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EXISTENCE OF STATES ON PSEUDOEFFECT ALGEBRAS

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Abstract. Pseudoeffect (PE-) algebras have been introduced as a non-commutative generalization of effect algebras. We study in this paper PE-algebras with the special property of having a non-empty state space.

To this end, we consider PE-algebras which are, in a certain sense, non-commutative only in the small. Such an PE-algebra is shown to possess a non-trivial commutative homomorphic image – which is equivalent to saying that there exist states. A typical example is given by an interval of the lexicographical product of two *po*-groups the first of which is abelian.

1. Introduction

MV-algebras, introduced by Chang in the 1950s, provide an algebraic semantics for the Łukasiewicz multi-valued logics [CiOtMu]. Not long ago, Georgulescu and Iorgulescu [GeIo] introduced a non-commutative counterpart of MV-algebras, the pseudo-MV algebras, heading towards the conception of a logic which in general does not allow to interchange the two arguments of a conjunction. MV- as well as pseudo-MV algebras are intervals of ℓ-groups, in the first case of the abelian ones [Mun1, Dvu1].

Effect algebras, introduced in 1993 [FoBe], generalize MV-algebras. They are partial additive algebras modeled upon the Hilbert space quantum effects, which in turn represent the positive outcomes of the yes-no tests performable at some physical system [BuGrLa]. Any interval of an abelian *po*-group, not necessarily a lattice-ordered one, gives rise to an effect algebra.

Now, pseudoeffect algebras – or PE-algebras for short – have been recently introduced in [DvVe1]. They generalize effect algebras in that

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they are no longer necessarily commutative; and the same time, they contain pseudo-MV algebras as a subclass. PE-algebras arise from intervals of *po*-groups of any kind; if this is the case, we talk about interval PE-algebras. In the present article, we are focussed exclusively on this kind of PE-algebra.

The dependences among the mentioned structures is illustrated by the following scheme. Note that there is a corresponding scheme for *po*-groups: containing *po*-groups, abelian *po*-groups, ℓ-groups, and abelian ℓ-groups.

$$\begin{array}{ccc}
\text{interval pseudoeffect} & \Leftarrow & \text{interval effect} \\
\text{(interval PE-)} & & \text{algebras} \\
\text{algebras} & & (E;+,0,1) \\
(E;+,0,1) & & \text{(commutative)} \\
\Uparrow & & \Uparrow \\
\text{pseudo-MV algebras} & \Leftarrow & \text{MV-algebras} \\
(E;\oplus,^-,^\sim,0,1) & & (E;\oplus,^\star,0,1) \\
\text{(lattice ordered,} & & \text{(lattice ordered,} \\
\text{fulfilling (RDP))} & & \text{commutative,} \\
 & & \text{fulfilling (RDP))}
\end{array}$$

The interval PE-algebras are probably not characterizable in a purely algebraic manner. But by [DvVe2], there is certain kind of Riesz decomposition property, called (RDP_1), which implies for a PE-algebra to be an interval PE-algebra.

We are working towards a structure theory for interval PE-algebras. In [DvVe3], congruences have been considered and in particular, states have been examined to a certain extent. This article may be understood as a continuation of this discussion.

Given an interval PE-algebra, the first question to be asked might be, if there is a state at all. This question has already been considered for the subclasses of PE-algebras shown in our scheme. States on MV-algebras were studied in [Cho, Mun2]; in particular, MV-algebras always possess states. The same is true for interval effect algebras [Goo], although it is not true in general for effect algebras [Gre, Rie]. In the non-commutative case, the situation is more difficult. States reflect the commutative part of an algebra only; so it is not amazing, although far from easy to see, that even a pseudo-MV algebra need not have any state [Dvu2]. An algebraic criterion which exactly would tell that at least one state exists, is not known.

This paper is organized as follows. After providing in Section 2 a condensed version of all the background material needed, we discuss in Section 3 in a first step the exact condition for a PE-algebra to possess a state. Here, we in particular generalize the result of Goodearl [Goo] about abelian interpolation groups.

In what follows, we are guided by the following idea. Rather than further considering the general case, we ask which classes of PE-algebras seem interesting and reasonable to be studied at all in the present context. In a first step, in Section 4, we will actually study a class of concrete examples – intervals in the lexicographical product of abelian *po*-groups and possibly non-abelian ones. For them, we give the exact conditions to fulfil the Riesz decomposition property.

These lexicographical products may be considered the typical examples of PE-algebras being non-commutative only in the small. In Section 5, this idea is developed on general algebraic grounds. We will define order-regular and nearly commutative PE-algebras. We will prove that algebras of this kind always possess a non-trivial interval effect algebra as a homomorphic image. As a particular consequence, the state space is then non-empty.

Furthermore, we introduce a slightly strengthened version of order-regularity, and based on it, we give an improved version of the state existence theorem. Namely, by assuming this property we may show that a PE-algebra is represented by a *po*-group whose commutator subgroup is purely infinitesimal.

2. Interval PE-algebras

PE-algebras have been introduced in [DvVe1] as a generalization of effect algebras. The axiom of commutativity, holding for the latter structures, is replaced by what is called here pseudocommutativity. Consequently, also the uniqueness of a complement is no longer assumed and replaced by the uniqueness of a left and of a right complement. We recall the basic definitions.

Definition 2.1. A structure $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudoeffect algebra*, or a *PE-algebra* for short, if, for all $a,b,c \in E$, the following holds.

(PE1) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$ (*strong law of associativity*).

(PE2) If $a+b$ exists, there are elements $d, e \in E$ such that $a+b = d+a = b+e$ (*pseudocommutativity*).

(PE3) There is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$ (*unique left and right complementation*).

(PE4) If $1+a$ or $a+1$ exists, then $a = 0$ (*zero-one law*).

Furthermore, for any $a, b \in E$, we define $a \leq b$ to hold if $a+c = b$ for some $c \in E$.

Finally, a PE-algebra E is called an *effect algebra* if, for all $a, b \in E$, $a+b$ is defined if and only if $b+a$ is defined, in which case $a+b = b+a$.

By $\leq$, a PE-algebra is partially ordered, and this order is, by the pseudocommutativity, two-sided.

In view of (PE3), we may introduce unary operations for the two complements. Furthermore, since a PE-algebra has the cancellation property [DvVe1, Lemma 1.4(v)], we may introduce a left and a right difference.

Definition 2.2. Let $(E; +, 0, 1)$ be a PE-algebra. Let ${}^\sim, {}^-$ be those unary operations on E such that, for any $a \in E$,

$$a + a^\sim = a^- + a = 1.$$

We call a PE-algebra *symmetrically complemented* if ${}^\sim = {}^-$.

Furthermore, let $/$, $\backslash$ be the partial binary operations on E such that, for $a, b \in E$, b / a is defined iff $b \backslash a$ is defined iff $a \leq b$, in which case we have

$$(b / a) + a = a + (b \backslash a) = b.$$

We then call $b \backslash a$ the *left* and b / a the *right difference* of b and a.

Remark 2.3. We note that for elements a, b of a symmetrically complemented PE-algebra, the existence of $a+b$ and of $b+a$ is equivalent.

We next clarify the notions of homomorphisms and congruences of PE-algebras; for further details, see [DvVe3].

Definition 2.4. Let $(E; +, 0, 1)$ and $(F; +, 0, 1)$ be PE-algebras.

(i) A mapping φ: $E \to F$ is called a *homomorphism* if the constants are preserved and, whenever $a+b$ is defined for $a, b \in E$, also $\varphi(a) + \varphi(b)$ is defined and equals $\varphi(a+b)$.

(ii) A homomorphism φ: $E \to F$ is called an *epimorphism* if its image $\varphi(E)$ generates the whole algebra F.

(iii) An equivalence relation $\sim$ on E is called a *congruence* if the following holds: For any $a, a', b, b' \in E$ such that $a \sim a'$, $b \sim b'$ and $a + b$, $a' + b'$ are both defined, we have $a + b \sim a' + b'$.

A congruence $\sim$ on E is called a *PE-algebra congruence* if the quotient algebra $[E]_\sim = \{[a]_\sim\colon\ a \in E\}$ is again a PE-algebra. Here, the addition on $[E]_\sim$ is defined according to rule

$$\begin{array}{ccc} [a]_\sim + [b]_\sim \text{ is defined} & \text{iff} & \text{for some } a' \sim a \text{ and } b' \sim b, \\ \text{and equals } [c]_\sim & & a' + b' \text{ is defined and } a' + b' \sim c \end{array}$$

for some $a, b, c \in E$, and the constants are $[0]_\sim$ and $[1]_\sim$.

A congruence does in general not lead again to a PE-algebra; but we have the following [DvVe3, Proposition 3.3(ii)].

Proposition 2.5. *Let $(E; +, 0, 1)$ be a PE-algebra, and let $\sim$ be a congruence on E such that, for any pair $a, b \in E$ the sum of which is defined, the following condition holds: For any $a' \sim a$, there is a $b' \sim b$, such that $a' + b'$ is defined; and for any $b'' \sim b$ there is an $a'' \sim a$, such that $a'' + b''$ is defined. Then $\sim$ is a PE-algebra congruence.*

Note that a congruence on a PE-algebra may identify any pair of elements; this results in an algebra with only one element. We shall call a PE-algebra $(E; +, 0, 1)$ *non-trivial* if this is not the case, that is, if $0 \neq 1$.

Now, as explained in the introduction, we are interested in those PE-algebras which arise from intervals of *po*-groups.

Definition 2.6. A structure $(G; +, \leq, u)$, where $(G; +, \leq)$ is a *po*-group and u is a strong unit of G, is called a *unital po-group*. We will usually refer to it simply by (G, u).

For a unital *po*-group (G, u), let $\Gamma(G, u) \stackrel{\text{def}}{=} \{g \in G\colon\ 0 \leq g \leq u\}$, and define $+$ to be the partial binary operation on $\Gamma(G, u)$ that is the restriction of the group addition to those pairs of elements whose sum lies in $\Gamma(G, u)$. Then the structure $(\Gamma(G, u); +, 0, u)$ is called an *interval PE-algebra.*

It is easily checked that an interval PE-algebra is actually a PE-algebra.

Among the unital *po*-groups representing a given interval PE-algebra, there is a canonical one, defined and constructed as follows [DvVe3, Definition 5.2, Proposition 5.3].

Definition 2.7. Let $(E;+,0,1)$ be a PE-algebra and let (G,u) be a unital *po*-group. Let $\iota\colon\ E \to \Gamma(G,u)$ be an injective epimorphism such that, for every mapping ψ from E into a group H preserving existing sums, there is a group homomorphism $h_\psi\colon\ G \to H$ such that $\psi = h_\psi \circ \iota$. Then (G,u) together with ι is called a *universal ambient group* for E.

Certainly, if a PE-algebra has a universal ambient group, then it is the only one up to isomorphism.

Proposition 2.8. *Let $(E;+,0,1)$ be an interval PE-algebra. Let $\mathcal{U}(E)$ be the group given by the generators E and the defining relations $a+b = c$ for $a,b,c \in E$ whenever this equation holds in E; let ι be the natural embedding of E in $\mathcal{U}(E)$; and let $u = \iota(1)$. We may then partially order $\mathcal{U}(E)$ by letting $\mathcal{U}(E)^+$ the subsemigroup generated by $\iota(E)$. Moreover, $((\mathcal{U}(E),u),\iota)$ is a universal ambient group for E.*

For an interval PE-algebra E, we will in the sequel identify E and $\Gamma(\mathcal{U}(E),u)$, which in particular means $u = 1$, and we will refer to $(\mathcal{U}(E),1)$, or $\mathcal{U}(E)$, simply as the extension of E to its universal ambient group.

Now, under the following condition a PE-algebra is always a *po*-group interval [DvVe2, Theorem 5.7].

Definition 2.9. We say that a PE-algebra $(E;+,0,1)$ fulfils (RDP) if for any $a,b,c,d \in E$ such that $a+b = c+d$, there are $e_1,e_2,e_3,e_4 \in E$ such that the following scheme holds:

$$\begin{array}{cccc} e_1 & e_2 & \to & a \\ e_3 & e_4 & \to & b \\ \downarrow & \downarrow & & \\ c & d & & \end{array} \tag{1}$$

Moreover, we say that E fulfils (RDP$_1$) if for any $a,b,c,d \in E$ such that $a+b = c+d$, there are $e_1,e_2,e_3,e_4 \in E$ such that (i) the scheme (1) holds and (ii) for every $e_2' \leq e_2$ and $e_3' \leq e_3$, we have $e_2' + e_3' = e_3' + e_2'$.

Herein, by the scheme (1) to hold, we mean that every row and every column sums up to what the arrow points to; see [DvVe1].

Theorem 2.10. *Any PE-algebra $(E;+,0,1)$ fulfilling* (RDP$_1$) *is an interval PE-algebra.*

It is clear how the representing *po*-groups of symmetrically complemented PE-algebras are to be characterized.

Recall that the *center* of a group is the subset of those group elements which commute with all other elements.

Lemma 2.11. *Let $(E;+,0,1)$ be an interval PE-algebra and $(\mathcal{U}(E),1)$ the extension of E to its universal ambient group. Then E is symmetrically complemented if and only if 1 as an element of $\mathcal{U}(E)$ is in the center of $\mathcal{U}(E)$.*

3. States on interval PE-algebras

To analyze the structure of some algebra, we may consider the set of all homomorphisms to a certain simple kind of the same type of algebra. The probably simplest non-trivial kind of a PE-algebra is given by the real unit interval, that is, by $\Gamma(\mathbb{R},1)$, where $\mathbb{R}$ is the additive group of real numbers. The homomorphisms to it are called states [DvVe3].

Definition 3.1. Let $(E;+,0,1)$ be a PE-algebra. Let $([0,1];+,0,1)$ be the PE-algebra whose ground set is the real unit interval and whose addition is the usual sum of real numbers whenever defined within $[0,1]$. Then a *state* on E is a homomorphism $s\colon\ E \to [0,1]$ of PE-algebras.

Moreover, let (G,u) be a unital *po*-group. Then a *state* on G is a homomorphism $s\colon\ G \to \mathbb{R}$ of the unital *po*-groups (G,u) and $(\mathbb{R},1)$.

In other words, a state on a PE-algebra E is a function from E to the real unit interval preserving any existing sum and mapping the constant 1 to 1. Similarly, a state on a *po*-group (G,u) is a function from G to the real numbers preserving addition and positivity and mapping u to 1.

So states on a PE-algebra E, which is possibly non-commutative, are homomorphisms to the commutative algebra $[0,1]$. Possible different elements $a+b$ and $b+a$ for some $a,b \in E$, are mapped to the same value. So one may wonder if there is an effect algebra underlying a PE-algebra having the same state space. This is, under certain assumptions, indeed the case [DvVe3].

Theorem 3.2. *Let $(E;+,0,1)$ be an interval PE-algebra possessing a state. Then there is an interval effect algebra $(F;+,0,1)$ and an epimorphism ε: $(E;+,0,1) \to (F;+,0,1)$ such that the following holds. Any state s on E is of the form $s = s^F \circ \varepsilon$ for some state s^F on F; so the mapping $s \mapsto s^F$ establishes a one-to-one correspondence between the states of E and the states of F.*

Now note that Theorem 3.2 makes one crucial assumption: At least one state on the PE-algebra under consideration must exist. Only if this is case, we may reduce a non-commutative PE-algebra to a commutative one in a way which preserves the states – and, by the way, also the other homomorphisms to commutative PE-algebras. Now, to find sufficient conditions under which this situation is given, is the motivation underlying the present article.

In this section, we formulate exact conditions for a state on a PE-algebra to exist.

Let us see first how the states on a PE-algebra are related to those on a representing group.

Proposition 3.3. *Let $(E;+,0,1)$ be an interval PE-algebra, and let $(\mathcal{U}(E),1)$ be the extension of E to its universal ambient group. Then any state on E may be extended uniquely to a state on $(\mathcal{U}(E),1)$; and the reduction of any state on $(\mathcal{U}(E),1)$ to E is a state on E.*

In particular, E possesses a state if and only if $(\mathcal{U}(E),1)$ possesses a state.

Proof. Let $s\colon\ E \to [0,1]$ be a state. Then s, viewed as a mapping from E to $\mathbb{R}$, preserves existing sums, whence by Definition 2.7, s may be extended to a group homomorphism $S\colon\ \mathcal{U}(E) \to \mathbb{R}$.

Then S is a state. Indeed, S is, as stated, a group homomorphism; furthermore, it maps $\mathcal{U}(E)^+$ to $\mathbb{R}^+$, because $S(E) = s(E) \subseteq [0,1]$ and $\mathcal{U}(E)^+$ is generated by E; and, certainly, we have $S(1) = s(1) = 1$. □

For a unital *po*-group, we have in turn the following exact criterion for a state to exist.

Definition 3.4. Let $(G;+,\leq)$ be a *po*-group. Let $G_{\rm nc}$ be the subgroup of G generated by the elements of the form $(a+b)-(b+a)$, $a,b \in G$, endowed with the order inherited from G. We call $G_{\rm nc}$ the *commutator sub-po-group* of G.

Moreover, let (G,u) be a unital *po*-group. Let $G_{{\rm nc},u}$ be the subgroup of G generated by u and $G_{\rm nc}$, endowed with the order inherited from G. We call $(G_{{\rm nc},u},u)$ the *unital commutator sub-po-group* of (G,u).

Remark 3.5. Given a *po*-group G, note how we may describe the elements of $G_{\rm nc}$. Namely, $G_{\rm nc}$ contains exactly all sums of elements or negated elements of G such that every element occurs the same number of times negated and non-negated. In particular, $G_{\rm nc}$ is a normal subgroup and thus, endowed with the order of G, a *po*-group.

More general, let H be a subgroup of G containing $G_{\rm nc}$. If then any sum of elements of G is contained in H, also any permutation of this sum is in H. So again, H is automatically normal and, endowed with the order from G, a *po*-group.

Finally, given a unital *po*-group (G, u) such that u is in the center of G, note how we may describe $G_{\mathrm{nc},u}$. Namely, we simply have $G_{\mathrm{nc},u} = G_{\mathrm{nc}} \cup \{k\,u\colon\ k \in \mathbb{Z}\}$.

Proposition 3.6. *Let (G, u) be a unital po-group. Then (G, u) possesses a state if and only if there is no $a \geq u$ in G_{nc}.*

Proof. Let s be a state on (G, u). Then $a \geq u$ for some $a \in G_{\mathrm{nc}}$ would mean $s(a) \geq 1$; but s is constantly 0 on the whole of G_{nc}. This proves one direction of the statement.

Let now $\mathrm{conv}(G_{\mathrm{nc}})$ be the convex hull of G_{nc} in the partially ordered set G; then $\mathrm{conv}(G_{\mathrm{nc}})$ is closed under the group operations and under conjugation, so it is a convex normal subgroup of G. So we may form the quotient group $[G]_{\mathrm{conv}(G_{\mathrm{nc}})}$ of G by $\mathrm{conv}(G_{\mathrm{nc}})$, which has $[u]_{\mathrm{conv}(G_{\mathrm{nc}})}$ as a strong unit.

If for no $a \in G_{\mathrm{nc}}$ we have $u \leq a$, then $u \notin \mathrm{conv}(G_{\mathrm{nc}})$, and so $([G]_{\mathrm{conv}(G_{\mathrm{nc}})}, [u]_{\mathrm{conv}(G_{\mathrm{nc}})})$ is non-trivial abelian unital *po*-group. Every such group possesses a state by [Goo, Corollary 4.4]. It follows that also (G, u) possesses a state. □

We will now relate the states on a unital *po*-group (G, u) to its unital commutator sub-*po*-group $(G_{\mathrm{nc},u}, u)$. For, we will see that, if there is a state on one of these groups, then there is one on the other one. What we prove is even more: any state on a subgroup of G containing $G_{\mathrm{nc},u}$ is extendible to the whole of G.

We note that this result is in perfect analogy to [Goo, Chapter 4] about abelian *po*-groups fulfilling (RDP) and, furthermore, in accordance with [Dvu3].

Proposition 3.7. *Let (G, u) be a unital po-group such that u is in the center of G.*

(i) *Let H be a normal sub-po-group of G containing $G_{\mathrm{nc},u}$. Let s be a state on (H, u). Then s is extendible to a state on (G, u).*

(ii) *(G, u) possesses a state if and only if $(G_{\mathrm{nc},u}, u)$ possesses a state.*

Proof. (i) Let $\mathcal{S}$ be the set of all extensions of the state $s\colon\ H \to [0, 1]$ to a state on a larger normal subgroup of G, and let $\mathcal{S}$ be partially ordered by the extension relation. Since then every chain in $\mathcal{S}$ possesses a supremum, there is, by Zorn's Lemma, a maximal element $s'\colon\ H' \to$

$[0,1]$. Let us assume that there is some $c \in G \setminus H'$. We will show that then s' is extendible to a state s'' on the normal subgroup H'' generated by H' and c; this contradicts the maximality of s', and it follows that s' is a state on the whole of G.

Since H' contains $G_{\rm nc}$, it follows from Remark 3.5 that the subgroup generated by H' and c is normal and thus equal to H''. Furthermore, we again conclude from $G_{\rm nc} \subseteq H'$ that $H'' = \{k + ic\colon\ k \in H' \text{ and } i \in \mathbb{Z}\}$.

Let now

$$\begin{aligned} p &= \sup\,\{\frac{s'(a)}{m}\colon\ a \in H',\ m \in \mathbb{N},\ a \leq mc\}, \\ r &= \inf\,\{\frac{s'(b)}{n}\colon\ b \in H',\ n \in \mathbb{N},\ nc \leq b\}. \end{aligned}$$

We claim that then $-\infty < p \leq r < \infty$. Indeed, choose $k \in \mathbb{N}$ such that $-ku \leq c \leq ku$; then $-\infty < -k = s'(-ku) \leq p$, and similarly, we have $r \leq k < \infty$. Furthermore, choose $a, b \in H'$ and $m, n \in \mathbb{N}$ such that $a \leq mc$ and $nc \leq b$. Then $na \leq mnc \leq mb$, and we conclude

$$\frac{s'(a)}{m} = \frac{s'(na)}{mn} \leq \frac{s'(mb)}{mn} = \frac{s'(b)}{n},$$

which means $p \leq r$.

Now choose $q \in [p, r]$. We claim that we may define

$$\begin{aligned} s''\colon\quad & H'' \to \mathbb{R}, \\ & k + ic \mapsto s'(k) + iq, \text{ where } k \in H' \text{ and } i \in \mathbb{Z}. \end{aligned}$$

Indeed, assume $k_1 + i_1 c = k_2 + i_2 c$ for some $k_1, k_2 \in H'$, $i_1, i_2 \in \mathbb{Z}, i_2 \leq i_1$; this means $-k_1 + k_2 = (i_1 - i_2)c$. In the case $i_1 = i_2$ we have $s'(k_1) = s'(k_2)$, hence $s'(k_1) + i_1 q = s'(k_2) + i_2 q$. Otherwise we have $\frac{s'(-k_1+k_2)}{i_1-i_2} \leq p \leq q \leq r \leq \frac{s'(-k_1+k_2)}{i_1-i_2}$, that is, $s'(-k_1 + k_2) = (i_1 - i_2)q$ or $s'(k_1) + i_1 q = s'(k_2) + i_2 q$.

Moreover, s'' is positive. Indeed, assume $k + ic \geq 0$ for some $k \in H'$ and $i \in \mathbb{Z}$. If then $i < 0$, we have $\frac{s'(k)}{-i} \geq r \geq q$, that is, $s''(k + ic) = s'(k) + iq \geq 0$. If $i = 0$, we have $k \geq 0$, whence $s''(k) = s'(k) \geq 0$. If $i > 0$, we have $\frac{s'(-k)}{i} \leq p \leq q$, that is, $s''(k + ic) = s'(k) + iq \geq 0$. This concludes the proof that s'' is a state on (H'', u).

(ii) This follows easily from part (i). □

Given a group G, let $G^0 = G$ and, for $n \geq 1$, $G^n = G^{n-1}_{\rm nc}$. Recall that G is called *solvable*, if for some n, $G^n = \{0\}$. An easy consequence of Proposition 3.7 is the following.

Proposition 3.8. *Let (G, u) be a unital po-group such that u is in the center, and let G be solvable. Then G possesses a state.*

Proof. We have $G_{\mathrm{nc},u} = G_{\mathrm{nc}} \cup \{k\,u\colon\ k \in \mathbb{Z}\}$ by Remark 3.5. Now, by repeated application of Proposition 3.7(ii), we conclude that (G, u) possesses a state iff this is the case for the sub-*po*-group $\{k\,u\colon\ k \in \mathbb{Z}\}$, which evidently is true. □

We shall finally apply our results to PE-algebras.

Theorem 3.9. *Let $(E; +, 0, 1)$ be a symmetrically complemented interval PE-algebra, and let $(\mathcal{U}(E), 1)$ be the extension of E to its universal ambient group. Then E the following statements are equivalent:*

(i) *E possesses a state.*

(ii) *$(\mathcal{U}(E), 1)$ possesses a state.*

(iii) *There is no $a \geq 1$ in $\mathcal{U}(E)_{\mathrm{nc}}$.*

(iv) *$(\mathcal{U}(E)_{\mathrm{nc},1}, 1)$ possesses a state.*

Proof. (i) ⇔ (ii). By Proposition 3.3, E has a state iff $(\mathcal{U}(E), 1)$ has a state.

(ii) ⇔ (iii) This is the content of Proposition 3.6.

(ii) ⇔ (iv) This holds by Proposition 3.7(ii), because, by Lemma 2.11, 1 is in the center of $\mathcal{U}(E)$. □

4. Lexicographical product of an effect algebra and a *po*-group

We now turn our attention to a kind of PE-algebras for which it is natural to study homomorphisms to effect algebras and thus to study states. Independently from that, these special PE-algebras are quite interesting in their own right. What we will consider are PE-algebras which are built up from a commutative part, namely an effect algebra, and a non-commutative part changing elements, so to say, only in the small. The construction resembles the lexicographical product of a pair of *po*-groups.

In the subsequent chapter, we will discuss certain algebraic conditions on PE-algebras, for which the PE-algebras introduced here provide the typical examples.

Definition 4.1. Let $(E;+,0,1)$ be a PE-algebra and $(H;+,\leq)$ a *po*-group. Let

$$E \times_{\text{lex}} H \stackrel{\text{def}}{=} \{(e,h) \in E \times H\colon \quad \begin{array}{l} e = 0 \text{ and } h \geq 0 \\ \text{or } 0 < e < 1 \\ \text{or } e = 1 \text{ and } h \leq 0\}; \end{array}$$

and define a partial addition on $E \times_{\text{lex}} H$ componentwise whenever this is possible and leads to a result in $E \times_{\text{lex}} H$. Then $(E \times_{\text{lex}} H;+,(0,0),(1,0))$ is called the *lexicographical product* of the PE-algebra E and the *po*-group H.

This definition is motivated by the following facts.

Lemma 4.2. *Let $(E;+,0,1)$ be a PE-algebra and $(H;+,\leq)$ be a po-group.*

(i) *The lexicographical product $(E \times_{\text{lex}} H;+,(0,0),(1,0))$ of E and H is a PE-algebra.*

(ii) *Let E be an interval PE-algebra. Then so is $E \times_{\text{lex}} H$. Indeed, let $(\mathcal{U}(E),1)$ be the extension of E to its universal ambient group, and let $\mathcal{U}(E) \times_{\text{lex}} H$ be the usual lexicographical product of* po-groups; *then $E \times_{\text{lex}} H$ is isomorphic to $\Gamma(\mathcal{U}(E) \times_{\text{lex}} H,(1,0))$.*

Moreover, let H be directed. Then $\mathcal{U}(E) \times_{\text{lex}} H$ is the extension of $E \times_{\text{lex}} H$ to its universal ambient group.

Proof. (i) It is not difficult to verify the axioms (PE1) to (PE4).

(ii) The first half of what is claimed is obvious.

Assume now that H is directed. We show first that $\mathcal{U}(E \times_{\text{lex}} H)$ and $\mathcal{U}(E) \times H$ are isomorphic groups. From Proposition 2.8 we know that the first group of the two is generated by all $(e,h) \in E \times_{\text{lex}} H$ and subject to the defining relations $(e_1,h_1)+(e_2,h_2)=(e_3,h_3)$ whenever this equation holds in $E \times_{\text{lex}} H$. Now, by the directedness of H we have $H = H^+ - H^+$; and in $\mathcal{U}(E \times_{\text{lex}} H)$ we have that any $(0,h)$, $h \in H^+$ commutes with any $(e,0)$, $e \in E$. This means that the group is actually generated by all $(e,0)$ for $e \in E$ and all $(0,h)$ for $h \in H^+$ under the conditions $(e_1,0)+(e_2,0)=(e_3,0)$ for $e_1,e_2,e_3 \in E$ such that $e_1+e_2=e_3$, and $(0,h_1)+(0,h_2)=(0,h_3)$ for $h_1,h_2,h_3 \in H^+$ such that $h_1+h_2=h_3$. But then it is clear that $\mathcal{U}(E \times_{\text{lex}} H)$ may be identified with the product of $\mathcal{U}(E)$ and H, that is, with the group $\mathcal{U}(E) \times H$.

Now, under this identification, $\mathcal{U}(E \times_{\text{lex}} H)^+$ is the by Proposition 2.8 the subsemigroup generated by the elements (e,h), where $e \in E$,

$h \in H$ and either $e > 0$ or else $h \in H^+$. If follows that $\mathcal{U}(E \times_{\text{lex}} H)^+$ is identified with $(\mathcal{U}(E) \times_{\text{lex}} H)^+$. □

We note that for lexicographical products of PE-algebras and *po*-groups, symmetric complementation is not unnatural. Besides, we see that a symmetrically complemented PE-algebra is not necessarily commutative.

Proposition 4.3. *The lexicographical product of an effect algebra and a po-group is symmetrically complemented.*

Proof. This is easily checked. □

We shall now establish the exact conditions under which the lexicographical product of a PE-algebra and a *po*-group is of the kind we are primarily interested in – by fulfilling the Riesz condition (RDP). Note that we will not be concerned with the condition (RDP_1) here. Indeed, (RDP_1) would have rather strong consequences; it would typically imply the abelianess of the *po*-group involved. Anyhow, the more general discussion of the subsequent Section 5 is about interval PE-algebras fulfilling (RDP); so it is the latter property which is of primary interest in the present context.

We note that we generalize [Goo, Corollary 2.12] to the non-commutative case.

Theorem 4.4. *Let* $(E \times_{\text{lex}} H; +, (0,0), (1,0))$ *be the lexicographical product of an at least three-element PE-algebra* E *and a po-group* H*. Then* $E \times_{\text{lex}} H$ *fulfils* (RDP) *if and only if* E *as well as* H *fulfil* (RDP) *and one of the following conditions holds.*

(α) *E is atomless, and for any pair of non-comparable elements $a, b \in E$ whose sum $a + b$ exists, there is a non-zero $x < a, b$ such that*

$$a \,/\, x \,+\, b = b \,/\, x \,+\, a.$$

(β) *For any $g, h, k \in H$ there is an $f \geq g, h$ commuting with k.*

(γ) *E has a smallest non-zero element. Furthermore, for any $g \in H$ there is a positive $f \geq g$ commuting with g.*

Proof. Assume first that $E \times_{\text{lex}} H$ fulfils (RDP). It is clear that then E as well as H fulfil (RDP).

Assume now that (α) does not hold for the reason that there is an atom a. Then either there is a $b \in E$ such that $a \wedge b = 0$, or a is the only atom and lies below all non-zero elements.

In the first case, (β) follows. Indeed, we have $a+b=b+a$ by [DvVe1, Lemma 3.2(ii)]. For $g,h,k \in H$, we may apply (RDP) to the equation

$$(a,-g)+(b,k-h)=(b,-g+k)+(a,-h),$$

to get $f^1,\ldots,f^4 \in H$ such that

$$\begin{array}{ccccc} (0,f^1) & (a,f^2) & \to & (a,-g) \\ (b,f^3) & (0,f^4) & \to & (b,k-h) \\ \downarrow & \downarrow & & \\ (b,-g+k) & (a,-h). & & \end{array}$$

Because then $f^1,f^4 \geq 0$, we have $f^2 \leq -g,-h$ and so $f \stackrel{\text{def}}{=} -f^2 \geq g,h$; and f^2 commutes with f^3, so f commutes with $f^3-f^2=(f^3+f^4)-(f^2+f^4)=k$.

In the second case, (γ) follows. Indeed, let $g \in H$. The sum $a+a$ exists, because otherwise E would contain only the two elements 0 and $1=a$. So we may apply (RDP) to the equation

$$(a,0)+(a,0)=(a,g)+(a,-g),$$

to get $f^1,f^2 \in H$ such that either the scheme

$$\begin{array}{ccccc} (0,f^1) & (a,f^2) & \to & (a,-g) \\ (a,-f^1) & (0,-f^2) & \to & (a,g) \\ \downarrow & \downarrow & & \\ (a,0) & (a,0). & & \end{array}$$

holds, in which case we have $f \stackrel{\text{def}}{=} -f^2 \geq 0,g$ and $f+g=-f^2-f^1-f^2=g+f$; or

$$\begin{array}{ccccc} (a,f^1) & (0,f^2) & \to & (a,-g) \\ (0,-f^1) & (a,-f^2) & \to & (a,g) \\ \downarrow & \downarrow & & \\ (a,0) & (a,0). & & \end{array}$$

holds, in which case we have $f \stackrel{\text{def}}{=} -f^1 \geq 0,g$ and $f+g=g+f$.

Assume now that (α) does not hold for the reason that there is are non-comparable, summable $a,b \in E$ such that for no non-zero $x \leq a,b$, we have $a / x+b=b / x+a$. Choose then $a' \in E$ such that $a+b=b+a'$, and let $e^1,e^2 \in E$ such that

$$\begin{array}{cccc} e^1 & a \setminus e^1 & \to & a \\ b \setminus e^1 & e^2 & \to & b \\ \downarrow & \downarrow & & \\ b & a' & & \end{array}$$

holds. This means $a + b \setminus e^1 + e^2 = b + a \setminus e^1 + e^2$, that is, $a / e^1 + b = b / e^1 + a$; so we have by assumption $e^1 = 0$ and thus $e^2 = 0$ and $a' = a$. We may now proceed as above to prove (β).

This finishes the proof of one direction of the claimed equivalence.

Assume now that E and H fulfil (RDP), and let (α) or (β) or (γ) hold. We are going to prove that (RDP) holds in $E \times_{\mathrm{lex}} H$; so let the equation $(a_1, a_2) + (b_1, b_2) = (c_1, c_2) + (d_1, d_2)$ hold in $E \times_{\mathrm{lex}} H$. Since (RDP) holds in E, we may easily determine the first components of the four elements we are looking for; choose $e_1, \ldots, e_4 \in E$ such that

$$\begin{array}{ccll} e^1 & e^2 & \to & a_1 \\ e^3 & e^4 & \to & b_1 \\ \downarrow & \downarrow & & \\ c_1 & d_1. & & \end{array} \tag{2}$$

To determine the appropriate second components from H, we will distinguish several cases, dependent on at which places in (2) zeros appear.

The case that all entries in (2) are 0 is easily handled by the fact that (RDP) holds in H.

Assume now that one whole row or column in (2) is 0; a typical example would be $e^1 = e^2 = 0$. If then, furthermore, $e^3 = c_1 > 0$ the scheme

$$\begin{array}{ccll} (0, a_2) & (0, 0) & \to & (0, a_2) \\ (c_1, -a_2 + c_2) & (d_1, d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2) & & \end{array}$$

fulfils the requirements. If, on the other hand, $e^3 = c_1 = 0$ and $e^4 = b_1 = d_1 > 0$, we have $a_2, c_2 \geq 0$. We may then apply (RDP) to the equation $a_2 + c_2 = c_2 + (-c_2 + a_2 + c_2)$ to get $f^1, \ldots, f^4 \in H$ such that $f^1 + f^2 = a_2$, $f^3 + f^4 = f^1 + f^3 = c_2$, and $f^2 + f^4 = -c_2 + a_2 + c_2$. Set $f'^4 = -f^3 + b_2 = -f^2 + d_2$, and consider the scheme

$$\begin{array}{ccll} (0, f^1) & (0, f^2) & \to & (0, a_2) \\ (0, f^3) & (b_1, f'^4) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (0, c_2) & (b_1, d_2). & & \end{array}$$

Assume next that in (2), there is a zero in the e^1-e^4 diagonal, for instance $e^1 = 0$, and that $e^2 = a_1 > 0$ and $e^3 = c_1 > 0$.

If (α) holds and a_1, c_1 are either non-comparable or equal, there is some $x \in E$ such that $0 < x < a_1, c_1$ and $a_1 / x + c_1 = c_1 / x + a_1$. Then the scheme

$$\begin{array}{ccccc} & (x, a_2) & (a_1 \setminus x, 0) & \to & (a_1, a_2) \\ (3) \qquad & (c_1 \setminus x, -a_2 + c_2) & (e'^4, d_2) & \to & (b_1, b_2) \\ & \downarrow & \downarrow & & \\ & (c_1, c_2) & (d_1, d_2) & & \end{array}$$

fulfils, for some non-zero $e'^4 \in E$, the requirements.

If (α) holds and $a_1 < c_1$, set $x = a_1$ and $e'^4 = d_1$ in (3). Similarly, we proceed in the case $c_1 < a_1$.

If (β) holds, then there is an $f^1 \leq a_2, d_2$ such that f^1 commutes with $f^2 \stackrel{\text{def}}{=} f^1 + b_2 - d_2$. We get

$$\begin{array}{cccc} (0, a_2 - f^1) & (a_1, f^1) & \to & (a_1, a_2) \\ (c_1, f^2) & (e^4, -f^1 + d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2). & & \end{array}$$

If (γ) holds, let a be the atom. Then a commutes with all $b < 1$; indeed, choose a', a'' such that $a + b = b + a'$ and $b + a = a'' + b$; then $a + b = b + a + (a' \setminus a) = (a'' \,/\, a) + a + b + (a' \setminus a)$, whence $a = a' = a''$. So if $c_1 > a$, set $x = a$ and $e'^4 = x + e^4$ in (3). Similarly, we proceed in the case $c_1 = a$, but $a_1 > a$. If $a_1 = c_1 = a$, choose an $f \in H^+$ such that $f \geq -c_2 + a_2$ and f commutes with $-c_2 + a_2$, and consider the scheme

$$\begin{array}{ccccc} & (a, a_2 - f) & (0, f) & \to & (a, a_2) \\ (4) \qquad & (0, f - a_2 + c_2) & (b_1, -f + d_2) & \to & (b_1, b_2) \\ & \downarrow & \downarrow & & \\ & (a, c_2) & (b_1, d_2). & & \end{array}$$

Assume next that in (2), there is a zero in the e^2-e^3 diagonal, for instance $e^2 = 0$, and that $e^1 = a_1 > 0$ and $e^4 = d_1 > 0$.

If, moreover, $e^3 > 0$, we have

$$\begin{array}{cccc} (a_1, a_2) & (0, 0) & \to & (a_1, a_2) \\ (e^3, -a_2 + c_2) & (d_1, d_2) & \to & (b_1, b_2) \\ \downarrow & \downarrow & & \\ (c_1, c_2) & (d_1, d_2), & & \end{array}$$

So let $e^3 = 0$.

If (α) holds, we choose some $x \in E$ such that $0 < x < a_1, d_1$, and we have

$$\begin{array}{ccc} (a_1 \,/\, x, a_2) & (x, 0) & \to \ (a_1, a_2) \\ (x, -a_2 + c_2) & (b_1 \setminus x, d_2) & \to \ (b_1, b_2) \\ \downarrow & \downarrow & \\ (a_1, c_2) & (b_1, d_2). & \end{array}$$

If (β) holds, choose $r, s \in H$ such that $r \geq -a_2, -c_2$ and $s \geq -b_2, -d_2$, and apply (RDP) to the equation $(r + a_2) + (b_2 + s) = (r + c_2) + (d_2 + s)$ to get $f^1, \dots, f^4 \in H^+$ such that $f^1 + f^2 = r + a_2$, $f^3 + f^4 = b_2 + s$, $f^1 + f^3 = r + c_2$, $f^2 + f^4 = d_2 + s$. Then we have the scheme

$$\begin{array}{ccc} (a_1, -r + f^1) & (0, f^2) & \to \ (a_1, a_2) \\ (0, f^3) & (b_1, f^4 - s) & \to \ (b_1, b_2) \\ \downarrow & \downarrow & \\ (a_1, c_2) & (b_1, d_2) & \end{array}$$

If (γ) holds, choose an $f \in H^+$ such that $f \geq -c_2 + a_2$ and f and $-c_2 + a_2$ commute, to get a scheme similar to (4).

If in (2) there appears no zero, we have

$$\begin{array}{ccc} (e^1, a_2) & (e^2, 0) & \to \ (a_1, a_2) \\ (e^3, -a_2 + c_2) & (e^4, d_2) & \to \ (b_1, b_2) \\ \downarrow & \downarrow & \\ (c_1, c_2) & (d_1, d_2). & \end{array}$$

This finishes the proof the the second half of the theorem. □

We have in particular the following.

Theorem 4.5. *Let* $(E \times_{\mathrm{lex}} H; +, (0,0), (1,0))$ *be the lexicographical product of a PE-algebra* E *and a po-group* H*. Then also* $E \times_{\mathrm{lex}} H$ *fulfils* (RDP) *if* E *as well as* H *fulfil* (RDP) *and one of the following conditions holds.*

- (α) *For any non-zero* $a, b \in E$ *such that* $a + b$ *exists, there is an* $x \in E$ *such that* $0 < x < a, b$ *and* $a \,/\, x$ *commutes with* $b \,/\, x$*.*
- (β) *Any pair of elements from* H *has an upper bound in the center of* H*.*

Now, although the lexicographical products discussed so far involve possibly non-commutative PE-algebras, the example we have in mind is the lexicographical product of an effect algebra and a, possibly non-abelian, *po*-group.

It is clear that considering states on such algebras means considering states on the underlying effect algebra. Compare Theorem 3.2.

Proposition 4.6. *Let* $(E \times_{\text{lex}} H; +, (0,0), (1,0))$ *be the lexicographical product of an effect algebra* E *and a directed po-group* H*. Let* ι: $E \times_{\text{lex}} H \to E, \;\; (a,0) \mapsto a$*. Then any state* s *on* $E \times_{\text{lex}} H$ *is of the form* $s = s^E \circ \iota$ *for some state* s^E *on* E*; so the mapping* $s \mapsto s^E$ *establishes a one-to-one correspondence between the states of* $E \times_{\text{lex}} H$ *and the states of* E*.*

Proof. For any state s: $E \times_{\text{lex}} H \to [0,1]$, we have $s((0,h)) = 0$ for all $h \in H$, because $n\, s((0,h)) = s((0, n\, h)) \leq s((1,0)) = 1$ for all $n \in \mathbb{N}$. Since H is directed, it follows that $s((a,h))$, $h \in H$, does not depend on h. □

5. Order-regular and nearly commutative PE-algebras

The lexicographical product of an effect algebra and a possibly non-commutative *po*-group, as studied in the previous section, is just the simplest example of a PE-algebra about which we may say that it is non-commutative only in the small. In the present section, we shall make precise this idea, by introducing the appropriate algebraic conditions.

Namely, we shall introduce order-regular and nearly commutative PE-algebras. These two properties imply, under certain further natural assumptions, that a PE-algebra possesses a non-trivial effect algebra as a homomorphic image. In view of Theorem 3.2, this is equivalent to the existence of states on the algebra.

Moreover, we will slightly strengthen the property of being order-regular. We will see that this has an amazingly far-reaching consequence; the commutator subgroup of the *po*-group representing the PE-algebra is then purely infinitesimal.

Definition 5.1. Let $(E; +, 0, 1)$ be a PE-algebra. We shall call $a \in E$ *infinitesimal* if $n\, a$ is defined for all $n \in \mathbb{N}$. We denote the set of all infinitesimal elements of E by E_{infts}.

Furthermore, let (G, u) be a unital *po*-group. We shall call $a \in G$ *infinitesimal* if $n\, a \leq u$ for all $n \in \mathbb{Z}$. We denote the set of all infinitesimal elements of G by G_{infts}.

Now, if the commutator subgroup of the *po*-group representing a PE-algebra is purely infinitesimal, the question if the algebra has a state is easily answered affirmatively.

Proposition 5.2. *Let $(E;+,0,1)$ be an interval PE-algebra, and let $\mathcal{U}(E)$ the extension of E to its universal ambient group. If then the commutator sub-po-group of $\mathcal{U}(E)$ contains only infinitesimal elements, that is, if*

$$\mathcal{U}(E)_{\mathrm{nc}} \subseteq \mathcal{U}(E)_{\mathrm{infts}},$$

then E possesses a state.

Proof. Under the given assumptions it is clear that there is no $a \geq u$ in $\mathcal{U}(E)_{\mathrm{nc}}$; so the claim follows from Propositions 3.6 and 3.3. □

For what follows, we need the following preparatory definitions.

Definition 5.3. Let $(E;+,0,1)$ be a PE-algebra.

(i) Let $a, b \in E$. We say that a is *essentially smaller* than b if $a < b$ and neither the left nor the right difference of b and a is infinitesimal. In this case, we write $a \mathrel{\bar{<}} b$.

(ii) Two elements $a, b \in E$ are called *close* if for any $c \in E$ we have $c \mathrel{\bar{<}} a$ if and only if $c \mathrel{\bar{<}} b$, and if for any $d \in E$ we have $a \mathrel{\bar{<}} d$ if and only if $b \mathrel{\bar{<}} d$. In this case, we write $a \approx b$.

Remark 5.4. It is easily verified that, for elements a, b of a symmetrically complemented interval PE-algebra E such that $a \leq b$, we have $a \mathrel{\bar{<}} b$ iff $a \setminus b$ is not infinitesimal iff $a \,/\, b$ is not infinitesimal.

Based on the notion of closeness, we newly introduce two properties of PE-algebras. The algebras of the special type discussed in Section 4 provide typical examples.

Definition 5.5. Let $(E;+,0,1)$ be a PE-algebra.

(i) E is called *order-regular* if any two comparable elements whose left or right difference is infinitesimal, are close.

(ii) E is called *nearly commutative* if, for any $a, b \in E$, $a + b$ exists if and only if $b + a$ exists, in which case we have $a + b \approx b + a$.

Proposition 5.6. *The lexicographical product of an archimedean effect algebra and a po-group is order-regular and nearly commutative.*

Let us first state the crucial property of an order-regular PE-algebra.

Recall [DvVe3, Definition 3.4(i)] that an *ideal* of a PE-algebra E is a set $I \subseteq E$ such that (i) for any $b \in I$ and $a \in E$, $a \leq b$ implies $a \in I$ and (ii) for any $a, b \in I$ such that $a + b$ exists, also $a + b \in I$. An ideal is called *normal* if for $a, r, s \in E$ such that $r + a = a + s$, we have $r \in I$ if and only if $s \in I$.

Proposition 5.7. *Let* $(E; +, 0, 1)$ *be an interval PE-algebra fulfilling* (RDP) *which is symmetrically complemented and order-regular. Then* E_{infts} *is a normal ideal of* E*. Moreover, any infinitesimal element of* E *lies below any non-infinitesimal one.*

Proof. Clearly, E_{infts} is closed under smaller elements. To see that E_{infts} is closed under sums, let $a, b \in E_{\text{infts}}$ such that $a + b$ exists. We then have $0 \approx a$, because E is order-regular, and $a \,\bar{\not<}\, a + b$; so it follows $0 \,\bar{\not<}\, a + b$, which means that $a + b \in E_{\text{infts}}$. So E_{infts} is an ideal. Because E is symmetrically complemented, it easily follows that E_{infts} is normal.

Let now $a \in E_{\text{infts}}$ and b non-infinitesimal. Then, in view of Remark 5.4, we have $a \,\bar{<}\, a + b$. On the other hand, we have by the order-regularity $b \approx a + b$, so it follows $a \,\bar{<}\, b$ and in particular $a < b$. $\square$

Note that, given a PE-algebra E as specified in Proposition 5.7, we may by [DvVe3, Proposition 3.6] form the quotient algebra $[E]_{E_{\text{infts}}}$, to get a PE-algebra which is archimedean.

Here, we are interested in the closeness relation $\approx$, which, as we will see now, proves to be a structure preserving congruence relation as well.

We will from now on assume tacitly that the PE-algebras we deal with are non-trivial.

Theorem 5.8. *Let* $(E; +, 0, 1)$ *be an interval PE-algebra fulfilling* (RDP) *which is symmetrically complemented and order-regular. Then the relation* $\approx$ *on* E *is a PE-algebra congruence; the quotient algebra* $[E]_{\approx}$ *is a non-trivial PE-algebra fulfilling* (RDP).

Proof. Let us first note the following. For $a, b \in E$ such that $a \approx b$, we easily see that $a^{\sim} \approx b^{\sim}$. Furthermore, for $a, b \in E$ such that $a + b$ exists, we have $b \approx a^{\sim}$ iff $a + b \approx 1$. Indeed, since $b \leq a^{\sim}$, we have $b \approx a^{\sim}$ iff $(a + b)^{\sim} = a^{\sim} \setminus b \in E_{\text{infts}}$ iff $(a + b)^{\sim} \approx 0$ iff $a + b \approx 1$.

We now prove that $\approx$ is a congruence according to Definition 2.4(iii). It is, first of all, clear that $\approx$ is an equivalence relation on E. Let now $a, a', b, b' \in E$ be given such that $a + b$ and $a' + b'$ exist and $a \approx a'$ and $b \approx b'$. We claim that $a + b \approx a' + b'$.

In case that $b \approx a^{\sim}$, we have, as shown, $a + b \approx 1$; and from $b' \approx b \approx a^{\sim} \approx a'^{\sim}$, it follows $a' + b' \approx 1 \approx a + b$. In the opposite case, we have $b \,\bar{<}\, a^{\sim}$, which implies by $a'^{\sim} \approx a^{\sim}$ that $b \,\bar{<}\, a'^{\sim}$ holds and thus

$a' + b$ exists. Let now $c \in E$ such that $c \mathrel{\bar{<}} a + b$; we shall see that $c \mathrel{\bar{<}} a' + b$. We then have $c = c_a + c_b$ for some $c_a \leq a,\ c_b \leq b$. If then $a \setminus c_a$ is not infinitesimal, we have $c_a \mathrel{\bar{<}} a$, and it follows $c_a \mathrel{\bar{<}} a'$ and so $c = c_a + c_b \mathrel{\bar{<}} a' + c_b \leq a' + b$. Otherwise, because $c = c_a + c_b \mathrel{\bar{<}} a + b = c_a + a \setminus c_a + b \mathbin{/} c_b + c_b$ and so $a \setminus c_a + b \mathbin{/} c_b \notin E_{\text{infts}}$ and because E_{infts} is by Proposition 5.7 an ideal, $b \mathbin{/} c_b$ is not infinitesimal. Then from $a' \mathrel{\bar{<}} a' + b \mathbin{/} c_b$ and $a' \approx a \approx c_a$ we conclude $c_a \mathrel{\bar{<}} a' + b \mathbin{/} c_b$ and so $c = c_a + c_b \mathrel{\bar{<}} a' + b$. In a similar manner, we proceed to show that $c \mathrel{\bar{<}} a' + b'$. Finally, we may prove by analogous reasoning that $c \mathrel{\bar{<}} a + b$ iff $c \mathrel{\bar{<}} a' + b'$, and, for any $d \in E$, that $a + b \mathrel{\bar{<}} d$ iff $a' + b' \mathrel{\bar{<}} d$.

We now prove that $\approx$ is actually a PE-algebra congruence, using Proposition 2.5. So let $a, a', b \in E$ such that $a + b$ exists and $a' \approx a$; we will show that there is a $b' \approx b$ such that $a' + b'$ exists. We have $a \leq b^-$. If even $a \mathrel{\bar{<}} b^-$, it follows $a' \mathrel{\bar{<}} b^-$ and thus $a' + b$ exists. Otherwise $b^- \setminus a$ is infinitesimal; so $a' \approx a \approx b^-$ and, setting $b' = a'^{\sim}$, we have $b' = a'^{\sim} \approx b$ and $a' + b' = 1$ exists. Analogously, we see that for any $b'' \approx b$, there is an $a'' \approx a$ such that $a'' + b''$ exists.

So we have proved that $[E]_{\approx}$ is a PE-algebra. Now, since the constants 0 and 1 are not close, $[0]_{\approx}$ and $[1]_{\approx}$ are different elements, that is, $[E]_{\approx}$ is non-trivial.

It remains to show that $[E]_{\approx}$ fulfils (RDP); so let $a, b, c, d \in E$ be given such that $[a]_{\approx} + [b]_{\approx} = [c]_{\approx} + [d]_{\approx}$. By what was just proved, there are $b' \approx b$ and $d' \approx d$ such that $a + b'$ and $c + d'$ are defined, and we have $a + b' \approx c + d'$. Now the case that b' is infinitesimal is easy, because then $[a]_{\approx} = [c]_{\approx} + [d]_{\approx}$. Otherwise, we have $b' \mathrel{\bar{>}} 0$, so $a \mathrel{\bar{<}} a + b'$ and consequently $a \mathrel{\bar{<}} c + d'$, that is, $a + b'' = c + d'$ for some $b'' \approx b'$. Thus (RDP), holding in E, enables us to choose the appropriate four elements from $[E]_{\approx}$, as required by Definition 2.9. □

We are finally ready to formulate our first state existing theorem.

Theorem 5.9. *Let* $(E; +, 0, 1)$ *be an interval PE-algebra fulfilling* (RDP) *which is symmetrically complemented, order-regular, and nearly commutative. Then the quotient algebra* $[E]_{\approx}$ *is a non-trivial interval effect algebra. In particular,* E *possesses a state.*

Proof. From Theorem 5.8, we know that $[E]_{\approx}$ is a non-trivial PE-algebra fulfilling (RDP).

Now, by Remark 2.3, for any $a, b \in E$, $a+b$ is defined iff $b+a$ is defined, and it follows that $[a]_{\approx}+[b]_{\approx}$ is defined iff $[b]_{\approx}+[a]_{\approx}$ is defined. Because

E is nearly commutative, we have $a + b \approx b + a$ if these sums exist. It easily follows that $[E]_\approx$ is commutative.

So we have proved that $[E]_\approx$ is an effect algebra fulfilling (RDP). Since for effect algebras, (RDP) and (RDP$_1$) are equivalent, $[E]_\approx$ is by Theorem 2.10 an interval of an abelian *po*-group. Namely, $[E]_\approx$ is the unit interval of $(\mathcal{U}([E]_\approx), [1]_\approx)$.

$(\mathcal{U}([E]_\approx), [1]_\approx)$ possesses by [Goo, Corollary 4.4] a state, which reduces to a state on $[E]_\approx$. So also E possesses a state. □

Now, on our way to show the existence of states, we did not make use of the strong criterion used in Proposition 5.2, where it was assumed that the commutator sub-*po*-group $\mathcal{U}(E)_{\rm nc}$ is purely infinitesimal. But interestingly, we may force such a situation, by slightly strengthening one of our conditions.

Definition 5.10. Let $(E; +, 0, 1)$ be a PE-algebra. E is called *strongly order-regular* if E is order-regular and if for any non-infinitesimal close elements $a, b \in E$, there is a non-infinitesimal element $c \leq a, b$.

The typical example for strong order-regularity is given by those lexicographical products of PE-algebras and *po*-groups which fulfil the Riesz decomposition property – that is, by those PE-algebras which have been characterized in Theorem 4.4.

Proposition 5.11. *The lexicographical product of an archimedean at least five-element PE-algebra and a po-group such that this product fulfils* (RDP)*, is strongly order-regular.*

We need two preparatory lemmas.

Lemma 5.12. *Let $(E; +, 0, 1)$ be an interval PE-algebra fulfilling* (RDP) *which is symmetrically complemented, order-regular and nearly commutative. Let $\mathcal{U}(E)$ be the extension of E to its universal ambient group. Then there is a po-group congruence $\approx$ on $\mathcal{U}(E)$ whose restriction to E is the equally denoted relation on E.*

Proof. Let $\kappa\colon\ E \to [E]_\approx,\ \ a \mapsto [a]_\approx$ the natural homomorphism from E to $[E]_\approx$. By Theorem 5.9, $[E]_\approx$ is an interval effect algebra, represented by $\mathcal{U}([E]_\approx)$. We see from Definition 2.7 and Proposition 2.8 that κ extends to a *po*-group homomorphism from $\mathcal{U}(E)$ to $\mathcal{U}([E]_\approx)$. Now, κ induces a *po*-group congruence $\approx$ on $\mathcal{U}(E)$ which identifies two elements $a, b \in E$ if and only if $\kappa(a) = \kappa(b)$ if and only if $a \approx b$ in E. □

In what follows, we will use the essential order relation $\mathrel{\bar{<}}$ for $\mathcal{U}(E)$, the universal ambient group of some PE-algebra E, just in the same way as for E: Let, for a pair $a, b \in \mathcal{U}(E)$, $a \mathrel{\bar{<}} b$ hold if $a < b$ and $b \,/\, a,\ b \setminus a \notin \mathcal{U}(E)_{\text{infts}}$. If then $a, b \in E$, this relation has obviously the same meaning as with respect to the PE-algebra E.

Lemma 5.13. *Let* $(E; +, 0, 1)$ *be an interval PE-algebra fulfilling* (RDP) *which is symmetrically complemented, strongly order-regular, and nearly commutative. Let* $\mathcal{U}(E)$ *be its extension to its universal ambient group, and let* $\approx$ *be the extension of the closeness relation from* E *to* $\mathcal{U}(E)$*. Then, for* $a, b, c \in \mathcal{U}(E)$*,* $a \mathrel{\bar{<}} b$ *and* $b \approx c$ *imply* $a \mathrel{\bar{<}} c$*.*

Proof. We shall first prove the following preliminary statement with respect to $\mathcal{U}(E)$: For $b, c \in E$ and $d \in \mathcal{U}(E)^+$, $b \approx c$ and $d \mathrel{\bar{>}} 0$ imply $b \mathrel{\bar{<}} d + c$.

By replacing d by a smaller, but still non-infinitesimal element if necessary, we may assume that $d \in E$. Let $d = d_c + d_{c^-}$ such that $d_c \leq c$ and $d_{c^-} \leq c^-$. If $d_{c^-} \mathrel{\bar{>}} 0$, then from $c \mathrel{\bar{<}} d_{c^-} + c$ we have $b \mathrel{\bar{<}} d_{c^-} + c \leq d + c$. In the other case, we have $d_c \approx d$. If then, in addition $d_c \mathrel{\bar{<}} c$, it follows $d_c \mathrel{\bar{<}} b$, and we may conclude from $c^- \mathrel{\bar{<}} c^- + d_c$ that $c^- \mathrel{\bar{<}} b^- + d_c \leq b^- + d$, whence $b^- + b = c^- + c \mathrel{\bar{<}} b^- + d + c$ and $b \mathrel{\bar{<}} d + c$. If, otherwise, $d_c \approx c \approx d$ and, in addition $c \mathrel{\bar{<}} c^-$, we have $d_c \mathrel{\bar{<}} c^-$ and so $b \approx c \mathrel{\bar{<}} d_c + c$ and $b \mathrel{\bar{<}} d + c$. It remains the case $d_c \approx c \approx c^- \approx d$; then, by strong order-regularity, there is an $e \mathrel{\bar{>}} 0$ such that $e \leq d_c, c^-$, and we have $b \approx c \mathrel{\bar{<}} e + c$ and so $b \mathrel{\bar{<}} d + c$. This completes the proof of the preliminary statement.

Assume now $a \mathrel{\bar{<}} b$ and $b \approx c$, where $a, b, c \in \mathcal{U}(E)$; we shall prove $a \mathrel{\bar{<}} c$. Since $\mathcal{U}(E)$ is directed and the involved relations are translation-invariant, we may assume that $a, b, c \geq 0$. Furthermore, $b \approx c$ then holds exactly if $[b]_\approx = [c]_\approx$ in $\mathcal{U}([E]_\approx)^+$; and $\mathcal{U}([E]_\approx)^+$ is the semigroup freely generated by the elements of E with the defining relations $x + y = z$ for $x, y, z \in E$ such that this equation holds in E, and $x = y$ for $x, y \in E$ such that $x \approx y$ in E. So we may further assume that $b = b_1 + b_2 + b_3,\ c = b_1 + c_2 + b_3$ for $b_2, c_2 \in E$ such that $b_2 \approx c_2$; the general claim follows from this special case by induction.

Let $a = a_1 + a_2 + a_3$ such that $a_1 \leq b_1$, $a_2 \leq b_2$, and $a_3 \leq b_3$. If then $a_2 \mathrel{\bar{<}} b_2$, we have $a_2 \mathrel{\bar{<}} c_2$ and thus $a \mathrel{\bar{<}} c$. Otherwise we have $a_2 \approx b_2$, and it follows $a_1 \mathrel{\bar{<}} b_1$ or $a_3 \mathrel{\bar{<}} b_3$, because E_{infts} is by Proposition 5.7 a normal ideal; suppose e.g. that the first inequality holds. So we have $a_2 \approx c_2$ and $b_1 \setminus a_1 \mathrel{\bar{>}} 0$, and by what was proved above, we know $a_2 \mathrel{\bar{<}} (b_1 \setminus a_1) + c_2$, or $a_1 + a_2 \mathrel{\bar{<}} b_1 + c_2$. It follows $a \mathrel{\bar{<}} c$. □

In view of Proposition 5.2, the subsequent Theorem 5.14 may be considered the strengthened version of a state existence theorem.

Theorem 5.14. *Let* $(E; +, 0, 1)$ *be an interval PE-algebra fulfilling* (RDP) *which is symmetrically complemented, strongly order-regular, and nearly commutative. Let* $\mathcal{U}(E)$ *be the extension of* E *to its universal ambient group. Then the commutator sub-po-group of* $\mathcal{U}(E)$ *contains only infinitesimal elements, that is, we have* $\mathcal{U}(E)_{\mathrm{nc}} \subseteq \mathcal{U}(E)_{\mathrm{infts}}$.

Proof. Let $\approx$, as before, be the extension of the closeness relation from E to $\mathcal{U}(E)$, according to Lemma 5.12. Since $[E]_\approx$ is, by Theorem 5.9, an interval effect algebra, $\mathcal{U}([E]_\approx)$ is abelian, so we have for all $a, b \in \mathcal{U}(E)$ that $[a+b]_\approx = [b+a]_\approx$, or $a + b \approx b + a$. This means for all $g \in \mathcal{U}(E)_{\mathrm{nc}}$ that $g \approx 0$. By $0 \mathrel{\bar{<}} 1$ and Lemma 5.13, it follows $g \mathrel{\bar{<}} 1$, so in particular, $g \leq 1$. We conclude $\mathcal{U}(E)_{\mathrm{nc}} \subseteq \mathcal{U}(E)_{\mathrm{infts}}$. □

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