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CHAOS IN NON–AUTONOMOUS DIFFERENTIAL INCLUSIONS

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Abstract. The existence of a continuum many chaotic solutions are shown for certain differential inclusions which are small non-autonomous multivalued perturbations of ordinary differential equations possessing homoclinic solutions to hyperbolic fixed points. Applications are given to dry friction problems.

1. Introduction

Differential equations with non-smooth terms arise in many cases. For instance, as the effect of dry friction in mechanical systems, or as the case of impacts. Non-smooth functions are used in biological systems or electrical circuits to model switches. They also occur in control theory [1], [13], [14], [15]. To handle mathematically non-smooth differential equations, they are considered as differential inclusions [4].

Motivated by coupled several oscillators with small quasiperiodic forcing terms and with small dry friction effects (see (4.4) and (4.12) below), in this paper, we consider differential inclusions of the form

$$\dot{x}(t) \in f(x(t)) + \epsilon g(x(t), \epsilon, t) \quad \text{a.e. on} \quad \mathbb{R} \tag{1.1}$$

with $x \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}$. By a solution of any differential inclusion in this paper we mean a function which is absolutely continuous and satisfies the differential inclusion almost everywhere.

We make the following assumptions about (1.1):

(i) $f \in C^2(\mathbb{R}^n, \mathbb{R}^n)$ and $g\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}^n} \setminus \emptyset$, $i = 1, \cdots, k$ has a form $g(x, \epsilon, t) = F(x, \epsilon) + G(x, \epsilon, t)$, where mapping $F\colon \mathbb{R}^n \times \mathbb{R} \to 2^{\mathbb{R}^n} \setminus \emptyset$ is upper–semicontinuous [4, 21] with compact and convex values, and $G \in C(\mathbb{R}^{n+2}, \mathbb{R}^n)$. Moreover, for any bounded subset $\Omega \subset \mathbb{R}^{n+1}$, mapping $G(x, \epsilon, t)$ is bounded and uniformly continuous on $\Omega \times \mathbb{R}$.

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(ii) $f(0) = 0$ and the eigenvalues of $Df(0)$ lie off the imaginary axis.

(iii) The unperturbed equation has a homoclinic solution. That is, there exists a differentiable function $t \to \gamma(t) \neq 0$ such that $\lim_{t\to+\infty} \gamma(t) = \lim_{t\to-\infty} \gamma(t) = 0$ and $\dot\gamma(t) = f(\gamma(t))$.

The most interesting case is when

$$g(x, \epsilon, t) = q(x, \epsilon, \omega_1 t, \cdots, \omega_m t) \tag{1.2}$$

for $\omega_1, \cdots, \omega_m \in \mathbb{R}$ and the multivalued mapping $q\colon \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^m \to 2^{\mathbb{R}^n} \setminus \emptyset$ is upper–semicontinuous [4, 21] with compact and convex values. Moreover, the multivalued mapping $q(x, \epsilon, \theta_1, \cdots, \theta_m)$ is 1-periodic in each θ_i, $i = 1, 2, \cdots, m$. We note that our method of Section 4 of this paper is applicable to (1.2). When q is periodic in t, this problem is solved by J. Awrejcewicz and M.H. Holicke [2], and by the author [6]. Almost periodic ordinary differential equations are investigated by K.J. Palmer and D. Stoffer [18], K.R. Meyer and G.R. Sell [16], and S. Wiggins [22]. For partial differential equations, such problems are studied by M. Berti and C. Carminati [3].

The purpose of this paper is to find conditions ensuring that for any sequence $\{e_j\}_{j\in\mathbb{Z}}$, $e_j \in \{0, 1\}$ and $\epsilon > 0$ small, there is a bounded solution of (1.1) on $\mathbb{R}$. Moreover, we have different solutions for different sequences and in addition, sequences characterize (or count) turnings of corresponding solutions around γ. These chaotic solutions are in a narrow layer around γ. Consequently, we extend the deterministic chaos of [12] and [17] to (1.1). The proof of the main theorem of this paper is very similar to [6], but for the reader convenience we present it here.

In many cases, the assumption (iii) is replaced by

(iv) The unperturbed equation $\dot x = f(x)$ has a manifold of homoclinic solutions. That is, there exists a C^2–smooth mapping $\gamma\colon \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}^n$ such that $t \to \gamma(\theta, t)$ is a homoclinic solution of $\dot x = f(x)$ to 0 and $\left\{\frac{\partial\gamma}{\partial\theta_i}(\theta, t), \dot\gamma(\theta, t)\right\}_{i=1}^{d-1}$, $\theta = (\theta_1, \cdots, \theta_{d-1})$ form a basis of bounded solutions on $\mathbb{R}$ of the variational equation $\dot u(t) = Df(\gamma(\theta, t))u(t)$.

We finish this paper with almost periodically perturbed problems. We show like in [16], [18] and [20], that there are chaotic solutions over the hull [16] of the almost periodic perturbation function. We also present an example of two coupled nonlinear oscillators with multivalued and quasiperiodic perturbations when the unperturbed equation has a rotational symmetry creating a 2-dimensional homoclinic manifold. Such a system was not studied in [16], [18] and [20].

2. The Variational Equation

We repeat here some results from [6]. Let W^s, W^u be the stable and unstable manifolds, respectively, of the origin of the unperturbed equation

$$\dot x = f(x)\,. \tag{2.1}$$

By the variational equation of (2.1) along γ we mean the linear differential equation

$$\dot u(t) = Df(\gamma(t))u(t)\,. \tag{2.2}$$

We need the following result.

Theorem 2.1. ([11]) *Let $d_s = \dim(W^s)$, $d_u = \dim(W^u)$ for (2.1) and let I_s, I_u denote the identity matrices of order d_s, d_u respectively. There exists a fundamental solution U for (2.2) along with a non–singular matrix C, constants $M > 0$, $K_0 > 0$ and four projections P_{ss}, P_{su}, P_{us}, P_{uu} such that $P_{ss} + P_{su} + P_{us} + P_{uu} = I$ and that the following hold:*

(i) $|U(t)(P_{ss} + P_{us})U(s)^{-1}| \le K_0 e^{2M(s-t)}$ *for* $0 \le s \le t$,

(ii) $|U(t)(P_{su} + P_{uu})U(s)^{-1}| \le K_0 e^{2M(t-s)}$ *for* $0 \le t \le s$,

(iii) $|U(t)(P_{ss} + P_{su})U(s)^{-1}| \le K_0 e^{2M(t-s)}$ *for* $t \le s \le 0$,

(iv) $|U(t)(P_{us} + P_{uu})U(s)^{-1}| \le K_0 e^{2M(s-t)}$ *for* $s \le t \le 0$,

(v) $\lim\limits_{t\to+\infty} U(t)(P_{ss} + P_{us})U(t)^{-1} = C \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} C^{-1}$,

(vi) $\lim\limits_{t\to+\infty} U(t)(P_{su} + P_{uu})U(t)^{-1} = C \begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix} C^{-1}$,

(vii) $\lim\limits_{t\to-\infty} U(t)(P_{ss} + P_{su})U(t)^{-1} = C \begin{pmatrix} 0 & 0 \\ 0 & I_u \end{pmatrix} C^{-1}$,

(viii) $\lim\limits_{t\to-\infty} U(t)(P_{us} + P_{uu})U(t)^{-1} = C \begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix} C^{-1}$.

Also, there exists an integer d with $\operatorname{rank} P_{ss} = \operatorname{rank} P_{uu} = d$.

Let u_j denote column j of U and assume these are numbered so that

$$P_{uu} = \begin{pmatrix} I_d & 0_d & 0 \\ 0_d & 0_d & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad P_{ss} = \begin{pmatrix} 0_d & 0_d & 0 \\ 0_d & I_d & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, I_d denotes the $d \times d$ identity matrix and 0_d denotes the $d \times d$ zero matrix.

For each $i = 1, \cdots, n$ we define $u_i^{\perp}(t)$ by $\langle u_i^{\perp}(t), u_j(t)\rangle = \delta_{ij}$, where $\langle\cdot,\cdot\rangle$ is the inner product. The vectors $u_i^{\perp}$ can be computed from the formula $U^{\perp t} = U^{-1}$ where $U^{\perp}$ denotes the matrix with $u_j^{\perp}$ as column j. Differentiating $UU^{\perp t} = I$ we obtain $\dot{U}U^{\perp t} + U\dot{U}^{\perp t} = 0$ so that $\dot{U}^{\perp} = -(U^{-1}\dot{U}U^{\perp t})^t = -Df(\gamma)^t U^{\perp}$. Thus, $U^{\perp}$ is the adjoint of U. The function $\dot{\gamma}$ is always a solution to the variational equation (2.2). If $\dot{\gamma}$ appears as a column of U we shall assume $u_{2d} = \dot{\gamma}$. In general we shall always assume $< u_{2d}^{\perp}, \dot{\gamma} > \neq 0$.

Fix $\tau > 0$ and define the following Banach spaces:

$$Z_\tau = C\left([-\tau,\tau], \mathbb{R}^n\right),$$
$$Y_\tau = L_\infty\left([-\tau,\tau], \mathbb{R}^n\right).$$

We take as norm $\|\cdot\|_\tau$, the supremum for Z_τ, respectively $|\cdot|_\tau$, the usual L_∞ norm for Y_τ.

Integration of the inequalities in Theorem 2.1 yields the following result.

Theorem 2.2. ([10]) *Let U be the solution to (2.2) along with the projections P_{ss}, P_{su}, P_{us}, P_{uu} as in Theorem 2.1. Then there exists a constant $A>0$ such that for any $\tau>0$ and any $z\in Y_\tau$ the following hold:*

$$\text{(i)} \qquad \int_0^t |U(t)(P_{ss}+P_{us})U(s)^{-1}z(s)|\,ds \le A|z|_\tau \quad \text{for } t\in[0,\tau]\,,$$

$$\text{(ii)} \qquad \int_t^\tau |U(t)(P_{su}+P_{uu})U(s)^{-1}z(s)|\,ds \le A|z|_\tau \quad \text{for } t\in[0,\tau]\,,$$

$$\text{(iii)} \qquad \int_t^0 |U(t)(P_{ss}+P_{su})U(s)^{-1}z(s)|\,ds \le A|z|_\tau \quad \text{for } t\in[-\tau,0]\,,$$

$$\text{(iv)} \qquad \int_{-\tau}^t |U(t)(P_{us}+P_{uu})U(s)^{-1}z(s)|\,ds \le A|z|_\tau \quad \text{for } t\in[-\tau,0]\,.$$

Now we consider the linear equation

$$\dot v(t) = Df(0)v(t)\,, \tag{2.3}$$

and put

$$\begin{gathered} V(t)=\mathrm{e}^{tDf(0)}, \quad Q_{us}=C\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix}C^{-1}\\ Q_{ss}=0, \quad Q_{uu}=0, \quad Q_{su}=C\begin{pmatrix} 0 & 0\\ 0 & I_u\end{pmatrix}C^{-1}\,, \end{gathered}$$

where C is from Theorem 2.1.

Theorem 2.3. ([6]) *By considering in Theorems $2.1-2$ the exchanges*

$$U(t)\leftrightarrow V(t), \quad P_{ss}\leftrightarrow Q_{ss}, \quad P_{su}\leftrightarrow Q_{su}, \quad P_{us}\leftrightarrow Q_{us}, \quad P_{uu}\leftrightarrow Q_{uu}\,,$$

Theorems $2.1-2$ are valid for (2.3).

For any finite sequence $E=\{e_j\}_{j=-p}^p\in\{0,1\}^{2p+1}$, $p\in\mathbb{N}$ we put:
if $e_j=1$ then

$$\begin{gathered} A_j(t)=Df(\gamma(t)), \quad U_j=U, \quad P_{ss}^j=P_{ss}\\ P_{su}^j=P_{su}, \quad P_{us}^j=P_{us}, \quad P_{uu}^j=P_{uu}\,; \end{gathered}$$

if $e_j=0$ then

$$\begin{gathered} A_j(t)=Df(0), \quad U_j=V, \quad P_{ss}^j=Q_{ss}\\ P_{su}^j=Q_{su}, \quad P_{us}^j=Q_{us}, \quad P_{uu}^j=Q_{uu}\,. \end{gathered}$$

We put $I_p=\{-p,-p+1,\cdots,p\}$.

Now we fix a sequence $\{t_i\}_{i\in\mathbb{Z}}$, $t_i < t_{i+1}$ such that $t_i \to \pm\infty$ as $i \to \pm\infty$. For any $\alpha \in \mathcal{A}_E$, where the set $\mathcal{A}_E$ is defined by

$$\mathcal{A}_E = \Big\{(\alpha_{-p}, \cdots, \alpha_p) \in \mathbb{R}^{2p+1} : \ \alpha_j \in \mathbb{R} \quad \text{if} \quad e_j = 1 \quad \text{and} \quad \alpha_j = 0 \quad \text{if} \quad e_j = 0\Big\},$$

we consider the non–homogeneous coupled linear equations

$$\begin{aligned} \dot z_j &= A_j(t-\alpha_j)z_j + h_j, \quad j \in I_p, \quad h_j \in Y_{\tau_j} \\ z_j(\tau_j) &= z_{j+1}(-\tau_{j+1}) \quad \text{for} \quad -p \le j \le p-1 \\ z_p(\tau_p) &= z_{-p}(-\tau_{-p}), \end{aligned} \tag{2.4}$$

for $\tau_j = (t_{j+1} - t_j)/2$, $j \in I_p$. We put

$$\mathcal{Y}_p = \times_{j\in I_p} Y_{\tau_j}$$

with the norm $|h|_p = \max_{j\in I_p} |h_j|_{\tau_j}$, $h = (h_{-p}, \cdots, h_p)$, $h_j \in Y_{\tau_j}$, $j \in I_p$. We prove a Fredholm–like alternative result for (2.4).

Theorem 2.4. *For any $K > 0$, there exist $m_0 > 0$, $A > 0$, $B > 0$ such that if $\tau_j > m_0$ for every $j \in I_p$ and $\alpha \in \mathcal{A}_E$ such that $|\alpha| \le K$, then there exist linear functions $\mathcal{L}_{\alpha,j} : \mathcal{Y}_p \to \mathbb{R}^n$, $j \in I_p$ with $\|P^j_{uu}\mathcal{L}_{\alpha,j}\| \le A\mathrm{e}^{-2M\tau_j}$ and with the property that if $h \in \mathcal{Y}_p$ satisfies*

$$\int_{-\tau_j}^{\tau_j} P^j_{uu} U_j(t-\alpha_j)^{-1} h_j(t)\, dt + P^j_{uu}\mathcal{L}_{\alpha,j} h = 0$$

for every $j \in I_p$, then (2.4) has solutions in $z_j \in Z_{\tau_j}$ satisfying

$$P^j_{ss} U_j(-\alpha_j)^{-1} z_j(0) = 0 \quad \text{and} \quad \max_{j\in I_p} \|z_j\|_{\tau_j} \le B \max_{j\in I_p} |h_j|_{\tau_j}.$$

Moreover, these solutions z_j depend linearly on h and continuously on α as well.

Proof. We follow [6] by putting $\tilde U_j(t) = U_j(t-\alpha_j)$, $j \in I_p$. Given $h_j \in Y_{\tau_j}$, $j \in I_p$ we use variation of constants to construct the following two solutions to (2.4):

$$\begin{aligned} z_{1,j}(t) &= \tilde U_j(t) P^j_{su}\xi_{1,j} + \tilde U_j(t)(P^j_{us} + P^j_{uu})\tilde U_j(-\tau_j)^{-1}\varphi_{1,j} \\ &\quad + \tilde U_j(t)\int_0^t (P^j_{ss} + P^j_{su})\tilde U_j(s)^{-1} h_j(s)\, ds \\ &\quad + \tilde U_j(t)\int_{-\tau_j}^t (P^j_{us} + P^j_{uu})\tilde U_j(s)^{-1} h_j(s)\, ds, \\ z_{2,j}(t) &= \tilde U_j(t) P^j_{us}\xi_{2,j} + \tilde U_j(t)(P^j_{su} + P^j_{uu})\tilde U_j(\tau_j)^{-1}\varphi_{2,j} \\ &\quad + \tilde U_j(t)\int_0^t (P^j_{ss} + P^j_{us})\tilde U_j(s)^{-1} h_j(s)\, ds \\ &\quad - \tilde U_j(t)\int_t^{\tau_j} (P^j_{su} + P^j_{uu})\tilde U_j(s)^{-1} h_j(s)\, ds. \end{aligned}$$

Here $\xi_{1,j}$, $\xi_{2,j}$, $\varphi_{1,j}$, $\varphi_{2,j}$ are arbitrary. Each of these represents the general solution to (2.4) satisfying $P^j_{ss}\tilde U_j(0)^{-1}z_j(0) = 0$. We consider $z_{1,j}(t)$ for $t \in [-\tau_j, 0]$ and $z_{2,j}(t)$ for $t \in [0, \tau_j]$.

The conditions $z_{1,j}(0) - z_{2,j}(0) = 0$ decompose into the following three families of equations:

$$P^j_{su}\xi_{1,j} - P^j_{su}\tilde U_j(\tau_j)^{-1}\varphi_{2,j} + \int_0^{\tau_j} P^j_{su}\tilde U_j(s)^{-1}h_j(s)\,ds = 0\,, \tag{2.5a}$$

$$P^j_{us}\tilde U_j(-\tau_j)^{-1}\varphi_{1,j} + \int_{-\tau_j}^0 P^j_{us}\tilde U_j(s)^{-1}h_j(s)\,ds - P^j_{us}\xi_{2,j} = 0\,, \tag{2.5b}$$

$$P^j_{uu}\tilde U_j(-\tau_j)^{-1}\varphi_{1,j} - P^j_{uu}\tilde U_j(\tau_j)^{-1}\varphi_{2,j} + \int_{-\tau_j}^{\tau_j} P^j_{uu}\tilde U_j(s)^{-1}h_j(s)\,ds = 0\,. \tag{2.5c}$$

We can solve (2.5a), (2.5b) for $\xi_{1,j}$, $\xi_{2,j}$ respectively. If we write out the equations $z_{1,j+1}(-\tau_{j+1}) - z_{2,j}(\tau_j) = 0$, substitute the formulas for $\xi_{1,j+1}$, $\xi_{2,j}$ and rearrange terms we get the equations

$$\begin{aligned}&\left[\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix} + R_{1,j}(\alpha)\right] C^{-1}\varphi_{1,j+1} + \left[-\begin{pmatrix} 0 & 0\\ 0 & I_u\end{pmatrix} + R_{2,j}(\alpha)\right] C^{-1}\varphi_{2,j}\\ &+ R_{3,j}(\alpha)C^{-1}\varphi_{2,j+1} + R_{4,j}(\alpha)C^{-1}\varphi_{1,j} = \Psi_j(\alpha,h)\,,\end{aligned} \tag{2.6}$$

where the matrix C is taken from Theorem 2.1 and

$$\begin{aligned}
R_{1,j}(\alpha) &= C^{-1}\left[\tilde U_{j+1}(-\tau_{j+1})(P^{j+1}_{us} + P^{j+1}_{uu})\tilde U_{j+1}(-\tau_{j+1})^{-1} - C\begin{pmatrix} I_s & 0\\ 0 & 0\end{pmatrix}C^{-1}\right]C\\
R_{2,j}(\alpha) &= C^{-1}\left[-\tilde U_j(\tau_j)(P^j_{su} + P^j_{uu})\tilde U_j(\tau_j)^{-1} + C\begin{pmatrix} 0 & 0\\ 0 & I_u\end{pmatrix}C^{-1}\right]C\\
R_{3,j}(\alpha) &= C^{-1}\tilde U_{j+1}(-\tau_{j+1})P^{j+1}_{su}\tilde U_{j+1}(\tau_{j+1})^{-1}C\\
R_{4,j}(\alpha) &= -C^{-1}\tilde U_j(\tau_j)P^j_{us}\tilde U_j(-\tau_j)^{-1}C\\
C\Psi_j(\alpha,h) &= \tilde U_{j+1}(-\tau_{j+1})\int_{-\tau_{j+1}}^0 (P^{j+1}_{ss} + P^{j+1}_{su})\tilde U_{j+1}(s)^{-1}h_{j+1}(s)\,ds\\
&+ \tilde U_j(\tau_j)\int_0^{\tau_j}(P^j_{ss} + P^j_{us})\tilde U_j(s)^{-1}h_j(s)\,ds + \tilde U_j(\tau_j)\int_{-\tau_j}^0 P^j_{us}\tilde U_j(s)^{-1}h_j(s)\,ds\\
&+ \tilde U_{j+1}(-\tau_{j+1})\int_0^{\tau_{j+1}} P^{j+1}_{su}\tilde U_{j+1}(s)^{-1}h_{j+1}(s)\,ds\,.
\end{aligned}$$

Using Theorems 2.1–3 we see that each $|R_{i,j}(\alpha)| \to 0$ as $m_0 \to \infty$ and $\tau_j > m_0$ for any $j \in I_p$. Moreover, we get $|\Psi_j(\alpha,h)| = O(|h|_p)$ as well. If we write

$$C^{-1}\varphi_{1,j} = \begin{bmatrix} u_{1,j-1}\\ 0\end{bmatrix}, \qquad C^{-1}\varphi_{2,j} = -\begin{bmatrix} 0\\ u_{2,j}\end{bmatrix}, \qquad \tilde u_j = \begin{bmatrix} u_{1,j}\\ u_{2,j}\end{bmatrix},$$

where $u_{1,j}$, $u_{2,j}$ are of order s, u respectively, (2.6) become

$$\begin{aligned}&\left[I + R_{1,j}(\alpha)\begin{pmatrix} I_s & 0 \\ 0 & 0\end{pmatrix} - R_{2,j}(\alpha)\begin{pmatrix} 0 & 0 \\ 0 & I_u\end{pmatrix}\right]\tilde{u}_j \\ &- R_{3,j}(\alpha)\begin{bmatrix} 0 \\ u_{2,j+1}\end{bmatrix} + R_{4,j}(\alpha)\begin{bmatrix} u_{1,j-1} \\ 0\end{bmatrix} = \Psi_j(\alpha, h).\end{aligned}$$

There exists $m_0 > 0$ so that the coefficient matrices of $\tilde{u}_j$ in the preceding equations are invertible whenever $\tau_j > m_0$ for any $j \in I_p$. We note that m_0 is independent of p. In this case the equations can be solved for $\tilde{u}_j$ which leads to functions $\varphi_{i,j}(\alpha, h)$ such that $\max\limits_{j\in I_p} |\varphi_{i,j}| = O\left(|h|_p\right)$.

By (2.5c) we obtain

$$\int_{-\tau_j}^{\tau_j} P^j_{uu}\tilde{U}_j(s)^{-1}h_j(s)\,ds + P^j_{uu}\mathcal{L}_{\alpha,j}h = 0$$

for any $j \in I_p$, where

$$\mathcal{L}_{\alpha,j}h = \tilde{U}_j(-\tau_j)^{-1}\varphi_{1,j}(\alpha, h) - \tilde{U}_j(\tau_j)^{-1}\varphi_{2,j}(\alpha, h).$$

It follows from the properties of the $\varphi_{i,j}$ and Theorems 2.1–3 that

$$\|P^j_{uu}\mathcal{L}_{\alpha,j}h\| = |h|_p O\left(\mathrm{e}^{-2M\tau_j}\right).$$

□

Using the preceding result we define closed linear subspaces $Y_{\alpha,j,E} \subset \mathcal{Y}_p$ by

$$Y_{\alpha,j,E} = \left\{z \in \mathcal{Y}_p : \int_{-\tau_j}^{\tau_j} P^j_{uu}U_j(t-\alpha_j)^{-1}z_j(t)\,dt + P^j_{uu}\mathcal{L}_{\alpha,j}z = 0\right\}.$$

We can define a variation of constants map

$$K_{\alpha,E}\colon Y_{\alpha,E} = \bigcap_{j\in I_p} Y_{\alpha,j,E} \to \times_{j\in I_p} Z_{\tau_j} = \mathcal{Z}_p$$

by taking $K_{\alpha,E}(h)$, $h = (h_1, \cdots, h_p)$ to be the solution in $\mathcal{Z}_p$ to (2.4) from Theorem 2.4 satisfying $P^j_{ss}U_j(-\alpha_j)^{-1}K_{\alpha,E}(h)_j(0) = 0$ for any $j \in I_p$. Furthermore, the norm $\|K_{\alpha,E}\|$ is uniformly bounded with respect to E and $\alpha \in \mathcal{A}_E$ bounded. Moreover, $K_{\alpha,E}$ is a compact linear operator. We note that the norm on $\mathcal{Z}_p$ is defined as $\max_{j\in I_p} ||z_j||_{\tau_j}$.

Remark 2.5. We note that if $e_j = 0$ then $Y_{\alpha,j,E} = \mathcal{Y}_p$ and $P^j_{uu} = 0$.

3. Chaotic Solutions

We still follow [6]. We find chaotic solutions of (1.1) in this section. To this end, we first find solutions of (1.1) associated to any $E = \{e_j\}_{j\in I_p} \in \{0,1\}^{2p+1}$, $p \in \mathbb{N}$ and then by passing to the limit $p \to \infty$, we show the desired solutions.

We consider $\alpha \in \mathcal{A}_E$ and put:
if $e_j = 1$ then

$$\gamma_j(t) = \gamma(t-\alpha_j), \quad \beta_j = (\beta_{1,j}, \cdots, \beta_{d-1,j}) \in \mathbb{R}^{d-1}$$
$$u_{i+d,j}(t) = u_{i+d}(t-\alpha_j),\ i \in \{1,\cdots,d-1\};$$

if $e_j = 0$ then

$$\gamma_j(t) = 0, \quad \beta_j = (0,\cdots,0) \in \mathbb{R}^{d-1}, \quad u_{i+d,j} = 0,\ i \in \{1,\cdots,d-1\}.$$

We define the functions b_j, $j \in I_p$ by

$$\begin{aligned} &b_j(\alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}, r_1, r_2) \\ &= \gamma_{j+1}(-r_2) - \gamma_j(r_1) + \sum_{i=1}^{d-1}\Big(\beta_{i,j+1}u_{i+d,j+1}(-r_2) - \beta_{i,j}u_{i+d,j}(r_1)\Big). \end{aligned}$$

Note that

$$\begin{aligned} |b_j(\alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}, r_1, r_2)| = O(\mathrm{e}^{-2Mr}) \quad &\text{as} \quad r = \min\{r_1, r_2\} \to \infty \\ \text{uniformly with respect to bounded} \quad &\alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}. \end{aligned}$$

Now we take $s > 0$ and a sequence $\{t_i\}_{i\in\mathbb{Z}}$ such that $\tau_i = (t_{i+1} - t_i)/2 \geq 1/s$ for any $i \in \mathbb{Z}$. Then we consider (1.1) with $\epsilon = s^2$ on the interval $[t_{-p}, t_{p+1}]$ and we make on each interval $[t_j, t_{j+1}]$, $j \in I_p$ the changes of variables

$$\begin{aligned} x(t+\bar{t}_j) &= \gamma_j(t) + s^2 z_j(t) + \sum_{i=1}^{d-1} s\beta_{i,j}u_{i+d,j}(t) \\ &+ \frac{1}{2\tau_j} b_j(\alpha_j, \alpha_{j+1}, s\beta_j, s\beta_{j+1}, \tau_j, \tau_{j+1})(t+\tau_j), \quad z_j \in Z_{\tau_j}, \end{aligned} \tag{3.1}$$

where $\bar{t}_j = (t_{j+1} + t_j)/2$, $j \in I_p$ and we set $\tau_{p+1} = \tau_{-p}$. The functions b_j are constructed so that if $z_j(\tau_j) = z_{j+1}(-\tau_{j+1})$ for $-p \leq j \leq p-1$ and $z_p(\tau_p) = z_{-p}(-\tau_{-p})$, and z_j are continuous then x is continuously extended on $[t_{-p}, t_{p+1}]$.

The differential inclusions for z_j, $j \in I_p$ are

$$\begin{aligned} &\dot{z}_j(t) - Df(\gamma_j(t))z_j(t) \in g_{s,j,E}(z_j(t), \alpha_j, \alpha_{j+1}, \beta_j, \beta_{j+1}, t) \\ &\qquad\qquad \text{a.e. on} \quad [-\tau_j, \tau_j], \quad j \in I_p, \end{aligned} \tag{3.2}$$

where

$$
\begin{aligned}
&g_{s,j,E}(x,\alpha_j,\alpha_{j+1},\beta_j,\beta_{j+1},t) = \\
&\Big\{v \in \mathbb{R}^n :\, v \in \frac{1}{s^2}\Big\{f\Big(s^2x+\gamma_j(t)+s\sum_{i=1}^{d-1}\beta_{i,j}u_{i+d,j}(t) \\
&+\frac{1}{2\tau_j}b_j(\alpha_j,\alpha_{j+1},s\beta_j,s\beta_{j+1},\tau_j,\tau_{j+1})(t+\tau_j)\Big)-f(\gamma_j(t)) \\
&-s\sum_{i=1}^{d-1}\beta_{i,j}\dot u_{i+d,j}(t)-\frac{1}{2\tau_j}b_j(\alpha_j,\alpha_{j+1},s\beta_j,s\beta_{j+1},\tau_j,\tau_{j+1})-Df(\gamma_j(t))s^2x\Big\} \\
&+g\Big(s^2x+\gamma_j(t)+s\sum_{i=1}^{d-1}\beta_{i,j}u_{i+d,j}(t) \\
&+\frac{1}{2\tau_j}b_j(\alpha_j,\alpha_{j+1},s\beta_j,s\beta_{j+1},\tau_j,\tau_{j+1})(t+\tau_j),s^2,t+\bar t_j\Big)\Big\}.
\end{aligned}
$$

We note that $Df(\gamma_j(t)) = A_j(t-\alpha_j)$ in the notations of Section 2. Now we can repeat with the help of Theorem 2.4 the arguments of Section 3 in [7] to solve (3.2) in $\mathcal{Z}_p$. We omit details, since we can directly modify the proofs without any changes. For $j \in I_p$ such that $e_j = 1$, according to Section 3 of [7] (see (3.6) of [7]), (3.2) is homotopically associated to a mapping $\tilde M_j : \mathbb{R}^d \to 2^{\mathbb{R}^d} \setminus \emptyset$ given by

$$
\begin{aligned}
&\tilde M_j(\alpha_j,\beta_j) = M(\bar t_j+\alpha_j,\beta_j), \quad M=(M_1,\cdots,M_d)\,, \\
(3.3)\qquad &M_l(\alpha,\beta) = \Big\{\int_{-\infty}^{\infty}\langle h(s),u_l^{\perp}(s)\rangle\,ds \,:\, h\in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n) \text{ satisfying a.e. on } \mathbb{R} \\
&h(t)\in\Big(\frac{1}{2}\sum_{i,r=1}^{d-1}\beta_i\beta_r D^2f(\gamma(t))(u_{d+i}(t),u_{d+r}(t))+g(\gamma(t),0,t+\alpha)\Big)\Big\}.
\end{aligned}
$$

While for $j \in I_p$ such that $e_j = 0$, according to Section 3 of [7] and Remark 2.5 (see again (3.6) of [7]), (3.2) is homotopically solvable. We note that the mappings $\tilde M_j$ are upper–semicontinuous with compact convex values and they map bounded sets into bounded ones [7].

Consequently, the solvability of (3.2) is reduced to the solvability of the system of p_E inclusions

$$
\underbrace{0\in\tilde M_{j_1},\quad\cdots,\quad 0\in\tilde M_{j_{p_E}}}_{p_E \text{ times}}\,,
$$

where p_E is the number of 1 in E and each above $\tilde M_{j_k}$, $k=1,2,\cdots,p_E$ corresponds to such j_k that $e_{j_k}=1$.

Now let us take any infinite sequence $E=\{e_j\}_{j=-\infty}^{\infty}\in\{0,1\}^{\mathbb{Z}}$ and apply the above procedure to $E_p=\{e_j\}_{j=-p}^{p}$, $p\in\mathbb{N}$ for obtaining the solution x_{E_p} on $[t_{-p},t_{p+1}]$. By passing to the limit $p\to\infty$, then the proof of Theorem 7.1 of [7] immediately implies the following main results of this paper.

Theorem 3.1. *Let $(i)-(iii)$ hold and $d>1$. If there are non–empty open bounded sets $\mathcal{B}_j \subset \mathbb{R}^{d-1}$, $j \in \mathbb{Z}$ along with a sequence of intervals $\{(a_j, b_j)\}_{j\in\mathbb{Z}}$ such that*

(i) $a_j \to +\infty$ *as* $j \to +\infty$, *and* $b_j \to -\infty$ *as* $j \to -\infty$,

(ii) $\bigcup_{j\in\mathbb{Z}} \mathcal{B}_j$ *is bounded and* $\sup_{j\in\mathbb{Z}}(b_j - a_j) < \infty$,

(iii) $\inf\limits_{j\in\mathbb{Z}} \operatorname{dist}\big(0, M\big(\partial((a_j,b_j)\times\mathcal{B}_j)\big)\big) > 0\,,$

(iv) $\deg\left(M, (a_j,b_j)\times\mathcal{B}_j, 0\right) \neq 0\,,$

where deg *is the topological degree in the sense of* [4, pp. 154-155] *and* [21, p. 20].

Then there is a constant $K>0$ and for any sufficiently small $s>0$ there are increasing sequences $\{t_j\}_{j\in\mathbb{Z}}$, $t_{j+1}-t_j \geq 2/s$, $\forall j\in\mathbb{Z}$, and $\{k(j)\}_{j\in\mathbb{Z}}$, $k(j)\in\mathbb{Z}$, such that for $\epsilon = s^2$ and any infinite sequence $E=\{e_j\}_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, the differential inclusion (1.1) *possesses a solution $x_{E,s}$ satisfying, according to* (3.1),

$$\begin{aligned}
&\sup_{t_j\leq t\leq t_{j+1}} \left|x_{E,s}(t) - \gamma(t-\bar{\alpha}_{j,E,s})\right| \leq Ks \quad \textit{when} \quad e_j = 1\,,\\
&\sup_{t_j\leq t\leq t_{j+1}} \left|x_{E,s}(t)\right| \leq Ks^2 \quad \textit{when} \quad e_j = 0\,,
\end{aligned} \tag{3.4}$$

where $\bar{\alpha}_{j,E,s} \in (a_{k(j)}, b_{k(j)}) \subset (t_j, t_{j+1})$ for any $j\in\mathbb{Z}$ with $e_j = 1$. The mapping $E \to z_{E,s}$ is injective.

Proof. It remains to construct the sequence $\{t_j\}_{j\in\mathbb{Z}}$ in the above considerations. The idea is to find such $\{t_j\}_{j\in\mathbb{Z}}$ that

$$\bar{t}_j = \frac{t_{j+1}+t_j}{2} = \frac{a_{k(j)}+b_{k(j)}}{2}\,.$$

So we consider the sequence $S = \left\{\frac{a_k+b_k}{2}\right\}_{k\in\mathbb{Z}}$. From $\bar{t}_j = \frac{t_{j+1}+t_j}{2}$, we get

$$\begin{aligned}
&t_j = 2\bar{t}_{j-1} - 2\bar{t}_{j-2} + \cdots + (-1)^{j+1}2\bar{t}_0, \quad j\geq 1\\
&t_0 = 0\\
&t_j = 2\bar{t}_j - 2\bar{t}_{j+1} + \cdots + (-1)^{j+1}2\bar{t}_{-1}, \quad j\leq -1\,.
\end{aligned} \tag{3.5}$$

Now we take a small s with $\frac{2}{\sup_{j\in\mathbb{Z}}(b_j-a_j)} > s > 0$, and we find $\bar{t}_j \in S$ such that $t_{j+1}-t_j \geq 2/s$, i.e. by (3.5) we must have

$$\begin{aligned}
&\bar{t}_j \geq \frac{1}{s} + 2\bar{t}_{j-1} - 2\bar{t}_{j-2} + \cdots + (-1)^{j+1}2\bar{t}_0, \quad j\geq 0\\
&\bar{t}_j \leq -\frac{1}{s} + 2\bar{t}_{j+1} - 2\bar{t}_{j+2} + \cdots + (-1)^{j}2\bar{t}_{-1}, \quad j\leq -1\,.
\end{aligned} \tag{3.6}$$

So by using (3.6) and assumptions (i), (ii), we choose step by step an increasing sequence $\{k(j)\}_{j\in\mathbb{Z}}$, $k(j)\in\mathbb{Z}$ such that $\bar{t}_j = \frac{a_{k(j)}+b_{k(j)}}{2}$, $j\in\mathbb{Z}$ satisfying (3.6). Then we get by

(3.5) an increasing sequence $\{t_j\}_{j\in\mathbb{Z}}$ such that $t_{j+1}-t_j\geq 2/s$ and $\bar t_j=\frac{t_{j+1}+t_j}{2}$. Then we consider in (3.1) for any j with $e_j=1$ that $\beta_j\in\mathcal{B}_{k(j)}$ and $\alpha_j\in\Big(-\frac{b_{k(j)}-a_{k(j)}}{2},\frac{b_{k(j)}-a_{k(j)}}{2}\Big)$. Assumption (ii) implies that β_j,α_j are uniformly bounded with respect to $j\in\mathbb{Z}$. Consequently, we can perform the above considerations to finish the proof. □

Similarly we get the next result.

Theorem 3.2. *Let (i)–(iii) hold and $d=1$. If there is a sequence of intervals $\{(a_j,b_j)\}_{j\in\mathbb{Z}}$ along with a constant $\delta>0$ such that*

(i) *$a_j\to+\infty$ as $j\to+\infty$, $b_j\to-\infty$ as $j\to-\infty$, and $\sup_{j\in\mathbb{Z}}(b_j-a_j)<\infty$,*

(ii) *Either $M(a_j)\subset[\delta,\infty)$ and $M(b_j)\subset(-\infty,-\delta]$, or $M(a_j)\subset(-\infty,-\delta]$ and $M(b_j)\subset[\delta,\infty)$, for any $j\in\mathbb{Z}$,*

where $M\colon\mathbb{R}\to 2^{\mathbb{R}}\setminus\emptyset$ is defined by

$$M(\alpha)=\bigg\{\int_{-\infty}^{\infty}\langle h(s),u_1^{\perp}(s)\rangle\,ds\,:\,h\in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)\,,\\ h(t)\in g(\gamma(t),0,t+\alpha)\quad\text{a.e. on}\quad\mathbb{R}\bigg\}.$$

Then there is a constant $K>0$ and for any sufficiently small $s>0$, there are increasing sequences $\{t_j\}_{j\in\mathbb{Z}}$, $t_{j+1}-t_j\geq 2/s$, and $\{k(j)\}_{j\in\mathbb{Z}}$, $k(j)\in\mathbb{Z}$, such that for $\epsilon=\pm s^2$ and any infinite sequence $E=\{e_j\}_{j\in\mathbb{Z}}\in\{0,1\}^{\mathbb{Z}}$, the differential inclusion (1.1) possesses a solution $x_{E,s}$ satisfying, according to (3.1),

$$\sup_{t_j\leq t\leq t_{j+1}}\big|x_{E,s}(t)-\gamma(t-\bar\alpha_{j,E,s})\big|\leq Ks^2\quad\text{when}\quad e_j=1\,,\\ \sup_{t_j\leq t\leq t_{j+1}}\big|x_{E,s}(t)\big|\leq Ks^2\quad\text{when}\quad e_j=0\,, \tag{3.7}$$

where $\bar\alpha_{j,E,s}\in(a_{k(j)},b_{k(j)})\subset(t_j,t_{j+1})$ for any $j\in\mathbb{Z}$ with $e_j=1$. The mapping $E\to z_{E,s}$ is injective.

Hence for any $\epsilon>0$ sufficiently small and $E\in\{0,1\}^{\mathbb{Z}}$, (1.1) possesses a solution $z_{E,s}$ satisfying either (3.4) or (3.7). These estimates (3.4) and (3.7) give the chaotic behaviour of solutions $z_{E,s}$ for any $E\in\mathcal{E}=\big\{E\in\{0,1\}^{\mathbb{Z}}\big\}$.

Now we suppose condition (iv). Let $U(\theta,t)$ denote a fundamental solution of $\dot u(t)=Df(\gamma(\theta,t))u(t)$. Then Theorem 2.1 is valid for each parameter θ. Let $u_j(\theta,t)$ be the jth column of $U(\theta,t)$ and define $u_i^{\perp}(\theta,t)$ by $\langle u_i^{\perp}(\theta,t),u_j(\theta,t)\rangle=\delta_{ij}$. Now we can repeat the above procedure. Instead of (3.1), we make on each interval $[t_j,t_{j+1}]$, $j\in I_p$ the changes of variables

$$x\,(t+\bar t_j)=\gamma_j(\theta_j,t-\alpha_j)+\epsilon z_j(t)\\ +\frac{1}{2\tau_j}\Big(\gamma_{j+1}(\theta_{j+1},-\tau_{j+1}-\alpha_{j+1})-\gamma_j(\theta_j,\tau_j-\alpha_j)\Big)(t+\tau_j),\quad z_j\in Z_{\tau_j}\,, \tag{3.8}$$

where $\gamma_j(\theta,t)=\gamma(\theta,t)$ for $e_j=1$ and $\gamma_j(\theta,t)=0$ for $e_j=0$. In this way for $e_j=1$, we arrive at a mapping $\tilde M_j\colon \mathbb{R}^d\to 2^{\mathbb{R}^d}\setminus\emptyset$ given by

$$
\begin{aligned}
&\tilde M_j(\alpha_j,\theta_j)=M(\bar t_j+\alpha_j,\theta_j),\quad M=(M_1,\cdots,M_d)\,,\\
&M_l(\alpha,\theta)=\left\{\int_{-\infty}^{\infty}\langle h(s),u_l^{\perp}(\theta,s)\rangle\,ds\,:\ h\in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)\,,\right.\\
&\qquad\left. h(t)\in g(\gamma(\theta,t),0,t+\alpha)\quad\text{a.e. on}\quad\mathbb{R}\right\}.
\end{aligned}
\tag{3.9}
$$

Then like for Theorem 3.1, we get the next result.

Theorem 3.3. *Let $(i),(ii),(iv)$ hold. If there is a non–empty open bounded set $\mathcal{B}\subset\mathbb{R}^{d-1}$ along with a sequence of intervals $\{(a_j,b_j)\}_{j\in\mathbb{Z}}$ such that*

(i) $a_j\to+\infty$ *as* $j\to+\infty$, *and* $b_j\to-\infty$ *as* $j\to-\infty$,

(ii) $\sup_{j\in\mathbb{Z}}(b_j-a_j)<\infty$,

(iii) $\inf\limits_{j\in\mathbb{Z}}\operatorname{dist}\big(0,M\big(\partial((a_j,b_j)\times\mathcal{B})\big)\big)>0\,,$

(iv) $\deg\left(M,(a_j,b_j)\times\mathcal{B},0\right)\neq 0\,,$

where M is given by (3.9).

Then there is a constant $K>0$ and for any sufficiently small $\epsilon\neq 0$ there are increasing sequences $\{t_j\}_{j\in\mathbb{Z}}$, $t_{j+1}-t_j\geq 2/|\epsilon|$, $\forall j\in\mathbb{Z}$, and $\{k(j)\}_{j\in\mathbb{Z}}$, $k(j)\in\mathbb{Z}$, such that for any infinite sequence $E=\{e_j\}_{j\in\mathbb{Z}}\in\{0,1\}^{\mathbb{Z}}$, the differential inclusion (1.1) possesses a solution $x_{E,\epsilon}$ satisfying, according to (3.8),

$$
\begin{aligned}
&\sup_{t_j\leq t\leq t_{j+1}}\left|x_{E,\epsilon}(t)-\gamma(\theta_j,t-\bar\alpha_{j,E,\epsilon})\right|\leq K|\epsilon|\quad\textit{when}\quad e_j=1\,,\\
&\sup_{t_j\leq t\leq t_{j+1}}\left|x_{E,\epsilon}(t)\right|\leq K|\epsilon|\quad\textit{when}\quad e_j=0\,,
\end{aligned}
$$

where $\bar\alpha_{j,E,\epsilon}\in(a_{k(j)},b_{k(j)})\subset(t_j,t_{j+1})$ and $\theta_j\in\mathcal{B}$ for any $j\in\mathbb{Z}$ with $e_j=1$. The mapping $E\to z_{E,\epsilon}$ is injective.

Finally we note that like above, we can extend the results of Section 4 in [6] about singular periodic differential inclusions to singular nonautonomous differential inclusions.

4. Examples

We study the case when

$$
g(x,\epsilon,t)=\epsilon\mu_1F(x)+\epsilon\mu_2h(t)
\tag{4.1}
$$

with $\mu_{1,2} \in \mathbb{R}$, $F\colon \mathbb{R}^n \to 2^{\mathbb{R}^n} \setminus \emptyset$ is upper-semicontinuous with compact and convex values, and $h \in C(\mathbb{R}, \mathbb{R}^n)$ is almost periodic [16]. Then (3.3) has the form

$$M_l(\alpha,\beta) = \mu_1 \left\{ \int_{-\infty}^{\infty} \langle h(s), u_l^{\perp}(s)\rangle \, ds \, : \, h \in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)\,, \right. \\ \left. h(t) \in F(\gamma(t)) \quad \text{a.e. on} \quad \mathbb{R} \right\} + \frac{1}{2}\sum_{i,r=1}^{d-1} \beta_i \beta_r b_{irl} + \mu_2 a_l(\alpha)\,, \tag{4.2}$$

where

$$b_{irl} = \int_{-\infty}^{\infty} \langle D^2 f(\gamma(t))(u_{d+i}(t), u_{d+r}(t)), u_l^{\perp}(t)\rangle \, dt$$

$$a_l(\alpha) = \int_{-\infty}^{\infty} \langle h(t+\alpha), u_l^{\perp}(t)\rangle \, dt\,,$$

while (3.9) has the form

$$M_l(\alpha,\theta) = \mu_1 \left\{ \int_{-\infty}^{\infty} \langle h(s), u_l^{\perp}(\theta,s)\rangle \, ds \, : \, h \in L_{2,\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)\,, \right. \\ \left. h(t) \in F(\gamma(\theta,t)) \quad \text{a.e. on} \quad \mathbb{R} \right\} + \mu_2 a_l(\alpha,\theta)\,, \tag{4.3}$$

where

$$a_l(\alpha,\theta) = \int_{-\infty}^{\infty} \langle h(t+\alpha), u_l^{\perp}(\theta,t)\rangle \, dt\,.$$

The functions $a(\alpha) = (a_1(\alpha), \cdots, a_d(\alpha))$ and $a(\alpha,\theta) = (a_1(\alpha,\theta), \cdots, a_d(\alpha,\theta))$ are also almost periodic in α, while $a(\alpha,\theta)$ uniformly on bounded sets of θ.

Theorem 4.1. *Let $d > 1$. If there is a non–empty open bounded set $\mathcal{B} \subset \mathbb{R}^{d-1}$ along with an interval $(a,b) \subset \mathbb{R}$ such that*

(i) $0 \notin M\big(\partial((a,b)\times\mathcal{B})\big)\,,$

(ii) $\deg\,(M, (a,b)\times\mathcal{B}, 0) \neq 0\,,$

where M is given by either (4.2) *or* (4.3)*. Then the statements of Theorems* 3.1 *and* 3.3*, respectively, hold for* (1.1) *with perturbation* (4.1)*.*

Proof. We consider the case (4.2), the case (4.3) is similar. We have to show the validity of the assumptions of Theorem 3.1. Since $a(\alpha)$ is almost periodic, for any $\delta > 0$ there is a sequence $\{c_k\}_{k\in\mathbb{Z}}$, $c_k \to \pm\infty$ as $k \to \pm\infty$ such that

$$|a(\alpha + c_k) - a(\alpha)| \leq \delta, \quad \forall\, \alpha \in \mathbb{R}\,.$$

For any $j \in \mathbb{Z}$, we put

$$\mathcal{B}_j = \mathcal{B}, \quad a_j = c_j + a, \quad b_j = c_j + b\,.$$

If δ is sufficiently small, then assumptions (i)-(iv) of Theorem 3.1 are satisfied. This gives the statement of Theorem. □

Similarly we have the next result.

Theorem 4.2. *Let $d = 1$. If there is an interval $(a, b) \subset \mathbb{R}$ such that*

$M(a)$ contains only positive (respectively negative) numbers and $M(b)$ contains only negative (respectively positive) ones,

where M is given by

$$M(\alpha) = \mu_1 \Big\{ \int_{-\infty}^{\infty} \langle h(s), u_1^\perp(s)\rangle\, ds \,:\, h \in L_{2,\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)\,,$$

$$h(t) \in F(\gamma(t)) \quad \text{a.e. on} \quad \mathbb{R} \Big\} + \mu_2 a_1(\alpha)\,.$$

Then the statement of Theorem 3.2 holds for (1.1) with perturbation (4.1).

Now we want to show that conditions of Theorems 4.1 and 4.2 imply the validity of these conditions also for any h^* in the hull $H(h)$ of h [16], [18], where we say $h^* \in H(h)$ if there is a sequence $\tau_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that

$$h(t + \tau_n) \to h^*(t) \quad \text{as} \quad n \to \infty$$

uniformly on $\mathbb{R}$. We note that h^* is also almost periodic. We again consider the case (4.2). We know that

$$a_l^h(\alpha) = \int_{-\infty}^{\infty} \langle h(t+\alpha), u_l^\perp(t)\rangle\, dt\,,$$

$$a_l^{h^*}(\alpha) = \int_{-\infty}^{\infty} \langle h^*(t+\alpha), u_l^\perp(t)\rangle\, dt$$

for $l = 1, 2, \cdots, d$. Theorem 2.1 implies that

$$|u_l^\perp(t)| \le K_0 e^{-2M|t|} \quad \forall\, t \in \mathbb{R}\,.$$

For any $\delta > 0$, there is an $\tau \in \mathbb{R}$ such that

$$|h(t+\tau) - h^*(t)| \le \delta \quad \forall\, t \in \mathbb{R}\,.$$

Hence

$$\int_{-\infty}^{\infty} \left|\langle h(t+\alpha) - h^*(t+\alpha-\tau), u_l^\perp(t)\rangle\right| dt \le \delta K_0/M\,.$$

This gives

$$|a_l^h(\alpha) - a_l^{h^*}(\alpha-\tau)| \le \delta K_0/M\,.$$

By taking $\delta > 0$ sufficiently small, the assumptions of Theorems 4.1 and 4.2 are satisfied for h^* in (4.1) instead of h with the set $\mathcal{B}$ and with an interval $(a-\tau, b-\tau)$. Consequently, Theorems 4.1 and 4.2 imply chaos of (1.1) with the perturbation (4.1) over any element of the hull $H(h)$. This result is a generalization of similar ones from [16], [18], [22] to multivalued perturbations. We state this result in the next theorem.

Theorem 4.3. *Under conditions of Theorem* 4.1 *or Theorem* 4.2*, differential inclusion* (1.1) *is also chaotic with perturbation* (4.1) *when h is replaced in perturbation* (4.1) *by any $h^* \in H(h)$.*

Clearly the above results hold also for continuous $h(x,t)$ in perturbation (4.1) when $h(x,t)$ is uniformly almost periodic in t on any bounded set of variable x [16].

Finally, we present a concrete example given by

$$\begin{aligned} \ddot{x} &= x + 2x(x^2+y^2) + \epsilon\mu_1 \cos\omega_1 t - \epsilon\delta_1 \operatorname{sgn}\dot{x}\,, \\ \ddot{y} &= y + 2y(x^2+y^2) + \epsilon\mu_2 \cos\omega_2 t - \epsilon\delta_2 \operatorname{sgn}\dot{y}\,, \end{aligned} \tag{4.4}$$

where $\omega_1 > \omega_2$, $\delta_{1,2}$ and $\mu_{1,2}$ are positive constants and $\epsilon > 0$ is a small parameter. By introducing the multivalued mapping

$$\operatorname{Sgn} r = \begin{cases} \operatorname{sgn} r & \text{for } r \neq 0\,, \\ [-1,1] & \text{for } r = 0\,, \end{cases}$$

we consider (4.4) as a differential inclusion given by

$$\begin{aligned} \ddot{x} - x - 2x(x^2+y^2) - \epsilon\mu_1\cos\omega_1 t &\in -\epsilon\delta_1 \operatorname{Sgn}\dot{x}\,, \\ \ddot{y} - y - 2y(x^2+y^2) - \epsilon\mu_2\cos\omega_2 t &\in -\epsilon\delta_2 \operatorname{Sgn}\dot{y}\,. \end{aligned}$$

The unperturbed equation of (4.4) with the form

$$\ddot{x} = x + 2x(x^2+y^2), \quad \ddot{y} = y + 2y(x^2+y^2) \tag{4.5}$$

has a rotational symmetry [11]. So (4.5) has a homoclinic manifold given by $\gamma(\theta,t) = \big(r(t)\cos\theta, \dot{r}(t)\cos\theta, r(t)\sin\theta, \dot{r}(t)\sin\theta\big)$ where $r(t) = \operatorname{sech} t$, and then [11]

$$\begin{aligned} u_1^\perp(\theta,t) &= \big(-\dot{r}(t)\sin\theta, r(t)\sin\theta, \dot{r}(t)\cos\theta, -r(t)\cos\theta\big), \\ u_2^\perp(\theta,t) &= \big(-\ddot{r}(t)\cos\theta, \dot{r}(t)\cos\theta, -\ddot{r}(t)\sin\theta, \dot{r}(t)\sin\theta\big)\,, \end{aligned}$$

where $\theta \in [0, 2\pi)$. Now (4.3) has for $\theta \neq 0, \pi/2, \pi, 3\pi/2$ the form

$$
\begin{aligned}
M_1(\alpha,\theta) &= \pi\mu_1 \operatorname{sech}\frac{\omega_1\pi}{2}\sin\theta\cos\omega_1\alpha - \pi\mu_2\operatorname{sech}\frac{\omega_2\pi}{2}\cos\theta\cos\omega_2\alpha\,,\\
M_2(\alpha,\theta) &= \pi\mu_1\omega_1 \operatorname{sech}\frac{\omega_1\pi}{2}\cos\theta\sin\omega_1\alpha + \pi\mu_2\omega_2\operatorname{sech}\frac{\omega_2\pi}{2}\sin\theta\sin\omega_2\alpha\\
&\quad -2\delta_1|\cos\theta| - 2\delta_2|\sin\theta|\,.
\end{aligned}
\tag{4.6}
$$

To handle (4.6), we consider that $\theta \in (0, \pi/2)$ and $\alpha \in \left(0, \frac{\pi}{2\omega_1}\right) = I_0$. Then equation $M(\alpha, \theta) = 0$ has the form

$$
\begin{aligned}
A_1(\alpha)\sin\theta + A_2(\alpha)\cos\theta &= 0\\
A_3(\alpha)\sin\theta + A_4(\alpha)\cos\theta &= 0\,,
\end{aligned}
\tag{4.7}
$$

where functions $A_i(\alpha)$, $i = 1, 2, 3, 4$ are given by the definition. Note that $A_1(\alpha) \neq 0$ and $A_2(\alpha) \neq 0$ on I_0. It is clear that (4.7) is equivalent to

$$
\begin{aligned}
B(\alpha) = A_1(\alpha)A_4(\alpha) - A_2(\alpha)A_3(\alpha) &= 0\\
A_1(\alpha)\sin\theta + A_2(\alpha)\cos\theta &= 0\,.
\end{aligned}
\tag{4.8}
$$

Since $B(\alpha)$ is nonzero and analytic, it could have only isolated zeroes with finite orders. Consequently, if $B(\alpha)$ changes the sign over I_0, then there are $\alpha_0 \in I_0$, $\eta > 0$ such that $B(\alpha)$ has the only zero $\alpha = \alpha_0$ in the interval $[\alpha_0 - \eta, \alpha_0 + \eta] \subset I_0$ and $B(\alpha_0 - \eta)B(\alpha_0 + \eta) < 0$. We take

$$\theta(\alpha) = -\arctan(A_2(\alpha)/A_1(\alpha))\,,$$

and put

$$\theta_0^+ = \frac{2\omega_1\theta(\alpha_0) + \pi}{4\omega_1}, \quad \theta_0^- = \theta(\alpha_0)/2\,.$$

Then on the set $\mathcal{I} = [\alpha_0 - \eta, \alpha_0 + \eta] \times [\theta_0^-, \theta_0^+] \subset I_0 \times (0, \pi/2)$, mapping $M(\alpha, \theta)$ has the only zero point $(\alpha_0, \theta(\alpha_0))$. To show the non-vanishing of the Brouwer index $I(\alpha_0)$ of this zero point, we consider the equation

$$
\begin{aligned}
A_1(\alpha)\sin\theta + A_2(\alpha)\cos\theta &= 0\\
A_3(\alpha)\sin\theta + A_4(\alpha)\cos\theta &= c_1\,,
\end{aligned}
\tag{4.9}
$$

for $c_1 \neq 0$ small and $(\alpha, \theta) \in \mathcal{I}$. (4.9) gives

$$c_1 = A_3(\alpha)\sin\theta(\alpha) + A_4(\alpha)\cos\theta(\alpha) = B(\alpha)\frac{\cos\theta(\alpha)}{A_1(\alpha)} = C(\alpha)\,. \tag{4.10}$$

A boring computation gives

$$\det L(\alpha) = C'(\alpha)\big(A_2(\alpha)\sin\theta(\alpha) - A_1(\alpha)\cos\theta(\alpha)\big)\,, \tag{4.11}$$

where $L(\alpha)$ is the linearization of the left hand side of (4.9) at point $(\alpha,\theta(\alpha))$. We note that $\big(A_2(\alpha)\sin\theta(\alpha)-A_1(\alpha)\cos\theta(\alpha)\big)\neq 0$ on $[\alpha_0-\eta,\alpha_0+\eta]$. Let $\big\{(\alpha_i,\theta(\alpha_i))\big\}_{i=1}^N$ be the solutions of (4.9) in $\mathcal{I}$. Then for c_1 sufficiently small such that $C'(\alpha_i)\neq 0$, $i=1,2,\cdots,N$, by the definition of the Brouwer degree and (4.11), we have

$$|I(\alpha_0)|=\Big|\sum_{i=1}^N \text{sgn det } L(\alpha_i)\Big|=\Big|\sum_{i=1}^N \text{sgn}\, C'(\alpha_i)\Big|= \\ \big|\deg\big(C(\alpha),(\alpha_0-\eta,\alpha_0+\eta),0\big)\big|=1\,,$$

since by (4.10), function $C(\alpha)$ has the only zero $\alpha=\alpha_0$ in $[\alpha_0-\eta,\alpha_0+\eta]$ and $C(\alpha_0-\eta)C(\alpha_0+\eta)<0$. Hence Theorem 4.1 can be used. It remains to ensure that function $B(\alpha)$ changes the sign over I_0. We have

$$B(\alpha)=\frac{\pi^2}{2}\mu_1^2\omega_1\Big(\text{sech}\,\frac{\omega_1\pi}{2}\Big)^2\sin 2\omega_1\alpha+\frac{\pi^2}{2}\mu_2^2\omega_2\Big(\text{sech}\,\frac{\omega_2\pi}{2}\Big)^2\sin 2\omega_2\alpha \\ -2\delta_1\mu_1\pi\text{sech}\,\frac{\omega_1\pi}{2}\cos\omega_1\alpha-2\delta_2\mu_2\pi\text{sech}\,\frac{\omega_2\pi}{2}\cos\omega_2\alpha\,.$$

Then clearly $B(0)<0$ and

$$B\Big(\frac{\pi}{4\omega_1}\Big)=\frac{\pi^2}{2}\mu_1^2\omega_1\Big(\text{sech}\,\frac{\omega_1\pi}{2}\Big)^2+\frac{\pi^2}{2}\mu_2^2\omega_2\Big(\text{sech}\,\frac{\omega_2\pi}{2}\Big)^2\sin\frac{\pi\omega_2}{2\omega_1} \\ -\sqrt{2}\delta_1\mu_1\pi\text{sech}\,\frac{\omega_1\pi}{2}-2\delta_2\mu_2\pi\text{sech}\,\frac{\omega_2\pi}{2}\cos\frac{\pi\omega_2}{4\omega_1}\,.$$

By assuming $B\Big(\frac{\pi}{4\omega_1}\Big)>0$, we get from Theorem 4.1 the following result.

Theorem 4.4. *If*

$$\frac{\pi^2}{2}\mu_1^2\omega_1\Big(\text{sech}\,\frac{\omega_1\pi}{2}\Big)^2+\frac{\pi^2}{2}\mu_2^2\omega_2\Big(\text{sech}\,\frac{\omega_2\pi}{2}\Big)^2\sin\frac{\pi\omega_2}{2\omega_1}> \\ \sqrt{2}\delta_1\mu_1\pi\text{sech}\,\frac{\omega_1\pi}{2}+2\delta_2\mu_2\pi\text{sech}\,\frac{\omega_2\pi}{2}\cos\frac{\pi\omega_2}{4\omega_1}\,,$$

then (4.4) *is chaotic for any* $\epsilon>0$ *small.*

The condition of Theorem 4.4 reads that the forcing terms must be sufficiently large in (4.4) with respect to damping terms, to get chaos in (4.4), and the rate is given by this condition.

We note that if the ratio ω_2/ω_1 is irrational, then [16] the hull of the function $h(t)=(\mu_1\cos\omega_1 t,\mu_2\cos\omega_2 t)$ is given by

$$H(h)=\Big\{(\mu_1\cos(\omega_1 t+\xi_1),\mu_2\cos(\omega_2 t+\xi_2))\mid \xi_1,\xi_2\in\mathbb{R}\quad\text{are defined mod}\quad 2\pi\Big\}\,.$$

So (4.4) considered over $H(h)$ has the form

$$
\begin{aligned}
\ddot{x} &= x + 2x(x^2+y^2) + \epsilon\mu_1 \cos\Omega_1 - \epsilon\delta_1 \operatorname{sgn}\dot{x}\,,\\
\ddot{y} &= y + 2y(x^2+y^2) + \epsilon\mu_2 \cos\Omega_2 - \epsilon\delta_2 \operatorname{sgn}\dot{y}\,,\\
\dot{\Omega}_1 &= \omega_1\,,\\
\dot{\Omega}_2 &= \omega_2\,.
\end{aligned}
\tag{4.12}
$$

Theorem 4.3 implies chaos of (4.12) under conditions of Theorem 4.4, which is a generalization of similar ones of [16] and [22] to multivalued and quasiperiodic perturbations. We stress that methods of [16] and [22] can not be used in our case.

Of course, other discontinuous perturbations than $\operatorname{sgn} r$ could be considered in (4.4). For instance [7], we can take for any $\eta \ge 0$, $\zeta \ge 1$, $0 < \kappa \le 1$ the multivalued function $\operatorname{Sgn}_{\eta,\zeta,\kappa} r$ defined by

$$
\operatorname{Sgn}_{\eta,\zeta,\kappa} r = \begin{cases} -1 & \text{for } r < -\eta\,,\\ [-\zeta,-\kappa] & \text{for } -\eta \le r < 0\,,\\ [-\zeta,\zeta] & \text{for } r = 0\,,\\ [\kappa,\zeta] & \text{for } 0 < r \le \eta\,,\\ 1 & \text{for } r > \eta\,. \end{cases}
$$

The term $\operatorname{Sgn}_{\eta,\zeta,\kappa}\dot{x}$ can be considered as an extension for modelling dry friction forces. We are motivated for considering $\operatorname{Sgn}_{\eta,\zeta,\kappa} r$ by the fact that the smooth approximation of $\operatorname{sgn} r$ given, for instance, by

$$
\Phi(r) = \frac{1}{\pi}\big(7\arctan 8kr - 5\arctan 4kr\big), \quad k \gg 1
$$

seems to be physically more relevant than the mathematically convenient approximation of the form

$$
r \to \frac{2}{\pi}\arctan kr, \quad k \gg 1\,.
$$

The function Φ has two symmetric spikes at $r = \pm\frac{\sqrt{6}}{8k}$ of the values $\pm 1,2261$. Moreover, $\Phi(r)$ is quickly near to 1, respectively -1, when $r > 0$, respectively $r < 0$, tends off 0. The derivation of the map $M(\alpha,\theta)$ and its analysis is much more difficult for $\operatorname{Sgn}_{\eta,\zeta,\kappa} r$ than for (4.6).

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