{ij} \in E$ such that $a = c_{00} + c_{01} + \cdots + c_{0n}$, $b_j = c_{j0} + c_{j1} + \cdots + c_{jn}$ for $j = 1,\ldots,m$, and $b = c_{00} + c_{10} + \cdots + c_{m0}$, $x_i = c_{0i} + c_{1i} + \cdots + c_{mi}$ for $i = 1,\ldots,n$. Then $c_{00} + c_{01} + \cdots + c_{0n} + c_{10} + \cdots + c_{m0} = a + b - c_{00}$. Therefore, $a - (a - c_{00}) = b - (b - c_{00})$, and setting $e =: a - c_{00} = c_{01} + \cdots + c_{0n} \in I$ and $f := b - c_{00} = c_{10} + \cdots + c_{m0} \in I$, i.e., $a - e = b - f$ which proves $[a]_I = [b]_I$.

Using Theorem 5.7, we have the assertion in question. $\square$

7 Subdirect Representation and Perfect Effect Algebras

We give a representation of perfect effect algebras as a subdirect product of antilattice perfect effect algebras with the (RDP), and we introduce the class of $\mathcal{BP}_0$ effect algebras, and we give their subdirect representation.

We start with a new proof of Ravindran's representation, [Rav, Thm 2.18], of effect algebras with the (RDP) as a subdirect product of antilattice effect algebras with the (RDP).

Let $\{E_i]_{i\in I}$ be an indexed system of effect algebras. The Cartesian product $\prod_{i\in I} E_i$ can be organized into an effect algebra with the partial addition defined by coordinates. Each E_i is with the (RDP) iff so is $\prod_i E_i$. We say that an effect algebra is a *subdirect product* of effect algebras $\{E_i\}_{i\in I}$ if there is an injective homomorphism $f: E \to \prod_{i\in I} E_i$ such that, for every $j \in I$,

$\pi_j \circ f$ is a surjective homomorphism from E onto E_j, where π_j is the j-th projection of $\prod_i E_i$ onto E_j.

At the first, we give another characterization of the finite irreducibility of effect algebras, or equivalently of antilattice effect algebras with the (RDP).

Proposition 7.1 *An effect algebra E with the* (RDP) *is finitely subdirectly irreducible if and only if E is a subdirect product of E_1 and E_2, then $E \cong E_1$ or $E \cong E_2$.*

Proof. Let E be finitely subdirectly reducible, i.e., there are two non-zero ideals A and B of E such that $A \cap B = \{0\}$. The mapping $f: E \to E/A \times E/B$ given by $f(a) = ([a]_A, [a]_B)$, $a \in E$, is a homomorphism of effect algebras. If $f(a) = f(b)$, then there are $e, f \in A$ and $u, v \in B$ such that $a - e = b - f$ and $a - u = b - v$. If we now take the addition and subtraction in the corresponding unital interpolation group (G, u) such that $E = \Gamma(G, u)$, then $a - b = e - f \in \phi(A)$ and $a - b = u - f \in \phi(B)$, i.e., $a - b = 0$. Hence E is a subdirect product of E/A and E/B, and $E \ncong E/A$ and $E \ncong E/B$.

Conversely, suppose E is a subdirect product of E_1 and E_2 and let $f: E \to E_1 \times E_2$ be an injective homomorphism such that every $\pi_i \circ f: E \to E_i$, $i = 1, 2$, is not injective. Define $A_i = \{a \in E: \pi_i \circ f(a) = 0\}$, $i = 1, 2$. Then A_1 and A_2 are non-zero ideals of E. Assume $x \in A_1 \cap A_2$, then $f(x) = (0, 0)$, and the injectivity of f gives $x = 0$, which proves $A_1 \cap A_2 = \{0\}$, i.e., E is not finitely subdirectly irreducible. $\square$

The following result is the above mentioned Ravindran's representation.

Theorem 7.2 *Every effect algebra E with the* (RDP) *is a subdirect product of antilattice effect algebras with the* (RDP)*, and all existing meets and joins in E are preserved in the subdirect product.*

Proof. We assert that there is a system $\mathcal{P}$ of proper prime ideals of E such that

$$\bigcap \{P: P \in \mathcal{P}\} = \{0\}.$$

Indeed, take a non-zero element a of E. Then any value V of a is a prime ideal, Proposition 3.5. Fix a value of a, say $V(a)$, then for $\mathcal{P} := \{V(a): a > 0\}$ we have $\bigcap \{P: P \in \mathcal{P}\} = \{0\}$.

The mapping $f: E \to \prod_{P \in \mathcal{P}} E/P$ defined by $f(a) = (a/P)_{P \in \mathcal{P}}$, is a homomorphism of the effect algebras.

Assume $f(a) = f(b)$. Then $[a]_P = [b]_P$ for any $P \in \mathcal{P}$. Hence, by Theorem 6.12, $a/\phi(P) = b/\phi(P)$. Due to Theorem 6.11, $\bigcap\{\phi(P) : P \in \mathcal{P}\} = \{0\}$, which gives $a - b$ taken in the corresponding unital interpolation po-group (G, u) with $E = \Gamma(G, u)$, is an element of $\bigcap\{\phi(P) : P \in \mathcal{P}\}$, i.e., $a = b$.

Assume now $a \wedge b$ is defined in E. Then by Proposition 6.10, for any $P \in \mathcal{P}$, we have $(a \wedge b)/P = (a/P) \wedge (b/P)$. Consequently, $f(a \wedge b) = f(a) \wedge f(b)$, and in a similar way, $f(a \vee b) = f(a) \vee f(b)$. $\square$

Proposition 7.3 *Let P be a proper ideal of a perfect effect algebra E. Then the quotient effect algebra E/P is perfect.*

Proof. By Proposition 6.1, E/P satisfies the (RDP). We claim that E/P has the Rad-property. In view of Proposition 4.1

$$\mathrm{Infinit}(E)/P \subseteq \mathrm{Infinit}(E/P) \subseteq \mathrm{Rad}(E/P).$$

Assume $x/P \in \mathrm{Rad}(E/P)$. Then either $x \in \mathrm{Rad}(E)$ or $x' \in \mathrm{Rad}(E)$. If $x \in \mathrm{Rad}(E) = \mathrm{Infinit}(E)$, then $x/P \in \mathrm{Infinit}(E/P) \subseteq \mathrm{Rad}(E/P)$. If $x' \in \mathrm{Rad}(E)$, then $x'/P \in \mathrm{Infinit}(E/P)$, i.e., $x'/P \in \mathrm{Rad}(E/P)$, a contradiction, which proves $x/P \in \mathrm{Infinit}(E/P)$ and $\mathrm{Rad}(E/P) = \mathrm{Infinit}(E/P)$.

If now x/P is an arbitrary element of E/P, then by the above argument, either $x \in \mathrm{Rad}(E)$ or $x' \in \mathrm{Rad}(E)$ which proves either $x/P \in \mathrm{Infinit}(E/P) \subseteq \mathrm{Infinit}(E/P)$ or $x' \in \mathrm{Rad}(E)$ giving $x'/P \in \mathrm{Infinit}(E/P) \subseteq \mathrm{Infinit}(E/P)$, i.e., E/P is perfect. $\square$

As a corollary, we have the following result.

Theorem 7.4 *Every perfect effect algebra E is a subdirect product of antilattice perfect effect algebras.*

Proof. By Theorem 7.2, E is a subdirect product of antilattice effect algebras E/P, where P is a proper prime ideal. In view of Proposition 7.3, E/P is perfect. $\square$

In the paper [DLS], the authors introduced the class $\mathcal{BP}_0$ of MV-algebras. While every MV-algebra is a lattice effect algebra with the (RDP), and vice versa, inspiring by that notion, we are saying that an effect algebra with the (RDP) belongs to the class $\mathcal{BP}_0$ of effect algebras iff, for every maximal ideal M of E, we have $E = M \cup M'$.

Proposition 7.5 *Let an effect algebra E belong to the class $\mathcal{BP}_0$. Let M be a maximal ideal and P a proper ideal of E. Then $P \subseteq M$ if and only if $M/P \neq E/P$.*

Proof. Assume $P \subseteq M$. If there exists $x \in M$ such that $x/P = 1/P$, then there are two elements e and f in P such that $x - e = 1 - f$. Hence $1 - x = f - e \in P \subseteq M$, which is a contradiction.

Conversely, let $M/P \neq E/P$ and take $x \in P$. If $x' \in M$, then $x'/P = 0$, i.e., $x/P = 1/P$ giving a contradiction. Therefore, $x \in M$. $\square$

Theorem 7.6 *Every effect algebra E from the class $\mathcal{BP}_0$ with the property, for any $x \in E$ and any $P \in \mathcal{P}(E)$, we have $x/P \leq (1-x)/P \leq x/P$ or $(1-x)/P \leq x/P$, is a subdirect product of antilattice perfect effect algebras.*

Proof. According to Theorem 7.2, E is a subdirect product of antilattice effect algebras $\{E/P : \ P \in \mathcal{P}(E),\ P \neq E\}$. In what follows, we show that every E/P is a perfect effect algebra. Let N be a maximal ideal of E containing P and let x be an arbitrary element in E. If $x \in N$, then $(1-x)/P \leq x$ or $x/P \leq (1-x)/P$. Since N is a maximal ideal of E, by Proposition 6.4 and Proposition 7.5, the first case would imply $(1-x)/P \in N/P$ which is a contradiction. Hence, $x/P \leq (1-x)/P$ which gives $x/P + x/P$ is defined in E/P. Therefore, there are two elements $x_1, x_2 \in N$ such that $x_0 = x_1 + x_2 \in N$ and $x_1, x_2 \in [x]_P$. Repeating this process with the element x_0, we have $x_0/P + x_0/P \in E/P$, i.e., $4(x/P)$ is defined in E/P, etc. Therefore, $2^n(x/P) \in E/P$ for any integer $n \geq 1$, which implies $x/P \in \text{Infinit}(E/P) \subseteq \text{Rad}(E/P)$. Summarizing, we have proved that for every $x \in E$, either $x/P \in \text{Infinit}(E/P)$ or $(1-x)/P \in \text{Infinit}(E/P)$.

In particular, if $x/P \in \text{Rad}(E/P)$, then $x \in \kappa(M)$, where $M \in \mathcal{M}(E/P)$ and $\kappa(M) \supseteq P$ is defined in Proposition 6.4, which proves by the above $x/P \in \text{Infinit}(E/P)$ proving that E/P has the Rad-property, and E/P is perfect. $\square$

We notify that, e.g., every effect algebra E with the (RDP) satisfies the property: for any $x \in E$ and any prime ideal P of E, $x/P \leq (1-x)/P$ or $(1-x)/P \leq x/P$. On the other hand, if E/P is perfect, by Proposition 5.3, the condition holds. It would be interesting to know whether is it necessary in Theorem 7.6 to assume the condition to have the subdirect product from Theorem 7.6.

References

[Cha] C.C. Chang, *Algebraic analysis of many valued logics,* Trans. Amer. Math. Soc. **88** (1958), 467–490.

[DiLe] A. Di Nola, A. Lettieri, *Perfect MV-algebras are categorical equivalent to abelian ℓ-groups,* Studia Logica **53** (1994), 417–432.

[DLS] A. Di Nola, F. Liguori, S. Sessa, *Using maximals ideals in the classification of MV-algebras,* Portugalie Math. **50** (1993), 87–102.

[DvPu] A. Dvurečenskij, S. Pulmannová, *"New Trends in Quantum Structures"*, Kluwer Acad. Publ., Dordrecht, Ister Science, Bratislava, 2000.

[FoBe] D.J. Foulis, M.K. Bennett, *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[Fuc] L. Fuchs, *"Partially Ordered Algebraic Systems",* Pergamon Press, Oxford-London-New York-Paris, 1963.

[Gla] A.M.W. Glass *"Partially Ordered Groups"*, World Scientific, Singapore, 1999.

[Goo] K.R. Goodearl, *"Partially Ordered Abelian Groups with Interpolation",* Math. Surveys and Monographs No. 20, Amer. Math. Soc., Providence, Rhode Island, 1986.

[KoCh] F. Kôpka, F. Chovanec, *D-posets*, Math. Slovaca **44** (1994), 21–34.

[MaL] S. Mac Lane, *"Categories for the Working Mathematician",* Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[Rav] K. Ravindran, *On a structure theory of effect algebras*, PhD theses, Kansas State Univ., Manhattan, Kansas, 1996.

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