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The 3-colorability problem on graphs with maximum degree 4

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THE 3-COLORABILITY PROBLEM ON GRAPHS WITH MAXIMUM DEGREE 4

MARTIN KOCHOL, VADIM LOZIN, AND BERT RANDERATH

Abstract. The 3-colorability problem is known to be NP-complete in the class of graphs with maximum degree four. On the other hand, due to the celebrated Theorem of Brooks, the problem has a polynomial time solution for graphs with maximum degree three. To make the complexity gap more precise, we study a family of intermediate graph classes between these two extremes, and classify all of them according to the computational complexity of the problem. In particular, we generalize Brooks' Theorem in case of 3-colorability to a larger class by showing that every connected graph in that class is 3-colorable, unless it is a complete graph on four vertices.

1. Introduction

A 3-*coloring* of a graph is a partition of its vertex set into three subsets, each of which contains no pair of adjacent vertices. By Brooks' Theorem [1] every connected graph with a maximum vertex degree of at most three has a 3-coloring or is isomorphic to a complete graph on four vertices K_4. Hence, the decision problem, whether a given graph G has a 3-coloring, is trivial for graphs with maximum degree three. On the other hand, the complexity of the problem jumps from triviality to NP-completeness when we turn to the class of graphs with maximum degree four (see [4]). To make the complexity gap more precise, we study a family of graph classes that are intermediate between these two extremes. Every class X in the family is associated with a subset $\mathcal{H}'$ of the set of graphs on four vertices and is defined as follows: a graph G belongs to X if and only if each vertex of G has degree at most four, and the neighborhood of each 4-degree vertex induces a graph isomorphic to a member of the set $\mathcal{H}'$. The main result of the paper is that dichotomy holds: for a given subset $\mathcal{H}'$, the problem either

(1) is NP-complete, or
(2) can be solved in polynomial time.

Subject to the assumption $P \neq NP$ both cases exclude each other. For the polynomial cases we present an algorithm, which not only decides existence, but also finds a 3-coloring in polynomial time, if there is one. One of the classes in the family is of particular interest because of the fact that every connected graph in that class is

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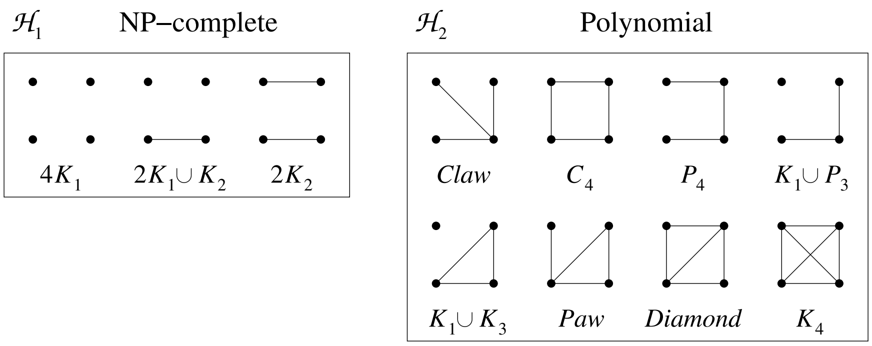

FIGURE 1. Problem-specific partition of $\mathcal{H}$

3-colorable, except for a K_4. Thereby, we extend Brooks' Theorem in the case of 3-colorability.

Our work is related to [6, 8, 9, 10], where the complexity of finding a 3-coloring in graphs defined by 'forbidden induced subgraph'-conditions is studied.

2. BASIC NOTATIONS AND MAIN RESULT

For graph theoretic terminology not defined in this paper, the reader is referred to [3]. All graphs considered are finite, undirected, without loops and multiple edges. The set of vertices and the set of edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. We let $n(G)$ denote the number of vertices of G, and $n_4(G)$ the number of vertices of degree four in G. For a vertex $v \in V(G)$, we write $N_G(v)$ for the neighborhood of v, i.e. the set of vertices adjacent to v. The closed neighborhood of v is defined as $N_G[v] := N(v) \cup \{v\}$. The degree of v is denoted by $d_G(v)$, and the maximum degree of a vertex in G by $\Delta(G)$. If no confusion arises, we omit the subscript G in the above notations.

Given a subset $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by the vertices in X, and let $G - X := G[V(G) - X]$. If $X = \{v\}$, we write $G - v$ for simplicity. We also write $G - e$ to denote the subgraph of G obtained by deleting an edge $e \in E(G)$. If G contains no induced subgraph isomorphic to a graph H, we say that G is H-free.

Throughout the paper, P_k and C_k stand respectively for a chordless path and a chordless cycle on k vertices, and K_r for a complete graph on r vertices. In particular, K_3 is a *triangle*. The set of all 4-vertex graphs will be denoted $\mathcal{H}$. This set contains eleven pairwise non-isomorphic graphs represented in Figure 1 along with their notations. By $\mathcal{H}_1$, $\mathcal{H}_2$, we refer to the subsets of $\mathcal{H}$ depicted in Figure 1.

A 4-vertex graph of particular interest is a *diamond* (see Figure 1). The nonadjacent vertices of a diamond will be called its *pick* vertices, and the other two vertices the *diagonal* vertices. Importance of this graph for the 3-colorability problem is due to the following simple observation.

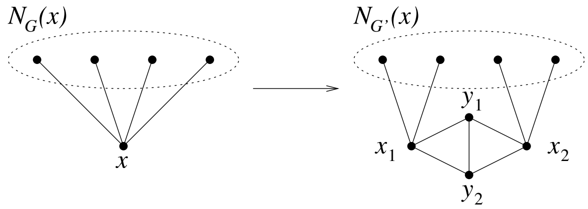

FIGURE 2. Reduction for the case $2K_1 \cup K_2 \in \mathcal{H}'$.

Proposition 1. *Let G be a 3-colorable graph containing a diamond as an induced subgraph. Then the pick vertices of the diamond have the same color in any 3-coloring of G.*

In this paper we study the following problem. Let $\mathcal{H}'$ be a fixed subset of $\mathcal{H}$.

$\mathcal{H}'$-ISOM-3-COL
Instance: A graph G with maximum degree four in which the neighborhood of each 4-degree vertex induces a graph isomorphic to a member of $\mathcal{H}'$.
Question: Does there exist a 3-coloring of G?

Obviously if $\mathcal{H}' = \emptyset$, we deal with the 3-colorability problem restricted to graphs with maximum degree three, which is polynomially solvable according to Brooks' Theorem. The main result of this paper asserts that $\mathcal{H}'$-ISOM-3-COL is NP-complete *only if* $\mathcal{H}'$ contains one of the graphs in the set $\mathcal{H}_1$.

Theorem 2. *The problem* $\mathcal{H}'$-ISOM-3-COL *is*
(i) NP-*complete if* $\mathcal{H}' \cap \mathcal{H}_1 \neq \emptyset$,
(ii) *solvable in linear time if* $\mathcal{H}' \cap \mathcal{H}_1 = \emptyset$.

Proof. Now we prove only (i). The rest is proved in the next two sections.

Obviously, the problem is in NP for any choice of $\mathcal{H}'$. If $4K_1 \in \mathcal{H}'$, then the statement holds by the result of Maffray and Preissmann [8], who proved that the 3-colorability problem is NP-complete for K_3-free graphs with maximum degree four.

Applying the transformation shown in Figure 2 to every vertex of degree four in a K_3-free graph G with $\Delta(G) \leq 4$, we obtain a graph G', which is obviously an instance of $\mathcal{H}'$-ISOM-3-COL in case that $2K_1 \cup K_2 \in \mathcal{H}'$. According to Proposition 1, G' is 3-colorable if and only if G is 3-colorable, which implies the statement in this case.

If $2K_2 \in \mathcal{H}'$, we refer to the result of Holyer [5], who has shown in particular that it is an NP-complete problem to decide whether a cubic K_3-free graph G is 3-*edge*-colorable. This result can be translated to the terminology of vertex colorability by associating with each graph G its line graph, denoted $L(G)$ and defined as follows: the vertex set of $L(G)$ is $E(G)$ and two vertices are adjacent in $L(G)$ if and only if the respective edges of G have a vertex in common. It is not hard to verify that, if G

is a triangle-free cubic graph, then $L(G)$ is a 4-regular diamond- and claw-free graph. Therefore, the result of Holyer implies NP-completeness of vertex 3-colorability in the class of 4-regular diamond- and claw-free graphs. Moreover, we may suppose that any graph G in this class is K_4-free, since a K_4 is not 3-colorable. But now G is an instance of the problem in question, and the conclusion holds. $\square$

3. Polynomial-time results

Let $\mathcal{R}$ denote the class of graphs every vertex of which has degree at most four and the neighborhood of each 4-degree vertex induces a graph isomorphic to a member of $\mathcal{H}_2$. If $G \in \mathcal{R}$, then $A(G)$ denotes the set of 4-degree vertices x from G such that $G[N(x)] \neq K_1 \cup P_2$.

Lemma 3. *Suppose we have a graph $G \in \mathcal{R}$ together with $A(G)$. If $A(G) \neq \emptyset$, then we can construct in a constant time a graph $G^* \in \mathcal{R}$ together with $A(G^*)$ such that $n(G^*) < n(G)$, G is 3-colorable iff G^* is 3-colorable, and every 3-coloring φ^* of G^* can be transformed to a 3-coloring φ of G in a constant time.*

Proof. In order to construct the desired graph G^* we use finitely many operations of vertex/edge deletion/addition or set $G^* = K_4$. A trivial consequence of this is that $A(G)$ can be transformed into $A(G^*)$ in a constant time as well.

Choose $x \in A(G)$. If $G[N(x)]$ has a triangle, G is not 3-colorable and we can set $G^* = K_4$. If $G[N(x)]$ has no triangle, we get one of the following cases.

Case 1: $G[N(x)]$ is a claw. Let $N(x) = \{y_1, y_2, y_3, x'\}$ and x' is the center. Denote by G^* the graph obtained from G by deleting the vertices of $N[x]$ and introducing a new vertex y adjacent to those vertices in $V(G) - N[x]$ that have a neighbor in $\{y_1, y_2, y_3\}$.

Clearly $n(G^*) < n(G)$. Assume φ is a 3-coloring of G. Then, according to Observation 1, y_1, y_2, y_3 have the same color, and we may transform φ into a 3-coloring of G^* by assigning to the vertex y the value $\varphi(y_1)$. The inverse direction follows by analogy. Thus G is 3-colorable iff G^* is so, and every 3-coloring φ^* of G^* can be transformed to a 3-coloring φ of G in a constant time.

To prove that $G^* \in \mathcal{R}$, let us notice that each y_j has at most one neighbor in $V(G) - N[x]$, since otherwise $N(y_j)$ would induce in G a member of $\mathcal{H}_1$. This implies that y has degree at most three in G^*. Obviously, the degree of any other vertex of G^* is at most four. Suppose now that G^* contains a vertex z of degree four such that $N_{G^*}(z)$ induces a graph $H \in \mathcal{H}_1$. Notice that $G[N_G(z)]$ either coincides with H, if $N_G(z) \cap \{y_1, y_2, y_3\} = \emptyset$, or is isomorphic to a spanning subgraph of H, otherwise. In both cases, $G[N_G(z)] \in \mathcal{H}_1$, contradicting the assumption.

Case 2: $G[N(x)] = C_4$. Let $N(x) = \{y_0, y_1, y_2, y_3\}$ such that y_i is adjacent with y_{i+1} for $i = 1, 2, 3$. Denote by G^* the graph obtained from G by deleting the vertices of $N[x]$, introducing a pair of adjacent vertices y_{odd} and y_{even}, and connecting y_{odd} (respectively, y_{even}) to those vertices in $V(G) - N[x]$ that have a neighbor in the set

$\{y_1, y_3\}$ (respectively, $\{y_0, y_2\}$) in G. Similarly as in Case 1 we can show that G^* has the desired properties.

Case 3: $G[N(x)] = P_4$. Let $N(x) = \{y_0, y_1, y_2, y_3\}$ such that y_i is adjacent with y_{i+1} for $i = 1, 2, 3$. Denote by G' the graph obtained from G by deleting the vertices of $N[x]$, introducing a pair of adjacent vertices y_{odd} and y_{even}, and connecting y_{odd} (respectively, y_{even}) to those vertices in $V(G) - N[x]$ that have a neighbor in the set $\{y_1, y_3\}$ (respectively, $\{y_0, y_2\}$) in G. Clearly G' has a 3-coloring iff G has 3-coloring and $n(G') < n(G)$. Furthermore,

(*) $\Delta(G') \leq 4$ and for every vertex $z \in V(G) - N[x]$ with $d_{G'}(z) = 4$, the neighborhood $N_{G'}(z)$ induces a member of $\mathcal{H}_2$.

If $G' \in \mathcal{R}$, then $G^* = G'$ has the desired properties.

Let $G' \notin \mathcal{R}$. Then by (*), y_{odd} or y_{even} induce in G' a graph isomorphic to a member of $\mathcal{H}_1$. Let us specify those conditions. Assume

(0) y_{even} have degree four in $G_{P_4,x}$ with $N(y_{even}) = \{y_{odd}, z_0, z_1, z_2\}$ for some vertices $z_0, z_1, z_2 \in V(G) - N[x]$. Without loss of generality let $N_G(y_0) = \{x, y_1, z_0, z_1\}$ and $N_G(y_2) = \{x, y_1, y_3, z_2\}$.

If the subgraph of G' induced by $N(y_{even})$ belongs to $\mathcal{H}_1$, then

(1) z_1 (or z_0) is adjacent to y_1 in G and hence to y_{odd} in G', since otherwise the neighborhood of y_0 would induce in G a graph isomorphic to a member of $\mathcal{H}_1$; thus $N_G(y_1) = \{x, y_0, y_2, z_1\}$.

(2) y_3 is adjacent neither to z_0 nor to z_2, since otherwise the subgraph of G' induced by $N(y_{even})$ does not belong to $\mathcal{H}_1$; for the same reason, z_1 is adjacent neither to z_0 nor to z_2.

(3) the subgraph of G' induced by $N(y_{odd})$ does not belong to $\mathcal{H}_1$, since otherwise z_2 should be adjacent to y_3 by analogy with (1).

Under conditions (0)-(3), let us define G^* to be the graph obtained from G by deleting the vertices of the set $U := \{x, y_1, y_3, z_1\}$, introducing a pair of adjacent vertices w and y_{odd}, and connecting y_{odd} (respectively, w) to those vertices in $V(G) - U$ that have a neighbor in the set $\{y_1, y_3\}$ (respectively, $\{x, z_1\}$) in G.

We check that $G^* \in \mathcal{R}$. It is obvious that for any vertex $z \in V(G) - U$, $d_{G^*}(z) \leq d_G(z)$. In order to show that w has degree at most four in G^*, let us observe that z_1 has degree at most three in G, since otherwise $N_G(z_1)$ would induce in G a graph isomorphic to a member of $\mathcal{H}_1$. Denoting a possible third neighbor of z_1 in G (different from y_0 and y_2) by v, we therefore conclude that w has at most four neighbors in G^*: y_0, y_2, y_{odd} and v. Similarly, y_3 has degree at most three in G, and hence y_{odd} is of degree at most four in G^*. In the subgraph H of G^* induced by $N(w)$ the vertex y_{odd} has degree at least two, and therefore $H \notin \mathcal{H}_1$. Analogously, the subgraph of G^* induced by $N(y_{odd})$ also is not a member of $\mathcal{H}_1$. For any other vertex z of the graph G^*, we use the same arguments as before: if z has degree four in G^*, then z has degree four in G as well. Moreover, the subgraph $G[N_G(z)]$ is isomorphic to a spanning subgraph

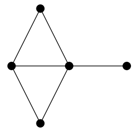

FIGURE 3. The Dart

of the graph induced by the neighborhood of z in G^*. Therefore, $G^* \notin \mathcal{R}$ would imply $G \notin \mathcal{R}$. To check the other desired properties for G^* is straightforward. □

4. DART GRAPHS

By means of reductions from Lemma 3, any graph $G \in \mathcal{R}$ can be transformed in polynomial time into a graph G' with $\Delta(G') \leq 4$ such that the neighborhood of each 4-degree vertex in G' induces a $K_1 \cup P_3$. In other words, each vertex of degree four together with its neighborhood induces the graph depicted in Figure 3. This graph is known in the literature under the name of *dart*. So, we call the graphs in the class under consideration *dart graphs*.

The class of dart graphs is an extension of the graphs with maximum degree three. According to the classical result of Brooks [1], every connected graph with vertex degree at most three is 3-colorable, unless it is a K_4. It turns out that this proposition remains valid in the larger class of dart graphs.

In a dart graph G, a vertex v will be called a diagonal (respectively, pick) vertex if v is a diagonal (respectively, pick) vertex of an induced diamond in G.

Lemma 4. *In a dart graph G, the subsets of pick and diagonal vertices are disjoint. As a result,*

(a) *in G, no pick vertex has degree more than 3,*
(b) *in the subgraph of G induced by the diagonal vertices, no vertex has degree more than 2.*

Proof. Let u be a pick vertex of an induced diamond D in G with the diagonal (the edge connecting the diagonal vertices of D) v_1v_2. By contradiction, assume uw is the diagonal of another induced diamond D' in G. Notice that w neither coincides with v_i nor is adjacent to v_i, since otherwise $G[N[v_i]]$ is not a dart ($i = 1, 2$). But then u has at lest 5 neighbors in G: v_1, v_2, w and the two pick vertices of D'. This contradiction shows that the subsets of pick and diagonal vertices are disjoint in G.

By definition, every vertex of degree 4 in G is a diagonal vertex, which implies (a). Every diagonal vertex has at least two non-diagonal (pick) neighbors, which implies (b). □

Consider a connected component C of the graph induced by the diagonal vertices of a dart graph G. Due to Lemma 4(b), C is either a chordless path or a chordless cycle. By adding to C all the diamonds containing the vertices of C we obtain a subgraph of G (not necessarily induced) that will be denoted by F. If F contains a vertex

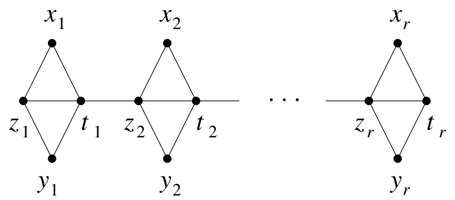

FIGURE 4. A Diamond Bracelet

which has degree 4 in G, we call F a *diamond bracelet.* The number of diamonds in the bracelet will be called the *size* of the bracelet. An example of a bracelet of size r is depicted in Figure 4. Notice that the vertices z_1 and t_r may have degree three in G. They also may be adjacent if C is a cycle. We denote the family of diamond bracelets of G by $B(G)$. Notice that $B(G) = \emptyset$ if and only if $n_4(G) = 0$.

Lemma 5. *Suppose we have a K_4-free dart graph G together with $B(G)$. If $r(G) > 0$, then we can construct in a constant time a K_4-free dart graph G^* together with $B(G^*)$ such that $n(G^*) \leq n(G)$, $n_4(G^*) < n_4(G)$, and every 3-coloring φ^* of G^* can be transformed to a 3-coloring φ of G in a constant time.*

Proof. In order to construct the desired graph G^* we use finitely many operations of vertex/edge deletion/addition. A trivial consequence of this is that $B(G)$ can be transformed into $B(G^*)$ in a constant time as well. In most cases it is straightforward to see that G^* is a K_4-free dart graph. When this fact is not that obvious, we provide necessary explanations.

The lemma is trivial if G contains a vertex of degree 4 with a neighbor v of degree at most 2: in this case $G^* = G - v$, and φ^* can be extended to φ by assigning to v a color missing in the neighborhood of v. So, in the course of the proof we shall assume that every neighbor of a 4-degree vertex has degree at least 3.

Since $n_4(G) > 0$, we may consider a diamond bracelet $F \in B(G)$ of size $r \geq 1$. We use the notation of vertices of F as indicated in Figure 4.

Case 1: $r \geq 3$. Let G^* be the K_4-free dart graph arising from G by deleting the edges t_1z_2, t_2z_3 and adding a new edge t_1z_3. Then $n(G^*) = n(G)$, $n_4(G^*) = n_4(G) - 2$ and for any 3-coloring φ^* of G^*, the colors $\varphi^*(z_2)$ and $\varphi^*(t_2)$ are different and can interchange. Moreover, the colors $\varphi^*(t_1)$ and $\varphi^*(z_3)$ also are different. Therefore, by possible switching of colors of the vertices z_2 and t_2 we can transform φ^* into φ so that $\varphi(t_1) \neq \varphi(z_2)$ and $\varphi(t_2) \neq \varphi(z_3)$, which means φ is a 3-coloring of G.

Case 2: $r \in \{1, 2\}$ and at least one of the vertices z_1 and t_r has degree 3. Assume t_r is of degree 3. Then z_r has degree 4 in G (otherwise F does not cover any vertex of degree 4). Let u denote the fourth neighbor of z_r, i.e. $u \neq t_r, x_r, y_r$. The vertices z_r and u are not the neighbors of a same 4-degree vertex. Therefore, $G^* = G - z_ru$ is a K_4-free dart graph, and furthermore $n(G^*) = n(G)$, $n_4(G^*) \leq n_4(G) - 1$. If φ^* is a 3-coloring of G^*, then, clearly, the colors $\varphi^*(z_r)$ and $\varphi^*(t_r)$ are different and can

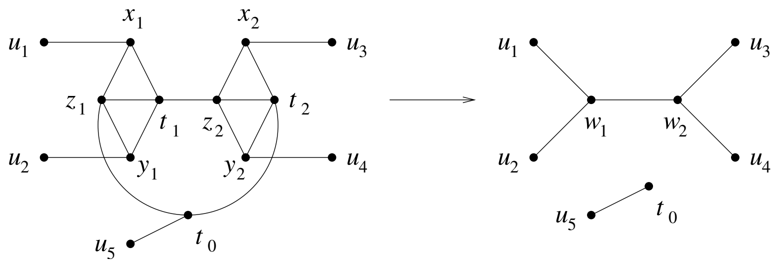

FIGURE 5

interchange, and hence we may transform φ^* into a 3-coloring φ of G in a constant time.

Case 3: $r = 2$ and $z_1t_2 \in E(G)$. Consider $G^* = G - \{z_1t_2, z_2t_1\}$, which is clearly a K_4-free dart graph with $n(G^*) = n(G)$, $n_4(G^*) = n_4(G) - 4$. Since the colors $\varphi^*(z_1)$ and $\varphi^*(t_1)$ (and similarly $\varphi^*(z_2)$ and $\varphi^*(t_2)$) are different and can interchange, we conclude that φ^* can be transformed into φ in a constant time.

From now on, let $N(z_1) = \{t_0, x_1, y_1, t_1\}$ and $N(t_r) = \{z_r, x_r, y_r, z_{r+1}\}$.

Case 4: $r = 2$, $z_1t_2 \notin E(G)$ and $t_0 \neq z_3$. If both t_0 and z_3 belong to F, say $t_0 = x_2$ and $z_3 = x_1$, consider the graph $G^* = G - \{z_1, t_1, x_1, z_2, t_2, x_2\}$. Then $n(G^*) = n(G) - 6$, $n_4(G^*) = n_4(G) - 4$, and φ^* can be easily extended to a 3-coloring of G in a constant time.

If $z_3 \notin F$, then we apply exactly the same arguments as in Case 1: transform G into G^* by deleting the edges t_1z_2, t_2z_3 and adding a new edge t_1z_3, and then transform φ^* into φ by color interchanging. Analogously if $t_0 \notin F$.

Case 5: $r = 2$ and $t_0 = z_3$. Remember that each of the vertices x_1, y_1, x_2, y_2, t_0 has degree exactly 3. Let us denote their respective neighbors different from z_1, t_1, z_2, t_2 by u_1, u_2, u_3, u_4, u_5 (see Figure 5). Notice that $t_0 \neq u_1, \dots, u_4$, but some of the vertices $u_1, \dots, u_5$ may coincide.

Transform G into a graph G' by deleting the vertices of F and introducing a pair of new vertices w_1, w_2 with $N(w_1) = \{u_1, u_2, w_2\}$ and $N(w_2) = \{u_3, u_4, w_1\}$ (see Figure 5).

Case 5.1: Assume first that G' is a K_4-free dart graph. Then set $G^* = G'$. Now $n(G^*) = n(G) - 6$, $n_4(G^*) = n_4(G) - 4$ and φ^* can be transformed into φ in the following way. If at least two values among $\varphi^*(u_1)$, $\varphi^*(u_2)$, $\varphi^*(u_5)$ are equal, we can assign to the vertices x_1, y_1, t_0 the same color which is not in the set $\{\varphi^*(u_1), \varphi^*(u_2), \varphi^*(u_5)\}$. The rest is trivial: define $\varphi(x_2) = \varphi(y_2) \notin \{\varphi^*(u_3), \varphi^*(u_4)\}$, $\varphi(t_2) \notin \{\varphi(x_2), \varphi(t_0)\}$, $\varphi(z_2) \notin \{\varphi(x_2), \varphi(t_2)\}$, $\varphi(t_1) \notin \{\varphi(z_2), \varphi(x_1)\}$, $\varphi(z_1) \notin \{\varphi(x_1), \varphi(t_1)\}$. If $\varphi^*(u_1)$, $\varphi^*(u_2)$, $\varphi^*(u_5)$ are pairwise different, then at least two values among $\varphi^*(u_3)$, $\varphi^*(u_4)$, $\varphi^*(u_5)$ must be equal (otherwise φ^* is not a 3-coloring of G^*) and we can apply symmetrical arguments.

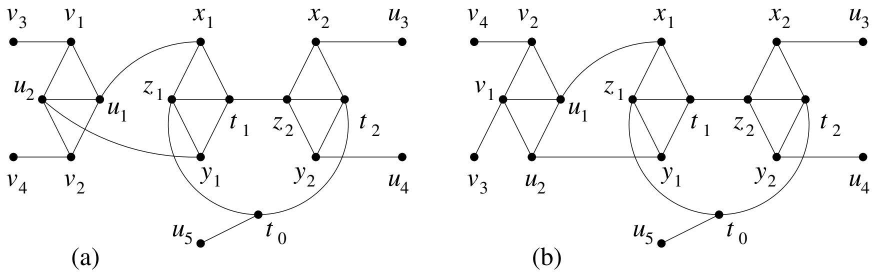

FIGURE 6

Case 5.2: Let G' be a dart graph containing a K_4. Then both new vertices w_1 and w_2 belong to the K_4 (indeed, $w_1 \notin K_4$ implies $w_2 \notin K_4$, since w_2 has at most two neighbors except w_1, but then G contains the same K_4). Therefore, up to symmetry, $u_1 = u_3$, $u_2 = u_4$ and $u_1u_2 \in E(G)$. Under this assumption, u_1 and u_2 cannot have degree 4 in G, and hence t_0 is a cut-point of G. Setting $G^* = G' - \{w_1, w_2, u_1, u_2\}$ we obtain a K_4-free dart graph G^* such that $n(G^*) = n(G) - 10$, $n_4(G^*) = n_4(G) - 4$. A 3-coloring φ^* of G^* can be extended to a 3-coloring φ of G as follows: $\varphi(z_1) = \varphi(z_2) = \varphi(u_1) \neq \varphi^*(t_0)$, $\varphi(t_1) = \varphi(t_2) = \varphi(u_2) \notin \{\varphi^*(t_0), \varphi(z_1)\}$, $\varphi(x_i) = \varphi(y_i) = \varphi^*(t_0)$, $i = 1, 2$.

Case 5.3: Suppose that G' is not a dart graph. Then at least one of the vertices $u_1, \ldots, u_4$, say u_1, has degree 4 in G', but $G'[N[u_1]]$ is not a dart. Therefore,

- u_1 has degree 4 in G;
- $u_1 \neq u_3, u_4$, otherwise $G[N[u_1]]$ is not a dart; as a result, $u_1w_2 \notin E(G')$;
- $u_1 \neq u_2$, and u_1 is adjacent to u_2, otherwise $G[N[u_1]]$ is not a dart;
- u_1 is not adjacent to t_0, otherwise $G[N[u_1]]$ is not a dart.

Denote the other two neighbors of u_1 in G, different from u_2 and x_1, by v_1 and v_2. A straightforward analysis reveals two possible combinations of the vertices in the neighborhood of u_1 in the graph G, shown in Figure 6. Notice that vertices v_3 and v_4 must exist in both cases. In case of Figure 6(a), this follows from our assumption that every vertex in the neighborhood of u_1 has degree at least 3. In case of Figure 6(b), the vertex v_3 exists, since otherwise we could apply the arguments of Case 2 to the diamond induced by u_1, u_2, v_1, v_2.

Assume first that $v_3 = t_0$ (or equivalently $v_1 = u_5$). Then the subgraph H of G induced by the vertices of F together with u_1, u_2, v_1, v_2, t_0 has at most 3 neighbors in the remaining part of G: u_3, u_4 and v_4. In this case, we define $G^* = G - V(H)$ and extend φ^* to φ according to the following rules: $\varphi(x_2) = \varphi(y_2) = \varphi(t_0) \notin \{\varphi^*(u_3), \varphi^*(u_4)\}$, $\varphi(x_1) = \varphi(y_1) = \varphi(v_2) \notin \{\varphi^*(v_4), \varphi(x_2)\}$. The rest is simple. The case $v_4 = t_0$ can be analyzed by analogy.

If neither v_3 nor v_4 coincides with t_0, then we define G^* to be the graph obtained from G by deleting the vertices u_1, u_2, v_1, v_2 and adding new edges x_1v_3 and y_1v_4. Let us show that G^* is a dart graph without a K_4. The K_4-freeness is obvious. It is also easy to see that every vertex of G^* has degree at most 4. Assume now that G^* contains a vertex w of degree 4 such that $G^*[N[w]]$ is not a dart. Then $w \notin \{v_3, v_4\}$, otherwise $G[N[w]]$ also is not a dart. Therefore, the dart $G[N[w]]$ has been destroyed under the transformation $G \to G^*$ because of a new edge connecting two vertices adjacent to w, say y_1 and v_4. However, the vertices y_1 and v_4 have no common neighbor of degree four in G, unless $v_4 = t_0$. Thus G^* is the desired graph with $n(G^*) = n(G) - 4$, $n_4(G^*) = n_4(G) - 2$. A 3-coloring φ^* of G^* can be extended to a 3-coloring φ of G in the following way. For the configuration in Figure 6(a), define $\varphi(v_1) = \varphi(v_2) = \varphi^*(x_1)$, $\varphi(u_1) \neq \varphi^*(x_1)$, and $\varphi(u_2) \notin \{\varphi(u_1), \varphi^*(x_1)\}$. For the configuration in Figure 6(b), define $\varphi(u_1) = \varphi^*(v_4)$, $\varphi(v_1) = \varphi^*(x_1)$, $\varphi(u_2) = \varphi(v_2) \notin \{\varphi^*(x_1), \varphi^*(v_4)\}$.

Case 6: $r = 1$ and t_0, z_2 have no common neighbor of degree four. If t_0, z_2 have two common neighbors v_1, v_2 of degree 3 which are adjacent to each other, then $G^* = G - \{t_0, z_2, v_1, v_2\}$ is a K_4-free dart graph with $n(G^*) = n(G) - 4$, $n_4(G^*) = n_4(G) - 2$, and we can trivially extend φ^* to φ in a constant time. Suppose that t_0, z_2 have no common adjacent neighbors of degree 3. Then deleting the edges t_0z_1, t_1z_2 and adding a new edge t_0z_2 (if $t_0z_2 \notin E(G)$) results in a K_4-free dart graph G^* such that $n(G^*) = n(G)$, $n_4(G^*) = n_4(G) - 2$. In any 3-coloring φ^* of G^* the colors of z_1, t_1 are different and can interchange, and the colors of t_0, z_2 are different. Therefore, φ^* can be transformed into a 3-coloring φ of G so that $\varphi(t_1) \neq \varphi(z_2)$ and $\varphi(z_1) \neq \varphi(t_0)$.

In the rest of the proof we shall assume that $r = 1$ and t_0, z_2 have a common neighbor of degree four. Therefore, there must exist another diamond bracelet F', containing that neighbor. Without loss of generality we may suppose that the size of F' is 1, since otherwise we can apply to F' the arguments in Cases 1 to 5. Moreover, as for F, we assume that both diagonal vertices of F' have degree four, and we denote the vertices of F' and the respective neighbors of its diagonal vertices by analogy with F, i.e. $z_1', t_1', x_1', y_1', t_0', z_2'$. Up to symmetry, we may suppose that the common neighbor of t_0 and z_2 is z_1', and z_2 coincides with x_1'. For t_0, there remain two possibilities: either $t_0 = y_1'$, in which case we say that F *strongly depends* on F', or $t_0 = t_0'$, in which case we say that F *weakly depends* on F'. To avoid Case 6 for F', we conclude that there is a bracelet F'' of size 1 (else apply Cases 1 to 5 to F'') such that F' depends on F'' either strongly or weakly.

Case 7: F depends on F' weakly. Observe that in this case F' must depend on F'' weakly. Indeed, if F' depends on F'' strongly, then, according to the definition, t_0' is a pick vertex of F'' and the neighborhood of t_0' contains z_1, z_1' and the two diagonal vertices of F'', contradicting Lemma 4.

Case 7.1: $z_2' \in F$. Then $F = F''$ (see Figure 7(a)). In this case we obtain G^* by deleting from G all vertices of F and F' together with t_0. Clearly G^* is a K_4-free dart graph with $n(G^*) = n(G) - 9$, $n_4(G^*) \leq n_4(G) - 4$. Given a 3-coloring φ^* of G^*, we can always extend it to a 3-coloring φ of G in such a way that $\varphi(y_1) = \varphi(y_1') \notin \{\varphi^*(u_1), \varphi^*(u_2)\}$ and $\varphi(t_0) \notin \{\varphi^*(u_3), \varphi(y_1)\}$. The rest is trivial.

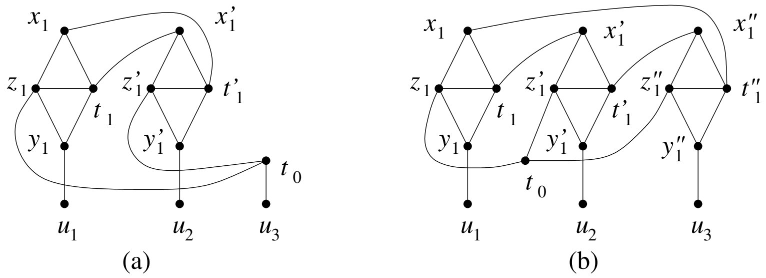

FIGURE 7

Case 7.2: $z'_2 \notin F$. With the help of Lemma 4 we conclude that F'' is disjoint from $F \cup F'$. Moreover, we may suppose that

- no diagonal vertex of F'' has a neighbor in F', otherwise we are in conditions of case 7.1 for the pair of bracelets F' and F'',
- one of the diagonal vertices of F'' is adjacent to t_0 by the definition of weak dependency,
- the other diagonal vertex of F'' is adjacent to a pick vertex of F, otherwise Case 6 can be applied to F''.

Therefore, we have the situation indicated in Figure 7(b). We define G^* to be the graph obtain from G by deleting all vertices of F, F' and F" along with t_0. Given a 3-coloring φ^* of G^*, we can always extend it to a 3-coloring φ of G in such a way that $\varphi(y_1) = \varphi(y'_1) \notin \{\varphi^*(u_1), \varphi^*(u_2)\}$, $\varphi(y''_1) \notin \{\varphi(y_1), \varphi^*(u_3)\}$, and $\varphi(t_0) \notin \{\varphi(y_1), \varphi(y''_1)\}$. The rest is trivial.

Case 8: F depends on F' strongly. If F' depends on F'' weakly, we are in conditions of Case 7 for the pair of bracelets F', F''. If F' depends on F'' strongly and $F \neq F''$, we delete from G the vertices of F' and connect the diagonal vertices of F to the pick vertices of F'', obtaining the desired K_4-free dart graph G^* with $n(G^*) = n(G) - 4$, $n_4(G^*) = n_4(G) - 2$. Any 3-coloring of G^* can be extended to a 3-coloring of G by coloring the vertices of F' analogously to the vertices of F. If $F = F''$, then $G[V(F \cup F')]$ is a 3-colorable component of G, and hence the graph $G^* = G - V(F \cup F')$ has the required properties. □

Theorem 6. *A dart graph G is 3-colorable if and only if it has no component isomorphic to a K_4. If G is 3-colorable, then a 3-coloring of G can be found in linear time.*

Proof. The necessity of the first part of the theorem is trivial. To see the sufficiency, observe that a dart graph is K_4-free if and only if it has no component isomorphic to a K_4. The same is true if G is a graph with vertex degree at most 3. Therefore, the sufficiency follows from Lemma 5 and Brooks' Theorem [1].

We can check whether a dart graph G is K_4-free in linear time. Analogously, the set of 4-degree vertices and the set $B(G)$ of diamond bracelets of G can be found in linear time. Consequently, by means of Lemma 5 we can create in linear time a K_4-free graph G' without vertices of degree more than 3 such that any 3-coloring of G' can be transformed into a 3-coloring of G in linear time. By [7], a 3-coloring of G' can be found in linear time. This proves the statement. □

End of the proof of Theorem 2. In linear time we can decide whether $G \in \mathcal{R}$ and construct $A(G)$. By Lemma 3, in linear time we can construct a graph G' such that $n(G') \leq n(G)$, $A(G') = \emptyset$, G' is 3-colorable iff G is 3-colorable and every 3-coloring of G' can be transformed to a 3-coloring of G in linear time. Now G' is a dart graph and the rest of the proof follows from Theorem 6. □

5. Outline of the algorithm for $\mathcal{H}_2$-ISOM-3-COL

Let us summarize the above arguments in the following procedure $3COL4MAX4$ that solves the 3-colorability problem for any graph $G \in \mathcal{R}$ in polynomial time.

Procedure 3COL4MAX4

Input: Graph $G \in \mathcal{R}$

Output: "YES" if G is 3-colorable; "NO" otherwise.

(1) **While** G has a vertex v such that $G[N(v)]$ is either a claw or a C_4 or a P_4, apply reductions from Lemma 3.

(2) **If** G contains a K_4 as an induced subgraph, then output "NO", **else** output "YES".

The correctness and a polynomial time bound of the procedure follow from the results of the preceding sections. Moreover, according to these results we can actually find a 3-coloring if there is one. If the graph in step (2) of the procedure is K_4-free, then it is a dart graph, and we may apply to it the reductions described in Lemma 5. This will produce a 3-colorable graph with vertex degree at most three. The algorithmic proof of Brooks' Theorem proposed by Lovász [7] suggests an idea how to find a 3-coloring in such graphs. To make the paper self-contained we present this proof below. V. Bryant [2] simplified this proof with the following characterization of cycles and complete graphs, highlighting thereby the exceptional role of the cycles and complete graphs in Brooks' Theorem.

Proposition 7. *Let G be a 2-connected graph. Then G is a cycle or a complete graph if and only if $G - \{u, v\}$ is not connected for every pair $\{u, v\}$ of vertices at distance two.*

Proof. ([9]) Let G be a 2-connected graph of order n. If G is a cycle or a complete graph, then obviously $G - \{u, v\}$ is not connected for any pair $\{u, v\}$ of vertices at distance two. Hence, assume that G is neither a cycle nor a complete graph and that $G - \{u, v\}$ is not connected for each pair $\{u, v\}$ of vertices at distance two. Since G is 2-connected, there must exist at least one cycle in G. Let C be a longest cycle in G. Assume that C is not a Hamiltonian cycle of G. Since C is a longest cycle and

G is connected, there exist vertices y, z of C and $x \in V(G) - V(C)$ such that z is adjacent to x and y, and x is not adjacent to y, i.e. $\mathrm{dist}_G(x, y) = 2$. Denote the path connecting x to y and containing all the vertices of C by P_1. According to our assumption, $G - \{x, y\}$ is not connected, and due to the 2-connectivity of G, there exists a second path P_2 connecting x to y, which is vertex disjoint with P_1, except for the endpoints. But then P_1 and P_2 form a cycle C' of length greater than C, a contradiction to the special choice of C. Thus, $C = v_0 v_1 \ldots v_{n-1} v_0$ is a Hamiltonian cycle. Now we consider a vertex v_i with $2 < d_G(v_i) < n - 1$, which must exist since G is neither a cycle nor a complete graph. Then there is $j \notin \{i-1, i+1\}$ such that v_i is adjacent to v_j but non-adjacent (without loss of generality) to v_{j-1}. Since $G - \{v_i, v_{j-1}\}$ is not connected, v_j is not adjacent to v_{j-2}. Therefore, $G - \{v_j, v_{j-2}\}$ is not connected, and hence $d_G(v_{j-1}) = 2$. Specifically, v_{j-1} is not adjacent to v_{j+1}. But now, on the one hand, $G - \{v_{j-1}, v_{j+1}\}$ is not connected according to our assumption, and on the other hand, v_j is adjacent to v_i. This contradiction completes the proof of the proposition. □

Theorem 8 (Brooks [1]). *Let G be a connected graph which is neither an odd cycle nor a complete graph. Then G is $\Delta(G)$-colorable.*

Algorithmic Proof of Brooks' Theorem [7]. Obviously, if the graph G with $\Delta(G) = \Delta$ is a cycle of even length, then G is 2-colorable. Hence, assume that G is neither a complete graph nor a cycle. Moreover, we shall assume that G is 2-connected, since otherwise the problem can be reduced to blocks of G. By Proposition 8 there exists a pair $\{u, v\}$ of vertices at distance two in graph G such that $G' = G - \{u, v\}$ is connected. Suppose that $w \in N_G(u) \cap N_G(v)$ and label the vertices of G as $v_1 = u$, $v_2 = v$ and the remainder as $v_3, v_4, \ldots, v_n = w$ in non-increasing order of their distance from w in G'. Then color the vertices $v_1, v_2, \ldots, v_n$ in that order using colors $1, 2, \ldots \Delta$ so that at each step the color used for the vertex v_i is the lowest-numbered color not yet used at a vertex adjacent to v_i. Hence, we apply a greedy algorithm along our vertex ordering. The labeling of the vertices ensures that at each stage, v_i $(1 \le i \le n)$ is adjacent to at least one higher numbered (and hence presently uncolored) vertex. Thus, v_i is adjacent to at most $\Delta - 1$ colored vertices and one of the Δ colors will be available for it. Finally, the vertex $v_n = w$ is adjacent to at most Δ vertices, at least two of which ($v_1 = u$ and $v_2 = v$) have the same color 1. Hence, there will be a color available for w. Therefore, the algorithm provides a vertex-coloring of G with at most Δ colors. □

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References

[1] R. L. Brooks, *On coloring the nodes of a network*, Proc. Cambrigde Phil. Soc. 37, (1941), 194–197.

[2] V. Bryant, *A characterisation of some 2-connected graphs and a comment on an algorithmic proof of Brooks' theorem*, Discrete Math. 158 (1996) 279–281.

[3] R. Diestel, *Graph Theory,* Springer, New York, (2000).

[4] M.R. Garey, D.S. Johnson, L. Stockmeyer, Some simplified NP-complete graph problems, Theor. Comput. Sci. 1 (1976), 237–267.

[5] I. Holyer, *The NP-completeness of edge-coloring,* SIAM J. Comput. 10 (1981), 718–720.

[6] D. Král, J. Kratochvíl, Zs. Tuza and G. Woeginger, *Complexity of coloring without forbidden induced subgraphs*, Lecture Notes in Computer Science 2204 (2001), 263–271.

[7] L. Lovász, *Three short proofs in graph theory,* J. Combin. Theory Ser. B 19 (1975) 269–271.

[8] F. Maffray and M. Preissmann, *On the NP-completeness of the k-colorability problem for triangle-free graphs*, Discrete Math. 162 (1996), no. 1-3, 313–317.

[9] B. Randerath and I. Schiermeyer, *A note on Brooks theorem for triangle-free graphs*, Australasian J. Combin. 26 (2002), 3–10.

[10] B. Randerath, I. Schiermeyer and M. Tewes, *Three-Colourability and forbidden subgraphs.II: polynomial algorithms*, Discrete Math. 251 (2002), no. 1-3, 137–153.

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