Mathematical Institute
Slovak Academy of Sciences, Bratislava

Preprint 7/2002

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August 6, 2002

PREPRINT SERIES

Between group variance component interval estimation for the unbalanced heteroscedastic one-way random effects model

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Abstract

The studied topic is motivated by the problem of interlaboratory comparisons. This paper is focused on the confidence interval estimation of the between group variance in the unbalanced heteroscedastic one-way random effects model. Several interval estimators have been proposed and compared, by means of the simulation study. Based on the study a very good properties have the estimator based on the generalized p-value method, however, the safest is the confidence interval based on Bonferroni's inequality.

Key words: Unbalanced heteroscedastic one-way random effects model; between group variance; confidence interval; interlaboratory comparisons; consensus mean.

AMS subject classification: 62F25, 62F11.

1 Introduction

The problem of interlaboratory comparisons is of particular interest for many applications in industrial and scientific practice, see e.g. Paule and Mandel (1982).

Consider measurements of the quantity of interest on virtually the same object which are made by k, $k \geq 2$, laboratories. The ith laboratory repeats its measurements n_i times, $n_i \geq 2$. The laboratories may exhibit the between laboratory variability, as well as different within laboratory variances (heteroscedasticity). Assume that the measurements follow normal distribution.

The results of a typical interlaboratory study (given in aggregated form) are presented in Table 1. Eberhardt et al. (1989) reported the data on Selenium in non-fat milk powder. The measurements are based on four independent measurement methods.

Method	n_i	$\bar{y}_i$ (mean)	s_i^2 (variance)
A	8	105.00	85.711
B	12	109.75	20.748
C	14	109.50	2.729
D	8	113.25	33.640

Table 1: Selenium in non-fat milk powder

Questions of fundamental importance in the analysis of such data are

1. how to form the best consensus mean, and what uncertainty to attach to this estimate, see e.g. Rukhin and Vangel (1998), and
2. how to detect and estimate the between laboratory variability.

By the consensus mean we understand a suitable estimate of the true (unknown) value of the quantity of interest. This fundamental questions are followed by a series of other questions regarding the statistical inference on the unknown common mean, e.g. (1) the choice of the appropriate model, (2) the choice of associated statistical methods, (3) the identification and verification of the model, (4) the influence of the between laboratory variability on the estimator of consensus mean, etc.

The problems, are not new in statistical literature, see e.g. Cochran (1937), Graybill and Deal (1959)), Fairweather (1972), Miller (1977), Paule and Mandel (1982), Searle et al. (1992), Rukhin and Vangel (1998), Witkovský (2001a), and Wimmer (2002). However, many questions still remain open.

As a basic tool for analysis of interlaboratory measurements we consider the one-way random effects model, which may be both unbalanced and heteroscedastic, that is

$$Y_{ij} = \mu + a_i + \varepsilon_{ij}. \tag{1}$$

Here μ is the true (unknown) measured value of the quantity of interest, a_i are the random group effects and $\varepsilon_{ij} \sim N(0, \sigma_i^2)$, $i = 1, \ldots, k$, $j = 1, \ldots, n_i$, are mutually independent errors. The result of (i,j)th measurement is y_{ij}; the realization of the random variable Y_{ij}.

Further, we assume that $a_i \sim N(0, \sigma_a^2)$ are mutually independent and independent with all ε_{ij}. The variance components σ_i^2 and σ_a^2 are the nuisance parameters: the within group and between group variances.

In this paper we focus on the interval estimation of the between group variance component in the one-way random effects model which characterizes the between laboratory variability. There are several interval estimators proposed for the between group variance component in the case of homoscedastic errors. However, in the heteroscedactic case such estimators were not known for us. So, in a quite natural way, we have considered these estimators also in the heteroscedastic case and investigated their behaviour under different scenarios.

Current state of the between group variance (σ_a^2) interval estimation in the unbalanced one-way random effects model (1) with equal within group variances (the homoscedastic case), i.e. $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$, is given in the paper by Hartung and Knapp (2000).

In this setup Thomas and Hultquist (1978) showed that

$$\frac{(k-1)MSA}{\sigma_a^2 + \sigma^2/\tilde{n}} \overset{\text{approx.}}{\sim} \chi^2_{k-1}, \tag{2}$$

where

$$MSA = \frac{1}{k-1}\sum_{i=1}^{k}(\bar{Y}_i - \bar{\bar{Y}})^2 \tag{3}$$

with $\bar{Y}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}$, $\bar{\bar{Y}} = \frac{1}{k}\sum_{i=1}^{k}\bar{Y}_i$ and $\tilde{n} = k/\sum_{i=1}^{k}\frac{1}{n_i}$. Adopting the ideas by Tukey (1951) and Williams (1962) thus proposed the following approximate $(1-\alpha)$ confidence interval $\langle L_{TH}, U_{TH}\rangle$ for σ_a^2;

$$\begin{aligned} L_{TH} &= \frac{(k-1)}{\chi^2_{k-1}(1-\alpha/2)}\left(MSA - \frac{F_{k-1,n-k}(1-\alpha/2)}{\tilde{n}}MSE\right), \\ U_{TH} &= \frac{(k-1)}{\chi^2_{k-1}(\alpha/2)}\left(MSA - \frac{F_{k-1,n-k}(\alpha/2)}{\tilde{n}}MSE\right), \end{aligned} \tag{4}$$

with

$$MSE = \frac{1}{n-k}\sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_i)^2 \tag{5}$$

and where $\chi^2_\nu(\gamma)$ denotes the γ-quantile of a χ^2-distribution with ν degrees of freedom, $F_{\nu_1,\nu_2}(\gamma)$ denotes the γ-quantile of a F-distribution with ν_1 and ν_2 degrees of freedom.

Burdick and Eickman (1986) investigated the approximate $(1-\alpha)$ confidence interval $\langle L_{BE}, U_{BE}\rangle$, where

$$\begin{aligned} L_{BE} &= \left(\frac{\tilde{n}l}{1+\tilde{n}l}\right)\frac{(k-1)}{\chi^2_{k-1}(1-\alpha/2)}MSA, \\ U_{BE} &= \left(\frac{\tilde{n}u}{1+\tilde{n}u}\right)\frac{(k-1)}{\chi^2_{k-1}(\alpha/2)}MSA, \end{aligned} \tag{6}$$

with

$$\begin{aligned} l &= \max\left\{0, \frac{1}{F_{k-1,n-k}(1-\alpha/2)}\frac{MSA}{MSE} - \frac{1}{n_{min}}\right\}, \\ u &= \max\left\{0, \frac{1}{F_{k-1,n-k}(\alpha/2)}\frac{MSA}{MSE} - \frac{1}{n_{max}}\right\}, \end{aligned} \tag{7}$$

$n_{min} = \min_{i=1,\ldots,k}\{n_i\}$, $n_{max} = \max_{i=1,\ldots,k}\{n_i\}$.

Let $\langle \eta_L, \eta_U\rangle$ be an exact $(1-\alpha)$ confidence interval for $\eta = \sigma_a^2/\sigma^2$ given in Wald (1940), see also Hartung and Knapp (2000). Hartung and Knapp proposed two approximate $(1-\alpha)$ confidence intervals $\langle L_{HK1}, U_{HK1}\rangle$ and $\langle L_{HK2}, U_{HK2}\rangle$ for σ_a^2 where

$$\begin{aligned} L_{HK1} &= \frac{n-k}{\chi^2_{n-k}(1-\alpha/2)}MSE\eta_L, \\ U_{HK1} &= \frac{n-k}{\chi^2_{n-k}(\alpha/2)}MSE\eta_U, \end{aligned} \tag{8}$$

and

$$\begin{aligned} L_{HK2} &= MSE\eta_L, \\ U_{HK2} &= MSE\eta_U. \end{aligned} \tag{9}$$

Hartung and Knapp (2000) performed a large simulation study where they investigated the properties of the above given approximate confidence intervals for σ_a^2.

2 Between group variance interval estimation under heteroscedasticity

We rewrite the model (1) in matrix notation

$$Y = \mathit{1}\mu + Za + \varepsilon, \tag{10}$$

where $Y = (Y_1', \ldots, Y_k')'$ with $Y_i = (Y_{i1}, \ldots, Y_{in_i})'$ is the n-vector of all measurements, $n = \sum n_i$. Further, $\mathit{1} = (\mathit{1}_1' \ldots, \mathit{1}_k')'$ with $\mathit{1}_i = (1, \ldots, 1)'$ (n_i-vector of ones); $a = (a_1, \ldots, a_k)'$ is the k-vector of group effects; $Z = Diag\{\mathit{1}_i\}$ is known $n \times k$-matrix; and $\varepsilon = (\varepsilon_1', \dots, \varepsilon_k')'$ with $\varepsilon_i = (\varepsilon_{i1}, \dots, \varepsilon_{in_i})'$ represents the random n-vector of within group errors.

Notice that under given assumptions the following statistical properties hold true: $E(Y) = 1\mu$, $Var(Y) = \sigma_a^2 ZZ' + Diag\{\sigma_i^2 I_{n_i}\}$, where I_{n_i} denotes $n_i \times n_i$-identity matrix.

Denoting $R = (Z'Z)^{-1}Z'$, we get

$$RY \sim N\left(1_k\mu, \sigma_a^2 I_k + Diag\{\frac{\sigma_i^2}{n_i}\}\right), \tag{11}$$

or equivalently $RY \sim N(1_k\mu, \sigma_a^2 I_k + \sigma_\varepsilon^2 \Delta)$, where $\sigma_\varepsilon^2 = \sum_{i=1}^k \sigma_i^2$ and $\Delta = Diag\{\frac{\varrho_i}{n_i}\}$ with $\varrho_i = \sigma_i^2 / \sum_{j=1}^k \sigma_j^2$, $i = 1, \dots, k$.

Let

$$H = \begin{pmatrix} \frac{1}{\sqrt{k}} 1_k' \\ H_2 \end{pmatrix} \tag{12}$$

be the k-th order Helmert matrix, see Searle (1971). This matrix is orthogonal and $Y'R'H_2'H_2RY = \sum_{i=1}^k \bar{Y}_i^2 - k\bar{\bar{Y}}^2$. So, $H_2RY \sim N(0_{k-1}, \sigma_a^2 I_{k-1} + \sigma_\varepsilon^2 H_2 \Delta H_2')$. Further, let P be an $(k-1) \times (k-1)$ orthogonal matrix which diagonalizes $H_2 \Delta H_2'$ and $P' = (P_1, P_2, \dots, P_{k-1})$. So,

$$PH_2\Delta H_2' P' = \Lambda = Diag\{\lambda_i\}, \tag{13}$$

and $PH_2RY \sim N(0_{k-1}, \sigma_a^2 I_{k-1} + \sigma_\varepsilon^2 \Lambda)$. The random variables

$$V_i = P_i' H_2 RY \sim N(0, \sigma_a^2 + \lambda_i \sigma_\varepsilon^2), \quad i = 1, \dots, k-1, \tag{14}$$

are independently distributed and hence,

$$W^* = \sum_{i=1}^{k-1} \frac{V_i^2}{\sigma_a^2 + \lambda_i \sigma_\varepsilon^2} \sim \chi_{k-1}^2. \tag{15}$$

We will approximate the random variable W^* by the random variable W, where

$$W = \sum_{i=1}^{k-1} \frac{V_i^2}{\sigma_a^2 + \bar{\lambda}\sigma_\varepsilon^2}, \tag{16}$$

with

$$\bar{\lambda} = \frac{1}{k-1} \sum_{i=1}^{k-1} \lambda_i = \frac{1}{k-1} tr(H_2 \Delta H_2') = \frac{1}{k} \frac{\sum_{i=1}^k \sigma_i^2 / n_i}{\sum_{i=1}^k \sigma_i^2}. \tag{17}$$

Similarly as in Thomas and Hultquist (1978), we suggest that the random variable W has approximately χ_{k-1}^2 distribution and can be used in constructing confidence intervals for σ_a^2. More details on suitability of this approximation could be find in the Appendix.

Since P is orthogonal, $\sum_{i=1}^{k-1} V_i^2 = (k-1)MSA$, and hence,

$$W = \frac{(k-1)MSA}{\sigma_a^2 + \frac{1}{k}\sum_{j=1}^k \frac{\sigma_j^2}{n_j}} \overset{\text{approx.}}{\sim} \chi_{k-1}^2. \tag{18}$$

2.1 Construction of the interval estimator based on the Bonferroni's inequality

From (18) we obtain

$$P\left\{\frac{(k-1)MSA}{\chi^2_{k-1}(1-\frac{\alpha}{2})} - \frac{1}{k}\sum_{j=1}^{k}\frac{\sigma_j^2}{n_j} < \sigma_a^2 < \frac{(k-1)MSA}{\chi^2_{k-1}(\frac{\alpha}{2})} - \frac{1}{k}\sum_{j=1}^{k}\frac{\sigma_j^2}{n_j}\right\} = 1-\frac{\alpha}{2}. \tag{19}$$

Of course,

$$P\left\{\frac{(n_j-1)S_j^2}{n_j\chi^2_{n_j-1}(1-\frac{\beta}{2})} < \frac{\sigma_j^2}{n_j} < \frac{(n_j-1)S_j^2}{n_j\chi^2_{n_j-1}(\frac{\beta}{2})}\right\} = 1-\beta, \tag{20}$$

$j=1,\ldots,k$, where $S_j^2 = \frac{1}{n_j-1}\sum_{i=1}^{n_j}(Y_{ji}-\bar{Y}_j)^2$, $\beta = 1-\sqrt[k]{1-\frac{\alpha}{2}}$, and so

$$P\left\{\frac{1}{k}\sum_{j=1}^{k}\frac{(n_j-1)S_j^2}{n_j\chi^2_{n_j-1}(1-\frac{\beta}{2})} < \frac{1}{k}\sum_{j=1}^{k}\frac{\sigma_j^2}{n_j} < \frac{1}{k}\sum_{j=1}^{k}\frac{(n_j-1)S_j^2}{n_j\chi^2_{n_j-1}(\frac{\beta}{2})}\right\} = 1-\frac{\alpha}{2}. \tag{21}$$

Using Bonferroni's inequality we obtain the desired confidence interval with confidence coefficient at least $(1-\alpha)$ given as $\langle L_{BO}, U_{BO}\rangle$, where $L_{BO} = \max\{0, l_{BO}\}$ and $U_{BO} = \max\{0, u_{BO}\}$, with

$$\begin{aligned} l_{BO} &= \frac{(k-1)MSA}{\chi^2_{k-1}(1-\frac{\alpha}{2})} - \frac{1}{k}\sum_{j=1}^{k}\frac{n_j-1}{n_j}\frac{S_j^2}{\chi^2_{n_j-1}(\frac{\beta}{2})} \\ u_{BO} &= \frac{(k-1)MSA}{\chi^2_{k-1}(\frac{\alpha}{2})} - \frac{1}{k}\sum_{j=1}^{k}\frac{n_j-1}{n_j}\frac{S_j^2}{\chi^2_{n_j-1}(1-\frac{\beta}{2})}. \end{aligned} \tag{22}$$

2.2 Construction of the interval estimator based on the generalized p-values

Based on the results given in Tsui and Weerahandi (1989), see also Weerahandi (1995) and Wimmer (2002), it is possible to construct the generalized $(1-\alpha)$ confidence interval for σ_a^2 based on the generalized p-values.

From (11), with $V = \sigma_a^2 I_k + Diag\{\frac{\sigma_i^2}{n_i}\}$, we get

$$\begin{aligned} SS &= Y'R'(V^{-1} - V^{-1}1(1'V^{-1}1)^{-1}1'V^{-1})RY \\ &= \sum_{i=1}^{k}\frac{1}{\sigma_a^2+\sigma_i^2/n_i}\left(\bar{Y}_i - \frac{\sum_{j=1}^{k}\frac{\bar{Y}_j}{\sigma_a^2+\sigma_j^2/n_j}}{\sum_{j=1}^{k}\frac{1}{\sigma_a^2+\sigma_j^2/n_j}}\right)^2 \sim \chi^2_{k-1}, \end{aligned} \tag{23}$$

and

$$\frac{n_i-1}{\sigma_i^2}S_i^2 \sim \chi^2_{n_i-1}, \quad i=1,\ldots,k. \tag{24}$$

Moreover, these random variables are mutually independent. To construct the generalized confidence interval estimate for σ_a^2 we consider the following potential test variable

$$T = \frac{Y'R'(V^{-1} - V^{-1}1(1'V^{-1}1)^{-1}1'V^{-1})RY}{y'R'(W^{-1} - W^{-1}1(1'W^{-1}1)^{-1}1'W^{-1})Ry}, \tag{25}$$

where W denotes a matrix with random elements,

$$W = \sigma_a^2 I_k + Diag\left\{\frac{\sigma_i^2}{n_i - 1}\frac{n_i - 1}{n_i}\frac{s_i^2}{S_i^2}\right\} \sim \sigma_a^2 I_k + Diag\left\{\frac{n_i - 1}{n_i}\frac{s_i^2}{\chi^2_{n_i - 1}}\right\}. \tag{26}$$

Note that the potential test variable fulfills the required properties, and the observed value $T_{obs} = 1$. Given σ_a^2, the generalized p-value is

$$p = P(T > T_{obs}|\sigma_a^2) = P(\chi^2_{k-1} > q(\sigma_a^2, \chi^2_{n_1-1}, \ldots, \chi^2_{n_k-1})), \tag{27}$$

where $q(\sigma_a^2, \chi^2_{n_1-1}, \ldots, \chi^2_{n_k-1}) = y'R'(W^{-1} - W^{-1}1(1'W^{-1}1)^{-1}1'W^{-1})Ry$ denotes random variable which depends on σ_a^2 and on the independent random variables with $\chi^2_{n_i-1}$ distribution, $i = 1, \ldots, k$.

The generalized $(1 - \alpha)$ confidence interval is then given as $\langle L_{PV}, U_{PV}\rangle$, where L_{PV}, U_{PV} are solutions of the following equations

$$\begin{aligned} \frac{\alpha}{2} &= P(\chi^2_{k-1} > q(L_{PV}, \chi^2_{n_1-1}, \ldots, \chi^2_{n_k-1})), \\ 1 - \frac{\alpha}{2} &= P(\chi^2_{k-1} > q(U_{PV}, \chi^2_{n_1-1}, \ldots, \chi^2_{n_k-1})). \end{aligned} \tag{28}$$

The evaluation requires k-dimensional numerical integration. However, in practical applications it is easy to evaluate the p-values approximately by Monte Carlo simulations. For that, one needs to generate the realizations of independent random variables $\chi^2_{k-1}, \chi^2_{n_1-1}, \ldots, \chi^2_{n_k-1}$ and count the relative frequencies of the cases in which $\chi^2_{k-1} > q(\sigma_a^2, \chi^2_{n_1-1}, \ldots, \chi^2_{n_k-1})$ for any chosen value σ_a^2.

An approximate $(1 - \alpha)$ confidence interval (based on the modified generalized p-values) for σ_a^2 inspired by the above mentioned method and (18) is $\langle L_{MPV}, U_{MPV}\rangle$. Here $L_{MPV} = \max\{0, l_{MPV}\}$, $U_{MPV} = \max\{0, u_{MPV}\}$, and l_{MPV} is the $\frac{\alpha}{2}$ quantile and u_{MPV} is the $(1 - \frac{\alpha}{2})$ quantile of the distribution

$$\left(\sum_{i=1}^{k}(\bar{y}_i - \bar{\bar{y}})^2\right)\frac{1}{\chi^2_{k-1}} - \sum_{j=1}^{k}\left(\frac{(n_j - 1)s_j^2}{kn_j}\right)\frac{1}{\chi^2_{n_j-1}}. \tag{29}$$

χ^2_{k-1} and $\chi^2_{n_j-1}$ are mutually independent chi-square distributed random variables with with $k-1$ and n_j-1 degrees of freedom, $j = 1, \ldots, k$. For the method of finding the quantiles of the linear combination of independent inverted chi-squared random variables see Witkovský (2001b).

2.3 Construction of the interval estimator based on plug-in estimates

It is easy to see that

$$\sum_{i=1}^{k} \frac{1}{\sigma_a^2 + \sigma_i^2/n_i} \left(\bar{Y}_i - \frac{\sum_{j=1}^{k} \frac{\bar{Y}_j}{\sigma_a^2+\sigma_j^2/n_j}}{\sum_{j=1}^{k} \frac{1}{\sigma_a^2+\sigma_j^2/n_j}} \right)^2 \sim \chi^2_{k-1}. \tag{30}$$

Replacing σ_i^2 by its estimate s_i^2, $i = 1, \ldots, k$, we obtain

$$\sum_{i=1}^{k} \frac{1}{\sigma_a^2 + s_i^2/n_i} \left(\bar{Y}_i - \frac{\sum_{j=1}^{k} \frac{\bar{Y}_j}{\sigma_a^2+s_j^2/n_j}}{\sum_{j=1}^{k} \frac{1}{\sigma_a^2+s_j/n_j}} \right)^2 \overset{\text{approx.}}{\sim} \chi^2_{k-1}, \tag{31}$$

what yields another approximate $(1-\alpha)$ confidence interval for σ_a^2, $\langle L_{PL}, U_{PL} \rangle$, with L_{PL} and U_{PL} being the solutions of the following equations

$$\begin{aligned}
\sum_{i=1}^{k} \frac{1}{L_{PL} + s_i^2/n_i} \left(\bar{Y}_i - \frac{\sum_{j=1}^{k} \frac{\bar{Y}_j}{L_{PL}+s_j^2/n_j}}{\sum_{j=1}^{k} \frac{1}{L_{PL}+s_j/n_j}} \right)^2 &= \chi^2_{k-1}(1-\frac{\alpha}{2}) \\
\sum_{i=1}^{k} \frac{1}{U_{PL} + s_i^2/n_i} \left(\bar{Y}_i - \frac{\sum_{j=1}^{k} \frac{\bar{Y}_j}{U_{PL}+s_j^2/n_j}}{\sum_{j=1}^{k} \frac{1}{U_{PL}+s_j/n_j}} \right)^2 &= \chi^2_{k-1}(\frac{\alpha}{2}).
\end{aligned} \tag{32}$$

2.4 Example

For illustration purposes, using the data from Table 1, we have calculated the 95% interval estimates for the between laboratory variance σ_a^2 by the methods mentioned in this paper, see Table 2.

Estimator	Equation	Lower bound	Upper bound
TH	(4)	0.3990	156.1074
BE	(6)	0.0000	156.1228
BO	(22)	0.0000	157.5982
PL	(32)	0.0000	151.2071
HK1	(8)	0.0000	260.7897
HK2	(9)	0.0000	157.0124
MPV	(29)	0.0000	153.4232
PV	(28)	0.0000	148.4149

Table 2: The 95% intervale estimates for the between laboratory variance σ_a^2.

3 The simulation study

An extensive simulation study was made in order to compare the confidence coefficients and the expected lengths of the five different confidence intervals for σ_a^2. We considered two-sided confidence intervals with $\alpha = 0.05$. All estimated confidence coefficients are based on 10 000 replications.

The simulated measurements come from different designs (e.g. with different number of groups k and different number of within group measurements $n_1, \ldots, n_k$) that were (as mentioned in Introduction) motivated by the interlaboratory comparisons. The designs are described by a triple A.B.C., where

$$\mathrm{A} = \begin{cases} 1 & \text{if } k = 5 \quad \text{(small number of laboratories)} \\ 2 & \text{if } k = 10 \quad \text{(moderate number of laboratories)} \\ 3 & \text{if } k = 20 \quad \text{(large number of laboratories)} \end{cases}$$

$$\mathrm{B} = \begin{cases} a & \text{if } n_1 = n_2 = \cdots = n_k \\ b & \text{if } n_1 \leq n_2 < \cdots < n_k \\ c & \text{if } n_1 > \cdots > n_{k-1} \geq n_k \end{cases}$$

$$\mathrm{C} = \begin{cases} \alpha & \text{if } \sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 \\ \beta & \text{if } \sigma_1^2 < \sigma_2^2 < \cdots < \sigma_k^2. \end{cases}$$

If the letters a, b, c, α, or β appear with the indices then the greater index means in general the greater values of $n_1, \ldots, n_k$ or $\sigma_1^2, \ldots, \sigma_k^2$ in the given subclass of designs with the same number of laboratories.

The complete results of simulations are presented in the Appendix B, where

- for example, $n_i = [2, 4, 6, 8, 10]$ means $k = 5$, $n_1 = 2$, $n_2 = 4$, $n_3 = 6$, $n_4 = 8$ and $n_5 = 10$,
- for example, $\sigma_i^2 = [0.1, 0.5, 1, 1.5, 2]$ means $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.5$, $\sigma_3^2 = 1$, $\sigma_4^2 = 1.5$ and $\sigma_5^2 = 2$,
- the columns are results of simulations with $\sigma_a^2 = 0$, $\sigma_a^2 = 0.1$, $\sigma_a^2 = 0.5$, $\sigma_a^2 = 1$, $\sigma_a^2 = 5$ and $\sigma_a^2 = 10$,
- BE means the results (the estimated confidence coefficient and the estimated expected length) for the Burdick and Eickman confidence interval (6),
- BO means the results for confidence interval based on the Bonferroni's inequality (22),
- PL means the results for the confidence interval based on using the plug-in estimates (32),

- HK means the results for the Hartung and Knapp confidence interval (9),
- MPV means the results for the confidence interval based on using the modified generalized p-value method (29).
- PV means the results for the confidence interval based on using the generalized p-value method (28).

In each table, the upper section gives the estimated confidence coefficients (the relative frequency of covering the true value σ_a^2 by the realized 0.95-confidence intervals calculated by different methods). The lower section gives the estimated expected length (the averaged lengths of realized intervals) of the confidence intervals.

The interval lengths for the MPV and PV methods were not calculated as it was very time consuming. (Some calculations based on small number of repetitions have been done and they didn't show any significant departure from the results of the other methods.)

3.1 Evaluation of the simulation results

– The results for $\sigma_a^2 = 0$ may seem strange. This is caused by the fact that the methods are restricted to result in the natural parameter space (i.e. $\sigma_a^2 \geq 0$) and in the case of negative unrestricted upper bound of the interval estimator the resulted value was restricted to be zero. So, when the probability of negative upper bound is greater than zero (BE, BO, and MPV estimators), all such results are concentrated to zero and so increase its coverage.

– The methods BE, PL, HK, and MPV fail in the unbalanced heteroscedastic case where n_i's are decreasing and σ_i^2's are growing (the designs $1.c.\beta_1.$, $1.c.\beta_2.$, $1.c.\beta_3.$, $2.c.\beta_1.$, $2.c.\beta_2.$, $2.c.\beta_3.$, $3.c.\beta_1.$, $3.c.\beta_2.$, $3.c.\beta_3.$).

– The PL method is not recommended in the case of small number of within laboratory measurements and/or if σ_a^2 is close to zero. This indicates that the approximation (31) is not suitable in such cases. Further research is needed to get a correction term to improve it.

– The performance of BE and the HK methods is not good especially in the case when the number of laboratories is moderate or large and σ_a^2 is small.

– There are some designs where the estimated confidence coefficient of the interval obtained by the MPV method differ from the desired confidence coefficient for several values of σ_a^2 (designs $1.c.\beta_3.$, $2.a_1.\beta_2.$, $2.a_1.\beta_3.$, $2.a_2.\beta_3.$, $2.b_2.\beta_3.$, $3.a_2.\beta_3.$, $3.c.\alpha.$, $3.a_1.\beta_1.$, $3.a_1.\beta_2.$, $3.b.\beta_3.$).

– The PV method behaves in general very well, however, when the number of laboratories is moderate or large and σ_a^2 is close to zero the coverage probabilities tends to be less than the nominal level.

If we accept the point of view of the metrologist who requires, in each situation, the confidence interval with the confidence coefficient on the level at least 0.95, we unambigously are led to the confidence interval given by the BO method (22), although, in some cases, it is somewhat larger than the intervals based on the other methods (e.g. designs $2.b.\beta_1.$, $2.b.\beta_2.$, $2.b.\beta_3.$, $3.b.\beta_1.$, $3.b.\beta_2.$, $3.b.\beta_3.$).

4 Appendix A. Approximation of the distribution of W

Denoting

$$\gamma_j = \frac{\sigma_a + \lambda_j \sum_{i=1}^k \sigma_i^2}{\sigma_a + \frac{1}{k} \sum_{s=1}^k \frac{\sigma_s^2}{n_s}}, \tag{33}$$

we obtain from (14) and (16)

$$W = \sum_{j=1}^{k-1} \gamma_j \frac{V_j}{(\sigma_a^2 + \lambda_j \sum_{s=1}^k \sigma_s^2)} \sim \sum_{j=1}^{k-1} \gamma_j \left({}^{(j)}\chi_1^2 \right), \tag{34}$$

where ${}^{(j)}\chi_1^2$, $j = 1, \ldots, k-1$ are independent random variables with characteristic functions $\Phi_{\,{}^{(j)}\chi_1^2}(t) = (1 - 2i\gamma_j t)^{-\frac{1}{2}}$. Therefore, the characteristic function of W is

$$\Phi_W(t) = \prod_{j=1}^{k-1} (1 - 2i\gamma_j t)^{-\frac{1}{2}}. \tag{35}$$

According to Imhof (1961) the distribution function of W is

$$F_W(x) = \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{\sin\theta(u)}{u\omega(u)} du, \tag{36}$$

where

$$\begin{aligned} \theta(u) &= -\frac{1}{2}xu + \frac{1}{2} \sum_{j=1}^{k-1} \arctan(\gamma_j u), \\ \omega(u) &= \prod_{j=1}^{k-1} (1 + \gamma_j^2 u^2)^{\frac{1}{4}} \end{aligned} \tag{37}$$

are real-valued functions.

Theorem 1 *The Taylor series expansion of the characteristic function* $\Phi_W(\cdot)$ *about the values* $\gamma_j^{(0)} = 1, j = 1, 2, \ldots, k-1$, *agrees with that of the characteristic function* $\Phi_W^*(\cdot)$ *of* W^* *up to terms of the first order.*

Proof. The Taylor series expansion of $\Phi_W(\cdot)$ about the values $\gamma_j^{(0)} = 1$, $j = 1, \ldots, k-1$ is

$$\Phi_W(\cdot) = (1-2it)^{-\frac{k-1}{2}} + it(1-2it)^{-\frac{k+1}{2}} \sum_{j=1}^{k-1} (\gamma_j - 1)$$
$$-t^2(1-2it)^{-\frac{k+3}{2}} \left\{ \sum_{j=1}^{k-1} (\gamma_j - 1)^2 + \sum_{j=1}^{k-1} (\gamma_j - 1) \sum_{s=1}^{k-1} (\gamma_s - 1) \right\} + R^* \quad (38)$$

(R^* being the reminder term). Using (17) it is easy to see that $\sum_{j=1}^{k-1}(\gamma_j - 1) = 0$. So,

$$\Phi_W(\cdot) = (1-2it)^{-\frac{k-1}{2}} - t^2(1-2it)^{-\frac{k+3}{2}} \sum_{j=1}^{k-1} (\gamma_j - 1)^2 + R^*. \quad (39)$$

□

Theorem 2 *The Taylor series expansion of the distribution function $F_W(\cdot)$ about the values $\gamma_j^{(0)} = 1$, $j = 1, \ldots, k-1$, agrees with that of the distribution function $F^*_W(\cdot)$ of W^* up to terms of the first order.*

Proof. The Taylor series expansion of $\frac{\sin\theta(u)}{u\omega(u)}$ about the values $\gamma_j^{(0)} = 1$, $j = 1, \ldots, k-1$ is

$$\begin{aligned}
\frac{\sin\theta(u)}{u\omega(u)} &= \frac{\sin\left(-\frac{1}{2}xu + \frac{k-1}{2}\arctan u\right)}{(1+u^2)^{\frac{k-1}{2}}} \\
&+ \left\{ \frac{u\cos\left(-\frac{1}{2}xu + \frac{k-1}{2}\arctan u\right)}{2(1+u^2)^{\frac{k+3}{4}}} \right. \\
&- \left. \frac{u^2\sin\left(-\frac{1}{2}xu + \frac{k-1}{2}\arctan u\right)}{2(1+u^2)^{\frac{k+3}{4}}} \right\} \sum_{j=1}^{k-1} (\gamma_j - 1) \\
&- \frac{u(1-u^2)\sin\left(-\frac{1}{2}xu + \frac{k-1}{2}\arctan u\right)}{4(1+u^2)^{\frac{k+7}{4}}} \sum_{j=1}^{k-1} (\gamma_j - 1)^2 \\
&- \frac{u^3\cos\left(-\frac{1}{2}xu + \frac{k-1}{2}\arctan u\right)}{2(1+u^2)^{\frac{k+7}{4}}} \sum_{j=1}^{k-1} (\gamma_j - 1)^2 + R^{**} \quad (40)
\end{aligned}$$

(here we have also utilized that $\sum_{j=1}^{k-1}(\gamma_j - 1) = 0$).

So we can write

$$F_W(x) =$$
$$F_{W^*}(x) + \frac{1}{\pi} \left\{ \int_0^\infty \left[\sum_{j=1}^{k-1} (\gamma_j - 1)^2 \left(\frac{u(1-u^2)\sin\left(-\frac{1}{2}xu + \frac{k-1}{2}\arctan u\right)}{4(1+u^2)^{\frac{k+7}{4}}} \right.\right.\right.$$

$$+ \quad \frac{u^3 \cos\left(-\frac{1}{2}xu + \frac{k-1}{2}\arctan u\right)}{2(1+u^2)^{\frac{k+7}{4}}}\Bigg) - R^{**}\Bigg] du \Bigg\}. \tag{41}$$

$\square$

Remark 1 *Relations (39) and (41) provide a good base for numerical analysis of the difference between the true and approximative distribution of* W*.*

5 Appendix B. The simulation results

Design 1.a_1.β_1. $n_i = [5,5,5,5,5]$, $\sigma_i^2 = [0.1, 0.5, 1, 1.5, 2]$						
BE	0.9594	0.9470	0.9569	0.9547	0.9505	0.9526
BO	1.0000	0.9935	0.9860	0.9818	0.9748	0.9732
PL	0.8473	0.9185	0.9497	0.9490	0.9500	0.9513
HK	0.9594	0.9550	0.9693	0.9710	0.9686	0.9699
MPV	0.9975	0.9536	0.9628	0.9595	0.9532	0.9546
PV	0.9541	0.9569	0.9601	0.9584	0.9529	0.9537
BE	1.4964	2.3175	5.5481	9.6294	41.0186	80.4729
BO	1.6440	2.4721	5.7711	9.9978	42.7142	83.4347
PL	1.4582	2.2814	5.4872	9.5556	40.9449	80.4039
HK	1.5519	2.4044	5.7698	10.0364	42.9144	84.2513

Design 1.a_1.β_2. $n_i = [5,5,5,5,5]$, $\sigma_i^2 = [1, 2, 5, 7, 10]$						
BE	0.9585	0.9323	0.9455	0.9487	0.9499	0.9502
BO	1.0000	0.9968	0.9919	0.9876	0.9800	0.9754
PL	0.8506	0.8818	0.9294	0.9370	0.9454	0.9467
HK	0.9585	0.9356	0.9546	0.9589	0.9689	0.9678
MPV	0.9968	0.9455	0.9503	0.9544	0.9584	0.9543
PV	0.9606	0.9486	0.9521	0.9546	0.9533	0.9529
BE	7.3373	8.1260	11.1808	15.4304	47.7580	86.7545
BO	8.0563	8.8462	11.9320	16.2390	49.5846	90.2277
PL	7.1886	7.9763	11.0267	15.2308	47.3988	86.3785
HK	7.6094	8.4272	11.5994	16.0183	49.7774	90.6070

Design 1.a_1.β_3. $n_i = [5,5,5,5,5]$, $\sigma_i^2 = [1, 5, 10, 15, 20]$						
BE	0.9576	0.9229	0.9378	0.9417	0.9536	0.9510
BO	1.0000	0.9968	0.9956	0.9925	0.9846	0.9795
PL	0.8449	0.8634	0.9096	0.9250	0.9443	0.9476
HK	0.9576	0.9247	0.9439	0.9504	0.9662	0.9676
MPV	0.9976	0.9402	0.9489	0.9514	0.9596	0.9596
PV	0.9597	0.9433	0.9529	0.9543	0.9551	0.9543
BE	14.9321	15.7139	18.8458	23.2473	55.7569	95.6319
BO	16.3995	17.1816	20.3337	24.7829	57.9998	99.2837
PL	14.6223	15.4158	18.5328	22.9129	55.1625	94.8852
HK	15.4853	16.2970	19.5474	24.1195	57.9904	99.6699

Design $1.a_2.\beta_1$. $n_i = [15,15,15,15,15]$, $\sigma_i^2 = [0.1,0.5,1,1.5,2]$						
BE	0.9598	0.9452	0.9484	0.9537	0.9516	0.9487
BO	0.9999	0.9826	0.9733	0.9681	0.9557	0.9512
PL	0.9438	0.9433	0.9483	0.9529	0.9520	0.9489
HK	0.9598	0.9501	0.9558	0.9607	0.9588	0.9566
MPV	0.9769	0.9489	0.9491	0.9543	0.9519	0.9486
PV	0.9690	0.9508	0.9508	0.9543	0.9524	0.9489
BE	0.4962	1.3084	4.4588	8.5194	39.5666	80.3057
BO	0.5348	1.3600	4.5953	8.7070	39.7945	80.5355
PL	0.4834	1.2981	4.4470	8.5094	39.5599	80.2997
HK	0.5016	1.3234	4.5166	8.6342	40.1188	81.4322

Design $1.a_2.\beta_2$. $n_i = [15,15,15,15,15]$, $\sigma_i^2 = [1,2,5,7,10]$						
BE	0.9616	0.9372	0.9490	0.9485	0.9503	0.9514
BO	1.0000	0.9903	0.9842	0.9785	0.9669	0.9603
PL	0.9481	0.9418	0.9484	0.9474	0.9496	0.9507
HK	0.9614	0.9394	0.9538	0.9527	0.9578	0.9577
MPV	0.9793	0.9447	0.9507	0.9483	0.9510	0.9516
PV	0.9722	0.9523	0.9527	0.9506	0.9513	0.9507
BE	2.3988	3.2542	6.4023	10.4116	42.2324	81.4150
BO	2.5875	3.4518	6.6549	10.7787	43.1489	82.4726
PL	2.3406	3.2075	6.3542	10.3564	42.1839	81.3747
HK	2.4247	3.2898	6.4756	10.5374	42.8017	82.5361

Design $1.a_2.\beta_3$. $n_i = [15,15,15,15,15]$, $\sigma_i^2 = [1,5,10,15,20]$						
BE	0.9586	0.9298	0.9432	0.9488	0.9542	0.9478
BO	0.9999	0.9910	0.9862	0.9838	0.9751	0.9649
PL	0.9423	0.9401	0.9449	0.9487	0.9541	0.9477
HK	0.9586	0.9308	0.9474	0.9524	0.9595	0.9569
MPV	0.9744	0.9361	0.9477	0.9509	0.9544	0.9478
PV	0.9704	0.9507	0.9480	0.9523	0.9556	0.9487
BE	4.9336	5.7255	8.9733	13.1027	44.5276	83.6570
BO	5.3199	6.1199	9.4061	13.6190	45.8969	85.5091
PL	4.8190	5.6280	8.8816	13.0024	44.4111	83.5566
HK	4.9872	5.7878	9.0731	13.2529	45.1047	84.7832

Design $1.b.\beta_1$. $n_i = [2,4,6,8,10]$, $\sigma_i^2 = [0.1, 0.5, 1, 1.5, 2]$						
BE	1.0000	0.9620	0.9640	0.9631	0.9527	0.9513
BO	1.0000	0.9910	0.9827	0.9781	0.9767	0.9737
PL	0.8578	0.9293	0.9485	0.9505	0.9495	0.9501
HK	0.9903	0.9430	0.9601	0.9642	0.9677	0.9654
MPV	0.9999	0.9549	0.9636	0.9634	0.9646	0.9594
PV	0.9707	0.9539	0.9550	0.9606	0.9553	0.9522
BE	0.9631	1.7571	5.0331	9.1099	40.7182	79.4375
BO	1.1706	1.9848	5.2900	9.4137	42.2877	82.7622
PL	1.0609	1.8628	5.0624	9.0219	40.4914	79.2586
HK	1.0009	1.8065	5.1440	9.2898	42.0315	82.3019

Design $1.b.\beta_2$. $n_i = [2,4,6,8,10]$, $\sigma_i^2 = [1, 2, 5, 7, 10]$						
BE	1.0000	0.9719	0.9670	0.9652	0.9630	0.9601
BO	1.0000	0.9965	0.9924	0.9852	0.9790	0.9785
PL	0.8342	0.8851	0.9328	0.9435	0.9480	0.9514
HK	0.9896	0.9364	0.9463	0.9532	0.9640	0.9679
MPV	0.9997	0.9499	0.9573	0.9576	0.9645	0.9660
PV	0.9702	0.9532	0.9545	0.9556	0.9581	0.9561
BE	4.9434	5.6893	9.0461	12.8696	45.7151	86.1668
BO	5.9659	6.7387	10.1658	14.0375	47.2255	88.8895
PL	5.4351	6.1921	9.5453	13.3157	45.2542	85.1611
HK	5.0633	5.8290	9.2657	13.1870	46.6246	88.2158

Design $1.b.\beta_3$. $n_i = [2,4,6,8,10]$, $\sigma_i^2 = [1, 5, 10, 15, 20]$						
BE	1.0000	0.9676	0.9653	0.9645	0.9654	0.9645
BO	1.0000	0.9960	0.9927	0.9926	0.9827	0.9796
PL	0.8511	0.8845	0.9221	0.9401	0.9472	0.9513
HK	0.9899	0.9234	0.9354	0.9447	0.9624	0.9654
MPV	0.9996	0.9457	0.9531	0.9561	0.9633	0.9657
PV	0.9696	0.9522	0.9564	0.9599	0.9598	0.9540
BE	9.6279	10.3214	13.7098	17.2657	50.4989	90.1990
BO	11.7080	12.4375	15.9011	19.5451	53.0538	93.2593
PL	10.6445	11.3643	14.7712	18.3405	50.7722	89.3775
HK	9.9823	10.6959	14.1323	17.7483	51.6140	92.0043

Design $1.c.\alpha$. $n_i = [10, 8, 6, 4, 2]$, $\sigma_i^2 = [1, 1, 1, 1, 1]$						
BE	0.9986	0.9812	0.9660	0.9595	0.9512	0.9493
BO	1.0000	0.9949	0.9857	0.9804	0.9744	0.9773
PL	0.7638	0.8841	0.9382	0.9457	0.9490	0.9490
HK	0.9736	0.9638	0.9666	0.9667	0.9657	0.9682
MPV	0.9987	0.9445	0.9570	0.9604	0.9646	0.9683
PV	0.9544	0.9520	0.9584	0.9555	0.9538	0.9546
BE	1.7015	2.5111	5.7817	9.7959	40.9104	81.3089
BO	1.8524	2.6704	5.9716	10.0776	42.5733	84.8312
PL	1.6634	2.4632	5.6651	9.6122	40.7383	81.1806
HK	1.6962	2.5405	5.8893	10.0010	42.3253	84.3225

Design $1.c.\beta_1$. $n_i = [10, 8, 6, 4, 2]$, $\sigma_i^2 = [0.1, 0.5, 1, 1.5, 2]$						
BE	0.9400	0.9274	0.9350	0.9390	0.9486	0.9513
BO	1.0000	0.9947	0.9884	0.9803	0.9776	0.9774
PL	0.7132	0.8469	0.9293	0.9380	0.9484	0.9515
HK	0.8393	0.8960	0.9537	0.9590	0.9662	0.9653
MPV	0.9978	0.9343	0.9488	0.9579	0.9649	0.9672
PV	0.9348	0.9518	0.9581	0.9567	0.9572	0.9574
BE	2.4968	3.3457	6.4943	10.4418	42.2561	81.6571
BO	2.5866	3.4469	6.6694	10.7642	44.0459	85.2501
PL	2.2459	3.0971	6.2348	10.1805	42.0909	81.5343
HK	2.5091	3.3938	6.6367	10.7201	43.8008	84.7407

Design $1.c.\beta_2$. $n_i = [10, 8, 6, 4, 2]$, $\sigma_i^2 = [1, 2, 5, 7, 10]$						
BE	0.9425	0.9358	0.9262	0.9304	0.9372	0.9468
BO	1.0000	0.9973	0.9925	0.9918	0.9800	0.9769
PL	0.7161	0.7694	0.8429	0.8969	0.9378	0.9477
HK	0.8477	0.8557	0.8967	0.9292	0.9559	0.9654
MPV	0.9972	0.9237	0.9371	0.9406	0.9548	0.9624
PV	0.9424	0.9384	0.9478	0.9494	0.9551	0.9542
BE	12.3901	13.2154	16.5384	20.3401	51.9776	91.6781
BO	12.8424	13.6794	17.0434	20.9094	53.5882	95.0755
PL	11.2264	12.0651	15.3710	19.1215	50.6646	90.5483
HK	12.4501	13.3231	16.7767	20.6993	53.3651	94.6313

Design $1.c.\beta_3$. $n_i = [10, 8, 6, 4, 2]$, $\sigma_i^2 = [1, 5, 10, 15, 20]$						
BE	0.9421	0.9317	0.9243	0.9234	0.9314	0.9445
BO	1.0000	0.9971	0.9963	0.9947	0.9872	0.9853
PL	0.7046	0.7342	0.8067	0.8468	0.9222	0.9455
HK	0.8355	0.8378	0.8683	0.8902	0.9496	0.9652
MPV	0.9976	0.9138	0.9227	0.9313	0.9523	0.9589
PV	0.9330	0.9279	0.9400	0.9476	0.9541	0.9553
BE	25.0820	26.1531	29.7500	33.5305	65.4652	105.5852
BO	25.9786	27.0685	30.7259	34.5531	67.2372	108.8482
PL	22.7441	23.8210	27.4321	31.1996	62.8966	102.9365
HK	25.2274	26.3370	30.1064	34.0226	66.9169	108.4133

Design $2.a_1.\beta_1$. $n_i = [5, 5, 5, 5, 5, 5, 5, 5, 5, 5]$ $\sigma_i^2 = [0.001, 0.01, 0.05, 0.1, 0.5, 1, 1.5, 2, 2.5, 3]$						
BE	0.9303	0.9148	0.9502	0.9491	0.9515	0.9497
BO	1.0000	0.9970	0.9894	0.9840	0.9775	0.9751
PL	0.7842	0.9024	0.9433	0.9427	0.9501	0.9497
HK	0.9303	0.9256	0.9660	0.9667	0.9710	0.9690
MPV	0.9993	0.9172	0.9491	0.9537	0.9540	0.9502
PV	0.8954	0.9487	0.9536	0.9525	0.9509	0.9497
BE	0.5131	0.8388	2.0502	3.5231	14.9246	29.1708
BO	0.6665	1.0064	2.3261	4.0158	17.2864	33.6133
PL	0.4427	0.7648	1.9465	3.4207	14.8570	29.1135
HK	0.5404	0.8847	2.1792	3.7697	16.1177	31.5492

Design $2.a_1.\beta_2$. $n_i = [5, 5, 5, 5, 5, 5, 5, 5, 5, 5]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 2, 5, 7, 10, 15, 20]$						
BE	0.9209	0.8579	0.8998	0.9259	0.9450	0.9474
BO	1.0000	0.9970	0.9955	0.9941	0.9844	0.9793
PL	0.7640	0.8402	0.9010	0.9272	0.9420	0.9465
HK	0.9209	0.8604	0.9099	0.9401	0.9627	0.9677
MPV	0.9976	0.8650	0.9073	0.9285	0.9524	0.9530
PV	0.8951	0.9288	0.9453	0.9483	0.9494	0.9529
BE	2.8273	3.1094	4.4987	6.0542	18.0201	32.1910
BO	3.6921	3.9953	5.4377	7.0751	20.5147	36.9683
PL	2.6141	2.8649	4.1236	5.5925	17.4134	31.6565
HK	2.9791	3.2764	4.7453	6.3955	19.2602	34.5987

Design $2.a_1.\beta_3$. $n_i = [5,5,5,5,5,5,5,5,5,5]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 5, 10, 15, 20, 30, 40]$						
BE	0.9269	0.8531	0.8776	0.9059	0.9440	0.9463
BO	1.0000	0.9984	0.9972	0.9961	0.9897	0.9836
PL	0.7617	0.8130	0.8794	0.9085	0.9418	0.9433
HK	0.9269	0.8542	0.8844	0.9143	0.9592	0.9651
MPV	0.9977	0.8619	0.8871	0.9084	0.9510	0.9509
PV	0.8957	0.9188	0.9366	0.9437	0.9566	0.9553
BE	5.7701	6.1412	7.3882	8.9273	21.3912	35.8693
BO	7.5147	7.9099	9.2185	10.8159	24.3170	40.8283
PL	5.3689	5.6937	6.7894	8.1908	20.1839	34.6448
HK	6.0793	6.4713	7.7880	9.4154	22.7158	38.3329

Design $2.a_2.\beta_1$. $n_i = [15,15,15,15,15,15,15,15,15,15]$ $\sigma_i^2 = [0.001, 0.01, 0.05, 0.1, 0.5, 1, 1.5, 2, 2.5, 3]$						
BE	0.9300	0.9373	0.9500	0.9466	0.9492	0.9521
BO	1.0000	0.9897	0.9810	0.9696	0.9564	0.9555
PL	0.9517	0.9464	0.9501	0.9460	0.9492	0.9517
HK	0.9300	0.9430	0.9569	0.9532	0.9571	0.9593
MPV	0.9649	0.9409	0.9503	0.9468	0.9489	0.9521
PV	0.9523	0.9488	0.9544	0.9478	0.9470	0.9481
BE	0.1677	0.4787	1.6438	3.0747	14.4229	28.7587
BO	0.2093	0.5401	1.8450	3.3356	14.7048	29.0407
PL	0.1333	0.4525	1.6253	3.0620	14.4167	28.7534
HK	0.1704	0.4871	1.6800	3.1462	14.7739	29.4626

Design $2.a_2.\beta_2$. $n_i = [15,15,15,15,15,15,15,15,15,15]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 2, 5, 7, 10, 15, 20]$						
BE	0.9216	0.8878	0.9312	0.9439	0.9495	0.9467
BO	1.0000	0.9929	0.9881	0.9865	0.9737	0.9648
PL	0.9450	0.9418	0.9452	0.9500	0.9500	0.9466
HK	0.9216	0.8908	0.9364	0.9493	0.9569	0.9532
MPV	0.9595	0.8999	0.9361	0.9442	0.9501	0.9472
PV	0.9550	0.9481	0.9489	0.9473	0.9540	0.9479
BE	0.9611	1.2718	2.4881	3.9788	15.4882	29.6288
BO	1.1969	1.5216	2.8150	4.4845	16.9224	31.1956
PL	0.8022	1.1165	2.3282	3.8192	15.4046	29.5696
HK	0.9763	1.2920	2.5311	4.0547	15.8456	30.3343

Design $2.a_2.\beta_3$. $n_i = [15, 15, 15, 15, 15, 15, 15, 15, 15, 15]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 5, 10, 15, 20, 30, 40]$						
BE	0.9300	0.8649	0.9177	0.9322	0.9492	0.9484
BO	1.0000	0.9923	0.9912	0.9883	0.9788	0.9729
PL	0.9496	0.9402	0.9505	0.9472	0.9454	0.9489
HK	0.9300	0.8667	0.9212	0.9376	0.9555	0.9555
MPV	0.9629	0.8817	0.9270	0.9355	0.9486	0.9484
PV	0.9589	0.9455	0.9520	0.9484	0.9509	0.9485
BE	1.9097	2.2214	3.4674	5.0186	16.5058	31.0096
BO	2.3828	2.7097	4.0084	5.6776	18.5934	33.8925
PL	1.6477	1.9421	3.1578	4.6877	16.2621	30.8398
HK	1.9396	2.2565	3.5238	5.1054	16.8643	31.7247

Design $2.b.\beta_1$. $n_i = [2, 4, 6, 8, 10, 12, 14, 16, 18, 20]$ $\sigma_i^2 = [0.001, 0.01, 0.05, 0.1, 0.5, 1, 1.5, 2, 2.5, 3]$						
BE	1.0000	0.9071	0.9503	0.9573	0.9557	0.9524
BO	1.0000	0.9905	0.9797	0.9775	0.9744	0.9723
PL	0.9279	0.9452	0.9504	0.9486	0.9534	0.9503
HK	0.9939	0.8795	0.9453	0.9535	0.9616	0.9616
MPV	0.9884	0.9518	0.9542	0.9534	0.9554	0.9523
PV	0.9586	0.9511	0.9500	0.9481	0.9504	0.9497
BE	0.0988	0.3827	1.6604	3.3155	14.6795	29.0411
BO	0.2004	0.5326	1.8473	3.5351	16.4986	32.5170
PL	0.1357	0.4520	1.5985	3.0625	14.5085	28.9601
HK	0.1223	0.4020	1.6057	3.1324	14.9930	29.9457

Design $2.b.\beta_2$. $n_i = [2, 4, 6, 8, 10, 12, 14, 16, 18, 20]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 2, 5, 7, 10, 15, 20]$						
BE	1.0000	0.8624	0.8946	0.9169	0.9579	0.9586
BO	1.0000	0.9945	0.9893	0.9861	0.9792	0.9765
PL	0.9154	0.9393	0.9435	0.9467	0.9522	0.9532
HK	0.9925	0.7774	0.8620	0.9046	0.9545	0.9607
MPV	0.9770	0.9208	0.9433	0.9506	0.9545	0.9557
PV	0.9563	0.9465	0.9485	0.9502	0.9537	0.9519
BE	0.5183	0.7622	1.9002	3.4887	16.6165	31.6490
BO	1.0876	1.4161	2.7450	4.4390	17.6983	33.6095
PL	0.7503	1.0748	2.3070	3.8122	15.4144	29.6479
HK	0.6517	0.8908	2.0086	3.5692	15.7327	30.5105

Design $2.b.\beta_3$. $n_i = [2,4,6,8,10,12,14,16,18,20]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 5, 10, 15, 20, 30, 40]$						
BE	1.0000	0.8662	0.8686	0.8860	0.9405	0.9583
BO	1.0000	0.9946	0.9926	0.9873	0.9779	0.9761
PL	0.9166	0.9404	0.9443	0.9412	0.9485	0.9488
HK	0.9912	0.7620	0.8185	0.8556	0.9362	0.9556
MPV	0.9744	0.9060	0.9305	0.9410	0.9507	0.9517
PV	0.9543	0.9505	0.9513	0.9463	0.9524	0.9519
BE	1.0497	1.2825	2.3630	3.7874	16.5055	33.1476
BO	2.1919	2.5200	3.8591	5.4778	18.6172	35.1765
PL	1.5463	1.8489	3.1129	4.5937	16.1294	30.7513
HK	1.3245	1.5543	2.6215	4.0129	16.0804	31.3710

Design $2.c.\alpha$. $n_i = [20,18,16,14,12,10,8,6,4,2]$ $\sigma_i^2 = [1,1,1,1,1,1,1,1,1,1]$						
BE	0.9999	0.9853	0.9698	0.9570	0.9543	0.9479
BO	1.0000	0.9961	0.9864	0.9811	0.9776	0.9732
PL	0.7451	0.8935	0.9412	0.9450	0.9537	0.9473
HK	0.9741	0.9579	0.9613	0.9573	0.9627	0.9570
MPV	0.9998	0.9324	0.9526	0.9570	0.9673	0.9619
PV	0.9242	0.9494	0.9509	0.9516	0.9522	0.9500
BE	0.3742	0.6953	2.0018	3.4943	14.8154	28.9939
BO	0.4546	0.7866	2.1263	3.7971	17.1635	33.7002
PL	0.3368	0.6434	1.8143	3.2575	14.7460	28.9542
HK	0.3120	0.6518	1.8848	3.3794	15.2587	29.9520

Design $2.c.\beta_1$. $n_i = [20,18,16,14,12,10,8,6,4,2]$ $\sigma_i^2 = [0.001, 0.01, 0.05, 0.1, 0.5, 1, 1.5, 2, 2.5, 3]$						
BE	0.8578	0.8370	0.8833	0.9156	0.9460	0.9466
BO	1.0000	0.9961	0.9913	0.9865	0.9785	0.9742
PL	0.7294	0.8348	0.9236	0.9376	0.9480	0.9465
HK	0.5458	0.7258	0.9017	0.9333	0.9585	0.9571
MPV	0.9983	0.8885	0.9374	0.9471	0.9629	0.9624
PV	0.9038	0.9329	0.9493	0.9529	0.9472	0.9528
BE	0.8460	1.1528	2.3054	3.7184	15.0457	29.2178
BO	0.8910	1.2193	2.5413	4.2191	17.4719	33.9916
PL	0.6097	0.8985	2.0692	3.5283	14.9558	29.1568
HK	0.7880	1.0988	2.2854	3.7786	15.5447	30.2081

Design $2.c.\beta_2$. $n_i = [20, 18, 16, 14, 12, 10, 8, 6, 4, 2]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 2, 5, 7, 10, 15, 20]$						
BE	0.8031	0.7970	0.7780	0.7855	0.8967	0.9285
BO	1.0000	0.9966	0.9951	0.9942	0.9859	0.9833
PL	0.7215	0.7424	0.8288	0.8641	0.9316	0.9412
HK	0.5035	0.5326	0.6648	0.7600	0.9195	0.9456
MPV	0.9980	0.8490	0.8797	0.9017	0.9396	0.9545
PV	0.9049	0.9125	0.9339	0.9416	0.9567	0.9532
BE	5.1135	5.4679	6.5520	8.1157	19.3605	33.6661
BO	5.4166	5.7927	6.9694	8.7041	21.9484	38.7865
PL	3.9301	4.2307	5.1901	6.6571	18.1700	32.7695
HK	4.8261	5.1932	6.2958	7.8641	19.6757	34.6325

Design $2.c.\beta_3$. $n_i = [20, 18, 16, 14, 12, 10, 8, 6, 4, 2]$ $\sigma_i^2 = [0.001, 0.01, 0.1, 1, 5, 10, 15, 20, 30, 40]$						
BE	0.8076	0.8058	0.7915	0.7769	0.8355	0.8946
BO	1.0000	0.9968	0.9971	0.9959	0.9901	0.9872
PL	0.7186	0.7254	0.7916	0.8264	0.9085	0.9322
HK	0.5030	0.5118	0.5957	0.6632	0.8662	0.9215
MPV	0.9982	0.8471	0.8576	0.8800	0.9305	0.9453
PV	0.9021	0.9117	0.9264	0.9305	0.9526	0.9524
BE	10.2903	10.5742	11.7298	13.3376	25.1522	39.2068
BO	10.8825	11.1834	12.4432	14.2007	27.7778	44.4627
PL	7.9581	8.1944	9.1645	10.6228	22.3409	36.8665
HK	9.7018	10.0011	11.2033	12.8225	25.0081	39.8610

Design $3.a_1.\beta_1$. $n_i = [5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5]$ $\sigma_i^2 = [10^{-6}, 5 \cdot 10^{-6}, 10^{-5}, 5 \cdot 10^{-5}, 10^{-4}, 5 \cdot 10^{-4}, 10^{-3}, 5 \cdot 10^{-3}, 10^{-2}, 5 \cdot 10^{-2}, 0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5]$						
BE	0.8972	0.8594	0.9358	0.9465	0.9496	0.9484
BO	1.0000	0.9989	0.9940	0.9885	0.9791	0.9771
PL	0.8507	0.8855	0.9378	0.9442	0.9483	0.9486
HK	0.8972	0.8700	0.9545	0.9658	0.9698	0.9706
MPV	0.9992	0.8373	0.9279	0.9466	0.9537	0.9516
PV	0.7954	0.9393	0.9506	0.9534	0.9499	0.9503
BE	0.2790	0.4690	1.1655	1.9489	8.1872	15.9284
BO	0.4503	0.6609	1.5166	2.5807	11.1380	21.7224
PL	0.2168	0.3757	1.0395	1.8402	8.1238	15.8773
HK	0.2935	0.4944	1.2469	2.1078	8.9740	17.4947

Design $3.a_1.\beta_2$. $n_i = [5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5]$ $\sigma_i^2 = [10^{-3}, 5\cdot 10^{-3}, 10^{-2}, 5\cdot 10^{-2}, 0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 7, 10, 13, 15, 17, 20]$						
BE	0.9060	0.8402	0.8850	0.9158	0.9450	0.9493
BO	1.0000	1.0000	0.9997	0.9973	0.9885	0.9846
PL	0.6293	0.7804	0.8946	0.9212	0.9445	0.9482
HK	0.9060	0.8442	0.8967	0.9308	0.9651	0.9710
MPV	0.9998	0.7897	0.8444	0.8840	0.9426	0.9530
PV	0.8001	0.8982	0.9361	0.9474	0.9541	0.9518
BE	1.2492	1.4481	2.2224	3.1528	9.5221	17.2688
BO	2.0303	2.2503	3.0918	4.1583	12.6380	23.2630
PL	1.2620	1.3840	1.9390	2.7342	9.1011	16.9305
HK	1.3130	1.5225	2.3423	3.3366	10.3109	18.8370

Design $3.a_1.\beta_3$. $n_i = [5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5,5]$ $\sigma_i^2 = [10^{-3}, 10^{-2}, 0.1, 1, 2, 3, 4, 5, 7, 10, 13, 15, 17, 20, 25, 30, 35, 40, 45, 50]$						
BE	0.9299	0.8495	0.8678	0.8929	0.9432	0.9480
BO	1.0000	1.0000	0.9998	0.9997	0.9969	0.9909
PL	0.5926	0.6730	0.8005	0.8566	0.9390	0.9425
HK	0.9299	0.8508	0.8749	0.8999	0.9578	0.9648
MPV	1.0000	0.7829	0.7991	0.8294	0.9143	0.9354
PV	0.7907	0.8606	0.9189	0.9311	0.9535	0.9501
BE	3.7608	4.0617	4.7960	5.8318	13.0045	20.8885
BO	6.1602	6.4762	7.2696	8.3773	16.8536	27.3891
PL	4.0144	4.2032	4.6735	5.3779	11.6699	19.6054
HK	3.9500	4.2671	5.0413	6.1353	13.8365	22.4774

Design $3.a_2.\beta_1$. $n_i = [15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15,15]$ $\sigma_i^2 = [10^{-6}, 5\cdot 10^{-6}, 10^{-5}, 5\cdot 10^{-5}, 10^{-4}, 5\cdot 10^{-4}, 10^{-3}, 5\cdot 10^{-3}, 10^{-2}, 5\cdot 10^{-2}, 0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5]$						
BE	0.8986	0.9143	0.9487	0.9488	0.9536	0.9496
BO	1.0000	0.9921	0.9836	0.9774	0.9652	0.9566
PL	0.9889	0.9437	0.9506	0.9480	0.9534	0.9490
HK	0.8986	0.9200	0.9542	0.9554	0.9604	0.9576
MPV	0.9578	0.9197	0.9490	0.9485	0.9536	0.9493
PV	0.9433	0.9499	0.9540	0.9524	0.9477	0.9516
BE	0.0898	0.2733	0.8958	1.6819	7.9121	15.7351
BO	0.1363	0.3494	1.1683	2.0262	8.2656	16.0884
PL	0.0377	0.2408	0.8735	1.6669	7.9058	15.7302
HK	0.0912	0.2784	0.9192	1.7288	8.1450	16.2017

Design $3.a_2.\beta_2$. $n_i = [15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15]$ $\sigma_i^2 = [10^{-3}, 5 \cdot 10^{-3}, 10^{-2}, 5 \cdot 10^{-2}, 0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 7, 10, 13, 15, 17, 20]$

BE	0.9133	0.8741	0.9312	0.9418	0.9479	0.9514
BO	1.0000	0.9976	0.9945	0.9905	0.9750	0.9706
PL	0.9399	0.9389	0.9447	0.9480	0.9472	0.9511
HK	0.9132	0.8762	0.9374	0.9482	0.9559	0.9583
MPV	0.9656	0.8808	0.9337	0.9448	0.9487	0.9517
PV	0.9381	0.9491	0.9502	0.9518	0.9473	0.9496
BE	0.4063	0.5954	1.3242	2.1185	8.2913	16.1042
BO	0.6189	0.8260	1.6910	2.7569	9.8635	17.7022
PL	0.3256	0.4849	1.1797	1.9967	8.2382	16.0677
HK	0.4124	0.6046	1.3495	2.1665	8.5237	16.5699

Design $3.a_2.\beta_3$. $n_i = [15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15, 15]$ $\sigma_i^2 = [10^{-3}, 10^{-2}, 0.1, 1, 2, 3, 4, 5, 7, 10, 13, 15, 17, 20, 25, 30, 35, 40, 45, 50]$

BE	0.9245	0.8684	0.9028	0.9272	0.9480	0.9496
BO	1.0000	0.9994	0.9979	0.9960	0.9857	0.9808
PL	0.9184	0.9341	0.9454	0.9462	0.9476	0.9482
HK	0.9245	0.8697	0.9068	0.9314	0.9545	0.9562
MPV	0.9710	0.8747	0.9045	0.9273	0.9474	0.9501
PV	0.9369	0.9401	0.9510	0.9535	0.9503	0.9488
BE	1.2499	1.4268	2.2198	3.1346	9.5140	17.2363
BO	1.9059	2.0955	2.9544	4.0160	12.5198	21.7354
PL	1.1871	1.3075	1.9350	2.7794	9.2654	17.0654
HK	1.2684	1.4481	2.2548	3.1889	9.7511	17.7042

Design $3.b.\beta_1$. $n_i = [2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]$ $\sigma_i^2 = [10^{-6}, 5 \cdot 10^{-6}, 10^{-5}, 5 \cdot 10^{-5}, 10^{-4}, 5 \cdot 10^{-4}, 10^{-3}, 5 \cdot 10^{-3}, 10^{-2}, 5 \cdot 10^{-2}, 0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5]$

BE	1.0000	0.7321	0.9066	0.9376	0.9539	0.9516
BO	1.0000	0.9919	0.9852	0.9777	0.9647	0.9583
PL	0.9846	0.9449	0.9481	0.9504	0.9488	0.9484
HK	0.9932	0.7120	0.9084	0.9381	0.9591	0.9603
MPV	0.9579	0.9281	0.9501	0.9517	0.9488	0.9487
PV	0.8978	0.9485	0.9482	0.9480	0.9460	0.9508
BE	0.0301	0.1736	0.9619	1.9901	8.0599	15.7169
BO	0.1199	0.3326	1.1708	2.0826	8.4228	16.1596
PL	0.0347	0.2368	0.8716	1.6526	7.8978	15.6402
HK	0.0442	0.1870	0.8721	1.6922	8.2274	16.3126

Design $3.b.\beta_2$. $n_i = [2,2,3,4,5,6,7,8,9,10,11,12,13,14,15,$ $16,17,18,19,20]$ $\sigma_i^2 = [10^{-3}, 5\cdot 10^{-3}, 10^{-2}, 5\cdot 10^{-2}, 0.1,$ $0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 7, 10, 13, 15, 17, 20]$						
BE	1.0000	0.7623	0.7946	0.8557	0.9392	0.9554
BO	1.0000	0.9989	0.9950	0.9894	0.9791	0.9781
PL	0.8535	0.9306	0.9449	0.9483	0.9479	0.9494
HK	0.9957	0.6610	0.7691	0.8477	0.9390	0.9561
MPV	0.9910	0.9089	0.9431	0.9543	0.9513	0.9538
PV	0.8976	0.9533	0.9522	0.9498	0.9515	0.9491
BE	0.1667	0.2929	0.9394	1.8787	9.9557	17.7817
BO	0.5928	0.8086	1.6581	2.7240	11.2083	21.7857
PL	0.3196	0.4999	1.1862	1.9982	8.2400	15.9942
HK	0.2152	0.3420	0.9895	1.8713	8.4575	16.5879

Design $3.b.\beta_3$. $n_i = [2,2,3,4,5,6,7,8,9,10,11,12,13,14,15,$ $16,17,18,19,20]$ $\sigma_i^2 = [10^{-3}, 10^{-2}, 0.1, 1, 2, 3, 4, 5, 7, 10,$ $13, 15, 17, 20, 25, 30, 35, 40, 45, 50]$						
BE	1.0000	0.8717	0.8098	0.8257	0.9040	0.9320
BO	1.0000	0.9993	0.9988	0.9973	0.9885	0.9822
PL	0.7818	0.8903	0.9365	0.9413	0.9518	0.9488
HK	0.9953	0.6983	0.7204	0.7772	0.9026	0.9331
MPV	0.9964	0.8964	0.9203	0.9366	0.9553	0.9549
PV	0.8889	0.9387	0.9498	0.9530	0.9518	0.9501
BE	0.6015	0.7555	1.3178	2.1519	9.8908	20.1372
BO	1.9344	2.1817	3.0127	4.1005	12.6176	23.1570
PL	1.2247	1.4043	2.0513	2.9186	9.3408	17.0831
HK	0.7016	0.8683	1.4375	2.2891	9.2147	17.3575

Design $3.c.\alpha$. $n_i = [20,19,18,17,16,15,14,13,12,11,10,9,8,7,$ $6,5,4,3,2,2]$ $\sigma_i^2 = [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1]$						
BE	1.0000	0.9906	0.9716	0.9598	0.9491	0.9494
BO	1.0000	0.9998	0.9918	0.9862	0.9775	0.9762
PL	0.5632	0.8501	0.9363	0.9459	0.9485	0.9481
HK	0.9733	0.9560	0.9561	0.9604	0.9594	0.9610
MPV	1.0000	0.8533	0.9122	0.9370	0.9614	0.9646
PV	0.8386	0.9410	0.9513	0.9511	0.9510	0.9499
BE	0.2197	0.4166	1.2249	2.0105	8.0740	15.7962
BO	0.3000	0.5099	1.3616	2.4328	10.9937	21.6085
PL	0.1814	0.3517	0.9851	1.7730	8.0257	15.7669
HK	0.1487	0.3551	1.0341	1.8584	8.3815	16.4569

Design $3.c.\beta_1$. $n_i = [20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 2]$ $\sigma_i^2 = [10^{-6}, 5 \cdot 10^{-6}, 10^{-5}, 5 \cdot 10^{-5}, 10^{-4}, 5 \cdot 10^{-4}, 10^{-3}, 5 \cdot 10^{-3}, 10^{-2}, 5 \cdot 10^{-2}, 0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5]$						
BE	0.4753	0.4355	0.6846	0.8250	0.9405	0.9457
BO	1.0000	0.9996	0.9956	0.9926	0.9811	0.9787
PL	0.6460	0.7118	0.8815	0.9149	0.9479	0.9498
HK	0.1934	0.3359	0.7385	0.8619	0.9554	0.9585
MPV	1.0000	0.7867	0.8502	0.8929	0.9439	0.9534
PV	0.7958	0.8998	0.9458	0.9505	0.9521	0.9523
BE	0.6525	0.8109	1.3973	2.1741	8.3376	16.0592
BO	0.7660	0.9817	1.8230	2.9089	11.3819	21.9772
PL	0.4378	0.5769	1.1897	1.9965	8.2329	15.9788
HK	0.6050	0.7734	1.4184	2.2463	8.6927	16.7518

Design $3.c.\beta_2$. $n_i = [20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 2]$ $\sigma_i^2 = [10^{-3}, 5 \cdot 10^{-3}, 10^{-2}, 5 \cdot 10^{-2}, 0.1, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4, 5, 7, 10, 13, 15, 17, 20]$						
BE	0.6408	0.6371	0.5953	0.6129	0.8678	0.9157
BO	1.0000	0.9995	0.9993	0.9985	0.9910	0.9864
PL	0.5402	0.6074	0.7329	0.8118	0.9262	0.9403
HK	0.2762	0.3039	0.4483	0.5971	0.8966	0.9385
MPV	1.0000	0.7540	0.7926	0.8161	0.8994	0.9287
PV	0.8022	0.8654	0.9116	0.9325	0.9501	0.9511
BE	2.8723	2.9996	3.7055	4.4606	10.3845	18.1932
BO	3.2665	3.4281	4.3356	5.3953	13.8620	24.6534
PL	2.0890	2.1716	2.7142	3.4201	9.6429	17.6057
HK	2.5379	2.6766	3.3919	4.2136	10.6869	18.9021

Design $3.c.\beta_3$. $n_i = [20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 2]$ $\sigma_i^2 = [10^{-3}, 10^{-2}, 0.1, 1, 2, 3, 4, 5, 7, 10, 13, 15, 17, 20, 25, 30, 35, 40, 45, 50]$						
BE	0.7975	0.7912	0.7717	0.7593	0.7724	0.8504
BO	1.0000	1.0000	0.9998	0.9996	0.9964	0.9946
PL	0.5228	0.5549	0.6429	0.7062	0.8700	0.9159
HK	0.3597	0.3696	0.4225	0.4866	0.7791	0.8782
MPV	0.9999	0.7550	0.7699	0.7876	0.8538	0.8858
PV	0.8095	0.8310	0.8757	0.9013	0.9444	0.9521
BE	8.1113	8.2700	9.0283	9.9617	16.0781	23.3304
BO	8.9789	9.1725	10.0668	11.1593	19.5425	30.2114
PL	5.8874	5.9962	6.4900	7.1325	12.9428	20.7937
HK	6.8438	7.0185	7.7750	8.6573	15.1785	23.3565

6 Acknowledgement

The research has been supported by the Grant VEGA 1/7295/20 from the Science Grant Agency of the Slovak Republic.

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