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Product Effect Algebras

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Abstract

We introduce a product on an effect algebra. We prove that every product effect algebra with the Riesz decomposition property, (RDP), is an interval in an Abelian unital interpolation po-ring, and we show that the category of product effect algebras with the (RDP) is categorically equivalent with the category of unital Abelian interpolation po-rings. In addition, we show that every product effect algebra with the (RDP) and with 1 as a product unity is a subdirect product of antilattice product effect algebras with the (RDP).

1 Introduction

Effect algebras entered mathematics in 1994 due to Foulis and Bennett [FoBe] as partial algebras with a partially defined addition $+$. They are additive counterparts to D-posets introduced by Kôpka and Chovanec [KoCh], where the subtraction of comparable elements is a primary notion. They met interest of mathematicians, physicists while they give a common base for algebraic as well as fuzzy set properties of the system $\mathcal{E}(H)$ of all effects of a Hilbert space H, i.e., of all Hermitian operators A on H such that $O \le A \le I$, where O and I are the null and the identity operators on H. In many cases, effect algebras are intervals in unital po-groups, e.g., $\mathcal{E}(H)$ is the interval in

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the po-group $\mathcal{B}(H)$ of all Hermitian operators on H; this group is of great importance for physics.

We recall that every MV-algebra of Chang [Cha] can be understood also as an effect algebra. For example, if M is a maximal set of mutually commuting effects on a separable Hilbert space H, then M can be converted into an MV-algebra: There is a system of Borel measurable functions $f_A: [0,1] \to [0,1]$, and a fixed effect operator $A_0 \in M$ such that $A = f_A(A_0)$. The MV-operations on M are then defined by $A \oplus B = (\max\{f_A + f_B, 1\})(A_0)$, $A' = I - A$, and moreover, we can define in a natural way the product $A \cdot B$ of A and B via $A \cdot B := (f_A \cdot f_B)(A)$, $A, B \in M$, where $f_A \cdot f_B$ is a usual product of functions f_A and f_B.

Inspiring that, in the present note we introduce product effect algebras. We use similar notions for MV-algebras which were studied in [DvRi] and [DiDv]. We show that every product effect algebra with the Riesz decomposition property is isomorphic with the interval in an Abelian unital interpolation po-ring, Section 2. We give some examples of products, including the product on perfect effect algebras, Section 3. In Section 4, we show that the category of product effect algebras with the (RDP) is categorically equivalent with the category of Abelian unital interpolation po-groups. Finally, we show that every product effect algebra with the (RDP) and with 1 as a product unity is a subdirect product of antilattice product effect algebras with the (RDP), Section 5.

2 Product Effect Algebras

An *effect algebra* is a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

(i) $a+b$ is defined in E iff $b+a$ is defined, and in such the case $a+b = b+a$;

(ii) $a+b, (a+b)+c$ are defined iff $b+c$ and $a+(b+c)$ are defined, and in such the case $(a+b)+c = a+(b+c)$;

(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a+a' = 1$;

(iv) if $a+1$ is defined in E, then $a = 0$.

If we define $a \leq b$ iff there exists an element $c \in E$ such that $a+c = b$, then $\leq$ is a partial ordering, and we write $c := b - a$.

Let a be any element of an effect algebra E and n an integer ($n \geq 0$). We define recurrently

$$0\,a := 0, \quad 1\,a = a, \quad (n+1)\,a = n\,a + a, \; n \geq 1,$$

supposing that $n\,a$ and $n\,a + a$ are defined in E.

For example, if (G, u) is an Abelian unital po-group with a strong unit u, [2] and if $\Gamma(G, u) := \{g \in G : \; 0 \leq g \leq u\}$ is endowed with the restriction of the group addition $+$, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra. More about effect algebras can be found in [DvPu].

Let E and F be two effect algebras. A mapping $h : \; E \to F$ is said to be a *homomorphism* if (i) $h(a+b) = h(a)+h(b)$ whenever $a+b$ is defined in E, and (ii) $h(1) = 1$. A bijective homomorphism h such that h^{-1} is homomorphism is said to be an *isomorphism* of E and F.

A *product* on an effect algebra $E = (E; +, 0, 1)$ is any total binary operation $\cdot$ on E such that, for all $a, b, c \in E$, the following holds: If $a + b$ is defined in E, then $a \cdot c + b \cdot c$ and $c \cdot a + c \cdot b$ exist in E and

$$\begin{aligned} (a+b) \cdot c &= a \cdot c + b \cdot c, \\ c \cdot (a+b) &= c \cdot a + c \cdot b, \end{aligned}$$

and we say that E with a product $\cdot$ is a *product effect algebra*, and we write $E = (E; +, \cdot, 0, 1)$. An element u of a product effect algebra E is said to be a *unity*, if $a \cdot u = u \cdot a = a$ for any $a \in E$.

A product $\cdot$ on E is

(i) *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $a, b, c \in E$;

(ii) *commutative* if $a \cdot b = b \cdot a$, $a, b \in E$.

It is worth saying that if $\cdot$ is a product on E, then

(iii) $a \cdot 0 = 0 = 0 \cdot a$;

(iv) if $a \leq b$, then for any $c \in E$, $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

Property (iii) follows easily from the following: $a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$, and the cancellation property gives $a \cdot 0 = 0$. Similarly, $0 \cdot a = 0$.

[2] An element $u \in G^+$ is said to be a *strong unit* for a po-group G, if given an element $g \in G$, there is an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

We recall that every effect algebra E possesses at least one commutative and associative product, namely the *zero* product, i.e. $a \cdot b = 0$ for all $a, b \in E$.

We recall that a *po-ring* is a ring $(R; +, \cdot, 0)$ such that (i) $(R; +, 0)$ is an additive Abelian po-group, and $c \cdot a \geq 0$ and $a \cdot a \geq 0$ for any $c, a \geq 0$. If u is a strong unit for R, i.e., for any $a \in R$ there is an integer $n \geq 1$ such that $a \leq nu$, and if $u \cdot u \leq u$, then the effect algebra

$$E = \Gamma(R, u)$$

is a product effect algebra with the product $\cdot$ which is the restriction of the ring product $\cdot$ to $E \times E$; the product $\cdot$ is commutative or associative on $\Gamma(R, u)$ whenever so is $\cdot$ on R.

For example, if $(\mathbb{R}; +, \cdot, 0)$ is the ring of the real numbers, then the standard interval $[0, 1] := \Gamma(\mathbb{R}, 1)$ is a product effect algebra.

We say that an effect algebra E satisfies (i) the *Riesz interpolation property*, (RIP) for short, if, for all x_1, x_2, y_1, y_2 in E, $x_i \leq y_j$ for all i, j implies there exists an element $z \in E$ such that $x_i \leq z \leq y_j$ for all i, j; (ii) the *Riesz decomposition property*, (RDP) for short, if $x \leq y_1 + y_2$ implies that there exist two elements $x_1, x_2 \in$ with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 + x_2$.

We recall that (1) if E is a lattice, then E has trivially the (RIP); the converse is not true as wee see below. (2) E has the (RDP) iff, [DvPu, Lem 1.7.5], $x_1 + x_2 = y_1 + y_2$ implies there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$, and $y_2 = c_{12} + c_{22}$. (3) the (RDP) implies the (RIP), but the converse is not true (e.g. if $E = L(H)$, the system of all closed subspaces of a Hilbert space H, then E is a complete lattice but without the (RDP)). On the other hand, every finite poset with the (RIP) is a lattice.

We recall that a poset $(E; \leq)$ is an *antilattice* if only comparable elements of E have a supremum (infimum). It is clear that any linearly ordered poset is an antilattice and every finite effect algebra with the (RIP) is a lattice.

There exists an effect algebra with (RIP) which is not a lattice:

Example 2.1 *Let G be the additive group $\mathbb{R}^2$ with the positive cone of all (x, y) such that either $x = y = 0$ or $x > 0$ and $y > 0$. Then $u = (1, 1)$ is a strong unit for G. The effect algebra $E = \Gamma(G, u)$ is an antilattice having the* (RIP) *and the* (RDP) *but E is not a lattice. Moreover, if we define $(x_1, y_1) \cdot (x_2, y_2) := (x_1 \cdot x_2, y_1 \cdot y_2)$, for $(x_1, y_1), (x_2, y_2) \in E$, then $\cdot$ is a commutative and associative product on E and with unity u.*

A partially ordered Abelian group $(G;+,0)$ is said to satisfy the *Riesz decomposition property* provided, given x, y_1, y_2 in G^+ such that $x \le y_1 + y_2$, there exist x_1, x_2 in G^+ such that $x = x_1 + x_2$ and $x_j \le y_j$ for each j. This condition is equivalent by [Goo, Prop. 2.1] with the following two equivalent conditions:

(a) Given x_1, x_2, y_1, y_2 in G such that $x_i \le y_j$ for all i, j, there exists z in G such that $x_i \le z \le y_j$ for all i, j.

(b) Given x_1, x_2, y_1, y_2 in G^+ such that $x_1 + x_2 = y_1 + y_2$, there exist $z_{11}, z_{12}, z_{21}, z_{22}$ in G^+ such that $x_i = z_{i1} + z_{i2}$ for each i and $y_j = z_{1j} + z_{2j}$ for each j.

According to [Goo], a group G with the Riesz decomposition property is said to be the *interpolation group*.

It is clear that if (G, u) is a unital interpolation group, then $E = \Gamma(G, u)$ is with the (RDP).

We recall that by a *universal group* for an effect algebra E we mean a pair (G, γ) consisting of an additive Abelian group G and a G-valued measure $\gamma : E \to G$ (i.e., $\gamma(a + b) = \gamma(a) + \gamma(b)$ whenever $a + b$ is defined in E) such that the following conditions hold: (i) $\gamma(E)$ generates G. (ii) If H is an additive Abelian group and $\phi : E \to H$ is an H-valued measure, then there is a group homomorphism $\phi^* : G \to H$ such that $\phi = \phi^* \circ \gamma$. Due to [FoBe], every effect algebra possesses a universal group.

Ravindran [Rav] ([DvPu, Thm 1.17.17]) proved the following important result.

Theorem 2.2 *Let E be an effect algebra with the Riesz decomposition property. Then there exists a unital interpolation group (G, u) with a strong unit u such that $\Gamma(G, u)$ is isomorphic with E, and there is a G-valued injective measure γ such that (G, γ) is a universal group for E.*

In [Dvu, Thm 5.8] we have proved that the category of effect algebras with the (RDP) is categorically equivalent to the category of unital Abelian interpolation po-groups: Let $\mathcal{EA}_{\mathcal{RDP}}$ be the category of effect algebras with the (RDP) whose objects are effect algebras and morphisms are homomorphisms of effect algebras, and let $\mathcal{UIG}$ be the category of unital interpolation po-groups (G, u) with a fixed strong unit u and morphisms are homomorphisms of unital po-groups, i.e., positive homomorphisms of unital po-groups which preserve fixed strong units.

Theorem 2.3 *The mapping* $\Gamma: \mathcal{UIG} \to \mathcal{EA}_{\mathcal{RDP}}$ *defines the categorical equivalence of the category* $\mathcal{UIG}$ *of unital interpolation po-groups and the category of effect algebras with the* (RDP).

In addition, suppose that $h: \Gamma(G,u) \to \Gamma(H,v)$ *is a homomorphism of effect algebras with the* (RDP), *then there is a unique homomorphism* $f: (G,u) \to (H,v)$ *of unital po-groups such that* $h = \Gamma(f)$, *and*

(i) *if* h *is surjective, so is* f;

(ii) *if* h *is injective, so is* f.

We present now a crucial result for our investigation.

Theorem 2.4 *Let* $(E;+,\cdot,0,1)$ *be a product effect algebra with the Riesz decomposition property. Then there exists a unique (up to isomorphism) unital po-ring* (R,u) *satisfying the Riesz decomposition property with the product* $\cdot$ *and with* $u\cdot u \le u$ *such that* $E \cong \Gamma(R,u)$ *and* $\phi(a\cdot b) = \phi(a)\cdot\phi(b)$, *where* ϕ *is an isomorphism of* E *onto* $\Gamma(R,u)$ *preserving the product* $\cdot$.

If $\cdot$ *is commutative or associative, so is* $\cdot$ *on* R.

Proof. Let E be an effect algebra with a product $\cdot$. According to Theorem 2.2, there is a unital group (R,u) with a strong unit u satisfying the Riesz decomposition property, and an isomorphism ϕ from E onto $\Gamma(R,u)$ We can define the product $\cdot$ on $\Gamma(R,u)$ as follows

$$\phi(a)\cdot\phi(b) := \phi(a\cdot b),\ a,b\in E.$$

Because $\Gamma(R,u)$ generates the positive cone R^+ of R, ϕ preserves all existing $+$ in R, we see that $\cdot$ is a product on $\Gamma(R,u)$.

Given $g \in R^+$ there exist $a_1,\dots,a_n \in E$ such that $g = \sum_{i=1}^n \phi(a_i)$. For any $\phi(c)$, where $c\in E$, we define

$$g\cdot\phi(c) = \phi(a_1\cdot c) + \cdots + \phi(a_n\cdot c). \tag{2.1}$$

We claim that (2.1) is defined unambiguously. Indeed, if $g = \sum_{j=1}^m \phi(b_j)$, for some $b_1,\dots,b_m \in E$, due to the Riesz decomposition property, there exist elements $c_{ij}\in E$ such that $a_i = \sum_{j=1}^m c_{ij}$ and $b_j = \sum_{i=1}^n c_{ij}$ for all i, $1\le i\le n$, and all j, $1\le j\le m$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \phi(a_i \cdot c) &= \sum_{i=1}^{n} \phi((\sum_{j=1}^{m} c_{ij}) \cdot c) = \sum_{i=1}^{n} \phi(\sum_{j=1}^{m} (c_{ij} \cdot c)) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \phi(c_{ij} \cdot c) = \sum_{j=1}^{m} \sum_{i=1}^{n} \phi(c_{ij} \cdot c) \\
&= \sum_{j=1}^{m} \phi(\sum_{i=1}^{n} c_{ij} \cdot c) = \sum_{j=1}^{m} \phi((\sum_{i=1}^{n} c_{ij}) \cdot c) \\
&= \sum_{j=1}^{m} \phi(b_j \cdot c),
\end{aligned}$$

which proves that $\cdot$ on $R^+ \times \Gamma(R,u)$ is correct. We now extend $\cdot$ to $R \times \Gamma(R,u)$ as follows: If $g = g_1 - g_2$, $g_1, g_2 \in R^+$, then

$$g \cdot \phi(c) := g_1 \cdot \phi(c) - g_2 \cdot \phi(c).$$

Since if $g_1 - g_2 = h_1 - h_2$ for $g_i, h_i \in R^+$, $i = 1,2$, then $g_1 + h_2 = h_1 + g_2$, by (2.1) we have

$$\begin{aligned}
(g_1 + h_2) \cdot \phi(c) &= (h_1 + g_2) \cdot \phi(c), \\
g_1 \cdot \phi(c) + h_2 \cdot \phi(c) &= h_1 \cdot \phi(c) + g_2 \cdot \phi(c), \\
g_1 \cdot \phi(c) - g_2 \cdot \phi(c) &= h_1 \cdot \phi(c) - h_2 \cdot \phi(c).
\end{aligned}$$

Let now $c \in R^+$. Then $c = \phi(c_1) + \cdots + \phi(c_s)$, where $c_t \in M$, $t = 1, \ldots, s$. We extend $\cdot$ to $R \times R^+$ as follows

$$g \cdot c := \sum_{t=1}^{s} g \cdot \phi(c_t), \; g \in G.$$

If $c = \sum_{w=1}^{v} \phi(d_w)$, using the Riesz decomposition property, we have as above

$$\sum_{t=1}^{s} g \cdot \phi(c_t) = \sum_{w=1}^{v} g \cdot \phi(d_w).$$

It is clear that the "multiplication" $\cdot$ can be extended to the whole $R \times R$: If $c = c_1 - c_2$, where $c_1, c_2 \in R^+$, let

$$g \cdot c := g \cdot c_1 - g \cdot c_2.$$

It is evident that if $g, h \in R^+$, then $g \cdot h \in R^+$, and $u \cdot u \leq u$, and due to (ii), $\cdot$ is associative on R, so that $(R; +, \cdot, 0)$ is a po-ring with a strong unit u, which proves Theorem.

The commutativity or associativity of the product on the ring R follows from its construction. $\square$

We present now some effect algebras which admit products.

Proposition 2.5 *If e is unity of a product effect algebra E, then $e = 1$. Moreover, 1 is unity for $\cdot$ if and only if, for any $a \in E$, $a \cdot 1 \geq a$ and $1 \cdot a \geq a$.*

Proof. Let e be unity for the product $\cdot$. Then $1 \cdot 1 \leq 1 = 1 \cdot e \leq 1 \cdot 1$ which proves $1 \cdot 1 = 1$. Let $a \in E$. Then $a \cdot 1 \geq a \cdot e = a$ and $1 \cdot a \geq e \cdot a = a$. In addition, $a + a' = 1 = 1 \cdot 1 = (a + a') \cdot 1 = a \cdot 1 + a' \cdot 1 \geq a + a'$, which entails $a \cdot 1 = a$. By symmetry, we have $1 \cdot a = a$ for any $a \in E$, and in addition $1 = 1 \cdot e = e$.

Assume now that for any $a \in E$, $a \cdot 1 \geq a$ and $1 \cdot a \geq a$. Then $1 = a + a' \leq a \cdot 1 + a' \cdot 1 = (a + a') \cdot 1 = 1 \cdot 1 \leq 1$ which proves that 1 is unity for $\cdot$ on E. $\square$

Proposition 2.6 *A finite effect algebra E which is a lattice and with the* (RDP) *admits a product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in E$ if and only if E is a Boolean algebra, i.e., $a \vee a' = 1$ for any $a \in E$. If this is the case, then $a \cdot b = a \wedge b \in E$.*

Proof. Every lattice effect algebra with the (RDP) can be converted into an MV-algebra. Therefore, the assertion follows from [DvPu, Thm 5.3.17]. $\square$

3 Product on Perfect Effect Algebras

In the present Section, we show that we can even define a product on perfect effect algebras. Such effect algebras were introduced and studied in [Dvu].

Let G be a directed Abelian po-group and define the lexicographical product

$$G(\mathbb{Z}) := \mathbb{Z} \times_{lex} G, \tag{3.1}$$

where $\mathbb{Z}$ is the group of all integers. Then the element $(1, 0)$ is a strong unit in the po-group $G(\mathbb{Z})$ and

$$E(G) := \Gamma(G(\mathbb{Z}), (1, 0)), \tag{3.2}$$

is an effect algebra. Every element $a \in E(G)$ is of the form either $a = (1, -g)$ or $a = (0, g)$, where $g \in G^+$. In addition, if G is a directed interpolation group, then $G(\mathbb{Z})$ is an interpolation group, [Goo, Cor. 2.12], and $E(G)$ satisfies the Riesz interpolation property.

An *ideal* of an effect algebra E is a non-empty subset I of E such that (i) $x \in E$, $y \in I$, $x \le y$ imply $x \in I$, and (ii) if $x, y \in I$ and $x + y$ is defined in E, then $x + y \in I$. An ideal I is said to be the *Riesz ideal* if $x \in I$, $a, b \in E$ and $x \le a + b$, there exist $a_1, b_1 \in I$ such that $x \le a_1 + b_1$ and $a_1 \le a$ and $b_1 \le b$.

For example, if E is with the (RDP), then any ideal of E is Riesz.

A proper ideal I of an effect algebra E is said to be *maximal* if is not a proper subset of another proper ideal of E. By Zorn's lemma E possesses at least one maximal ideal, and let $\mathcal{M}(E)$ be the set of all maximal ideals of E. We define the *radical* of E, $\mathrm{Rad}(E)$, via

$$\mathrm{Rad}(E) = \bigcap\{I : \ I \in \mathcal{M}(E)\}.$$

An element a is said to be *infinitesimal* if na is defined in E for any integer $n \ge 1$, and denote by $\mathrm{Infinit}(E)$ the set of all infinitesimals of E. Then (i) $0 \in \mathrm{Infinit}(E)$, (ii) if $b \in E$, $a \in \mathrm{Infinit}(E)$ and $b \le a$, then $b \in \mathrm{Infinit}(E)$, and (iii) $1 \notin \mathrm{Infinit}(E)$. By [Dvu, Prop 4.1] if E is an effect algebra satisfying the (RDP), then

$$\mathrm{Infinit}(E) \subseteq \mathrm{Rad}(E). \tag{3.3}$$

Since it can happen that in (3.3) we have the proper inclusion, according to [Dvu], an effect algebra E with the (RDP) is said to have the *Rad-property* if in (3.3) we have the equality. In [Dvu], we have introduced perfect effect algebras: We say that an effect algebra E with the Rad-property is *perfect* if, for any element $a \in E$ either $a \in \mathrm{Rad}(E)$ or $a' \in \mathrm{Rad}(E)$. According to [Dvu, Prop 5.3], every perfect effect algebra E is of the form (3.2) for some directed Abelian interpolation po-group G.

We say that G^+ of a po-group G is *Archimedean* if, for some $a, b \in G^+$ with $na \le b$ for any $n \ge 1$, $a = 0$.

Proposition 3.1 *Let E be a perfect effect algebra and define a binary operation $\cdot$ as follows. For $a, b \in \mathrm{Rad}(E)$, let*

$$
\begin{aligned}
a\cdot b &= 0,\\
a\cdot b' &= a,\\
a'\cdot b &= b,\\
a'\cdot b' &= (a+b)'.
\end{aligned} \tag{3.4}
$$

Then $\cdot$ is a commutative product on E such that $a\cdot 1 = a = 1\cdot a$ for any $a\in E$.

Let $E=\Gamma(\mathbb{Z}\times_{lex} G,(1,0))$ and let G^+ be Archimedean. Then on E there is a unique product $\cdot$ such that $a\cdot 1 = a = 1\cdot a$ for any $a\in E$; this product is defined by (3.4).

Proof. Since $\mathrm{Rad}(E)\cap \mathrm{Rad}'(E)=\emptyset$, we see that $\cdot$ is correctly defined by (3.4), and it is easy to verify that it is a product in question on E.

Let a and b be two element of $\mathrm{Rad}(E)$. Then nb is defined in E for any integer $n\geq 1$, and by [Dvu, Prop 5.1], we have

$$a = a\cdot 1 = a\cdot((nb)+(nb)') = n(a\cdot b) + a\cdot (nb)' \geq n(a\cdot b).$$

The Archimedeanicity of G^+ entails that $a\cdot b = 0$. Hence $a = a\cdot b + a\cdot b' = a\cdot b'$. Similarly $a = (b'\cdot a)+(b\cdot a) = b'\cdot a$, and $a' = a'\cdot b' + a'\cdot b = a'\cdot b' + b$ which gives $a'\cdot b' = (a+b)'$. $\square$

Proposition 3.2 *Let E be an effect algebra with the* (RDP) *satisfying the Rad-property such that* $\mathrm{Rad}(E)$ *is Archimedean, i.e., if $na\leq b$ for any $n\geq 1$ and for some $b\in \mathrm{Rad}(E)$, then $a=0$. If $\cdot$ is a product on E such that $a\cdot 1 = a = 1\cdot a$ for any $a\in E$, then, for any $a,b\in \mathrm{Rad}(E)$,* (3.4) *holds.*

Proof. The first three identities follow the same ideas as the proof of Proposition 3.1. For the last one we have

$$1 = (a+a')\cdot(b+b') = a\cdot b + a\cdot b' + a'\cdot b + a'\cdot b' = (a+b) + (a'\cdot b'),$$

which entails that $a'\cdot b' = (a+b)'$. $\square$

Remark 3.3 It is worth recalling that if an effect algebra E satisfying the (RDP) and the Rad-property has an Archimedean radical, then there is a unique binary operation $\cdot : (\mathrm{Rad}(E)\cup \mathrm{Rad}'(E))\times(\mathrm{Rad}(E)\cup\mathrm{Rad}'(E))\to \mathrm{Rad}(E)\cup\mathrm{Rad}(E)'$ which is commutative, associative, and for all $a,b,c\in \mathrm{Rad}(E)\cup\mathrm{Rad}(E)$ we have $(a+b)\cdot c = a\cdot c + b\cdot c$, $a\cdot 1 = a$; it is defined via (3.4). On the other hand it does not mean that $\cdot$ can be extended to a product on the whole E satisfying $a \cdot 1 = a = 1 \cdot a$ for any $a \in E$. For example, if E is a finite lattice, then $\mathrm{Rad}(E) = \{0\}$, and $\cdot$ can be extended iff E is a Boolean algebra, Proposition 2.6. An extension of $\cdot$ is possible, e.g., if E is a perfect effect algebra with an Archimedean radical, Proposition 3.1.

4 Product Effect Algebras and Categorical Equivalence

In this Section, we show that the category of product effect algebras with the (RDP) is categorically equivalent with the category of Abelian unital interpolation po-rings.

Denote by $\mathcal{PROD}_{\mathcal{EA}}$ the category of product effect algebras, i.e., its objects are product effect algebras, and morphisms are homomorphisms of effect algebras preserving also $\cdot$. We denote by $\mathcal{R}$ the category of associative unital interpolation po-rings (R, u) with a distinguished strong unit u such that $u \cdot u \le u$, and its morphisms are homomorphisms of po-groups which preserve $\cdot$ and the distinguished strong units. We denote by $\Gamma_{\mathcal{R}}$ a map from $\mathcal{R}$ into $\mathcal{PROD}_{\mathcal{EF}}$ defined by $\Gamma_{\mathcal{R}}((R;+,\cdot,0,\le,u)) := (\Gamma(R,u);+,\cdot,0,u)$, and $\Gamma_{\mathcal{R}}(f) := f|\Gamma(R,u)$.

Proposition 4.1 *$\Gamma_{\mathcal{R}}$ is a faithful and full functor from $\mathcal{R}$ to $\mathcal{PROD}_{\mathcal{EA}}$.*

Proof. Let h_1 and h_2 be two morphism from $(R;+,\cdot,0,\le,u)$ into $(R';+,\cdot,0,\le,u')$ such that $\Gamma_{\mathcal{R}}(h_1) = \Gamma_{\mathcal{R}}(h_2)$. Then $h_1(a) = h_2(a)$ for any $a \in \Gamma(R,u)$. Since $\Gamma(R,u)$ generates R^+ and R, we have that $h_1(g) = h_2(g)$ for any $g \in R$ which proves that $\Gamma_{\mathcal{R}}$ is faithful.

To prove that $\Gamma_{\mathcal{R}}$ is a full functor, suppose that f is a morphism from $\Gamma(R,u)$ into $\Gamma(R',u')$. Since $\Gamma(R,u)$ generates R, due to the Riesz decomposition property, f can be uniquely extended to a group-homomorphism $\hat{f}$ from R into R'.

We show that $\hat{f}$ preserves the product $\cdot$ in R.

Let $a, b \in R^+$. There exist $a_1,\dots,a_n$, $b_1,\dots,b_m \in \Gamma(R,u)$ such that $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_m$. Then $a \cdot b = \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j$ and $a_i \cdot b_j \in \Gamma(R,u)$. Calculate $\hat{f}(a\cdot b) = \sum_{i=1}^n \sum_{j=1}^m \hat{f}(a_i \cdot b_j) = \sum_{i=1}^n \sum_{j=1}^m \hat{f}(a_i) \cdot \hat{f}(b_j) = (\sum_{i=1}^n \hat{f}(a_i)) \cdot (\sum_{j=1}^m \hat{f}(b_j)) = \hat{f}(a) \cdot \hat{f}(b)$.

If now $a, b \in R$, then $a = a_1 - a_2$ and $b = b_1 - b_2$, where $a_1, a_2, b_1, b_2 \in R^+$. Then $\hat{f}(a \cdot b) = \hat{f}(a_1 \cdot b_1 - a_1 \cdot b_2 - a_2 \cdot b_1 + a_2 \cdot b_2) = \hat{f}(a_1 \cdot b_1) - \hat{f}(a_1 \cdot b_2) - \hat{f}(a_2 \cdot b_1) + \hat{f}(a_2 \cdot b_2) = \hat{f}(a_1) \cdot \hat{f}(b_1) - \hat{f}(a_1) \cdot \hat{f}(b_2) - \hat{f}(a_2) \cdot \hat{f}(b_1) + \hat{f}(a_2) \cdot \hat{f}(b_2) = \hat{f}(a) \cdot \hat{f}(b)$.

Consequently, we have proved that $\hat{f}$ is a morphism from $(R; +, \cdot, 0, \leq, u)$ into $(R'; +, \cdot, 0, \leq, u')$ such that $\Gamma_{\mathcal{R}}(\hat{f}) = f$. $\square$

We are now ready to present the main statement of the present section.

Theorem 4.2 *The functor $\Gamma_{\mathcal{R}}$ defines a categorical equivalence of the category $\mathcal{R}$ of unital associative interpolation po-rings with a strong unit u such that $u \cdot u \leq u$ and the category $\mathcal{PROD}_{\mathcal{EA}}$ of product effect algebras.*

Proof. According to [MaL, Thm IV.4.1], to prove that $\Gamma_{\mathcal{R}}$ is an equivalence of the categories in question, it is necessary and sufficient to show that $\Gamma_{\mathcal{R}}$ is faithful and full, and each object E from $\mathcal{PROD}_{\mathcal{EA}}$ is isomorphic to $\Gamma_{\mathcal{R}}((R; +, \cdot, 0, \leq, u))$ for some object $(R; +, \cdot, 0, \leq, u)$ in $\mathcal{R}$.

Due to Proposition 4.1, $\Gamma_{\mathcal{R}}$ is faithful and full, and by Theorem 2.4, there exists an object $(R; +, \cdot, 0, \leq, u)$ in $\mathcal{R}$ such that $\Gamma_{\mathcal{R}}((R; +, \cdot, 0, \leq, u))$ is isomorphic to E. That proves Theorem. $\square$

5 Subdirect Product of Product Effect Algebras

In the present Section, we show that every product effect algebra with the (RDP) and with 1 as a product unity is a subdirect product of antilattice product effect algebras with the (RDP). To show that, we give some results on ideals and quotient effect algebras.

Let $\cdot$ be a product on an effect algebra E. We say that an ideal I of E is a $\cdot$-*ideal* of E if $a \in I$ and $b \in E$ entail $a \cdot b \in I$ and $b \cdot a \in I$. If 1 is a product unity, then every ideal of E is a $\cdot$-ideal of E.

Let P be a proper ideal of E. We define a relation $\sim_P$ on E via $a \sim_P b$ iff $a - e = b - f$ for some $e, f \in P$. According to [DvPu, Sec. 3.1.2], we have that $\sim_P$ is an equivalence such that (i) $a + b \in E$, $a_1 + b_1 \in E$, $a \sim_P a_1$, $b \sim_P b_1$ imply $(a+b) \sim_P (a_1 + b_1)$, (ii) $a \sim_P b$ implies $a' \sim_P b'$, (iii) $a + b \in E$, $c \sim_P a$ imply there exists an element $d \in E$ such that $d \sim_P b$ and $d + c \in E$, (iv) $a + b, a_1 + b_1 \in E$, $a_1 \sim_P a, (a_1 + b_1) \sim_P (a + b)$ imply $b_1 \sim_P b$. If we define $a/P := [a] := [a]_P := \{b \in E : \ b \sim_P a\}$, then $E/P := \{[a]_P : \ a \in E\}$

is an effect algebra, where $[a]+[b]=[c]$ iff there exist $a_1 \in [a], b_1 \in [b], c_1 \in [c]$ such that $a_1+b_1=c_1$. As the constant elements in E/P we take $[0]$ and $[1]$. We recall that

$$[a]_P \leq [b]_P \text{ in } E/P \quad \Leftrightarrow \quad \text{there exists } a_1 \in [a]_P \text{ such that } a_1 \leq b. \tag{5.1}$$

We say that an ideal P of an effect algebra E with the (RDP) is *prime* if, for all ideals I and J of E, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. We denote by $\mathcal{P}(E)$ the set of all prime ideals of E.

Let a be a non-zero element of E. By Zorn's lemma, there exists an ideal V not containing a and which is maximal with respect to this property. Such an ideal is said to be a *value* of a in E.

According to [Dvu], (i) every value is a prime ideal of E, (ii) E/P is an effect algebra with the (RDP) whenever P is a proper ideal of E with the (RDP), (iii) E/P is antilattice effect algebra if P is prime.

Proposition 5.1 *Let P be a proper ideal of a product effect algebra $(E;+,\cdot,0,1)$ with the* (RDP)*. If $a_1 \sim_P a_2$ and $b_1 \sim_P b_2$, then $a_1 \cdot b_1 \sim_P a_2 \cdot b_2$. In addition, E/P is a product effect algebra with the induced product $\cdot$ defined via*

$$[a]_P \cdot [b]_P := [a \cdot b]_P, \quad (a, b \in E), \tag{5.2}$$

and the canonical mapping $f:\ E \to E/P$ given by $f(a)=[a]_P$, $a \in E$, is a surjective homomorphism preserving all existing joins, meets and the product in E.

Proof. There are $e, f, u, v \in P$ such that $a_1-e=a_2-f$ and $b_1-u=b_2-v$. Then

$$\begin{aligned} (a_1-e)\cdot(b_1-u) &= (a_2-f)\cdot(b_2-v),\\ a_1\cdot(b_1-u)-e\cdot(b_1-u) &= a_2\cdot(b_2-v)-f\cdot(b_2-v),\\ a_1\cdot b_1-(a_1\cdot u+e\cdot(b_1-u)) &= a_2\cdot b_2-(a_2\cdot v+f\cdot(b_2-v)). \end{aligned}$$

Since $x:=a_1\cdot u+e\cdot(b_1-u)\in P$ and $y:=a_2\cdot v+f\cdot(b_2-v)\in P$, we have $a_1\cdot b_1-x=a_2\cdot b_2-y$, i.e., $a_1\cdot b_1 \sim_P a_2\cdot b_2$. Therefore, the product $\cdot$ on E/P defined by (5.2) is a product in question.

The rest is now clear, see also [Dvu, Prop 6.9]. $\square$

Let $\{E_i\}_{i\in I}$ be an indexed system of effect algebras. The Cartesian product $\prod_{i\in I} E_i$ can be organized into an effet algebra with the partial addition defined by coordinates. Each E_i is with the (RDP) iff so is $\prod_i E_i$. If every E_i is a product effect algebra, then so is $\prod_i E_i$, where the product is defined by coordinates.

We say that an effect algebra is a *subdirect product* of effect algebras $\{E_i\}_{i\in I}$ if there is an injective homomorphism $f:\ E \to \prod_{i\in I} E_i$ such that, for every $j \in I$, $\pi_j \circ f$ is a surjective homomorphism from E onto E_j, where π_j is the j-th projection of $\prod_i E_i$ onto E_j. In an analogical manner we define a subdirect product of product effect algebras assuming that the function f preserves also the product in E, and every $\pi_j \circ f$ is a surjective homomorphism of effect algebras preserving product.

Theorem 5.2 *Every product effect algebra* $(E;+,\cdot,0,1)$ *with the* (RDP) *having* 1 *as a product unity is a subdirect product of antilattice product effect algebras with the* (RDP). *In addition, we can find an injective homomorphism into the subdirect product which preserves all existing meets, joins and the product in* E.

Proof. According to [Dvu, Thm 7.2], every effect algebra with the (RDP) is a subdirect product of antillatice effect algebras $\{E/P:\ P \in \mathcal{P}(E),\ P \neq E\}$ and each of them is with the (RDP). The corresponding injective mapping $f:\ E \to \prod_P E/P$ is defined via $f(a) = ([a]_P)_{P\in\mathcal{P}(E)}$, $a \in E$.

Due to Proposition 5.1, every E/P is a product effect algebra. In addition, the mapping f preserves all existing meets, joins (see [Dvu, Thm 7.2]), and the product in E, and $\pi_P \circ f$ is a surjective homomorphism from E onto E/P preserving $\cdot$. $\square$

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