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MOMENTS OF VECTOR MEASURES AND PETTIS INTEGRABLE FUNCTIONS

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Abstract. There are investigated conditions under which the elements of a locally convex vector space are the moments of a regular vector-valued measure and of a Pettis integrable function, respectively, both with values in a locally convex vector space.

1. Introduction

The Hausdorff moment problem [3Ha, 4HS, 5H, 8SC, 9Wi] is the following: given a prescribed set of real numbers $\{a_n\}_0^\infty$, find a bounded non-decreasing function $u(t)$ on the closed interval $[0,1]$ such that its moments are equal to the prescribed values; that is,

$$\int_{[0,1]} t^n du(t) = a_n, \qquad n = 0, 1, 2, \dots .$$

The integral is a Riemann-Stieltjes integral. Equivalently, find a nonnegative measure μ on borelian subsets in $[0,1]$ with

$$\int_{[0,1]} t^n d\mu(t) = a_n, \qquad n = 0, 1, 2, \dots .$$

We shall need the operator $\nabla^k \quad (k = 0, 1, 2, \dots)$ defined by

$$\nabla^0 a_n = a_n, \qquad \nabla^1 a_n = a_n - a_{n+1},$$

$$\nabla^k a_n = a_n - \binom{k}{1} a_{n+1} + \binom{k}{2} a_{n+2} - \dots + (-1)^k a_{n+k}, \qquad n = 1, 2, \dots .$$

for any sequence of real or complex numbers $\{a_n\}_0^\infty$.

Remark that we confine ourselves to interval $[0,1]$ for simplicity, this is, however, no limitation of generality, for any bounded interval $[a,b]$.

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Definition. For each $f \in L_1([0,1])$, the elements

$$a_n = \int_0^1 f(t)t^n\, dt, n - 1, 2, \dots$$

are called the moments of f. For a measure μ on $[0,1]$, the elements

$$a_n = \int_0^1 t^n(t)\mu(dt), n = 0, 1, 2, \dots \tag{3}$$

are called the moments of μ.

Put

$$l_{k,m} = \binom{k}{m} \nabla^{k-m} a_m,\ (k, m = 0, 1, 2, \dots)$$

See [9Wi] where $\Delta^k a_n = (-1)^k \nabla^k a_m$ is used.

For the sequence of (scalar, eventually vector) elements $\{a_n\}$, define an operator $L_N(t)[a]$ by

$$L_N(t)[a] = (1+N) l_{N([Nt])}(a), (N = 1, 2, , \dots)$$

$[Nt]$ means the largest integer contained in Nt.

We have

$$\int_0^1 |L_N(t)[a]| dt = \frac{N+1}{N} \sum_{m=0}^{N-1} |l_{N,m}(a)|, N = 1, 2, \dots$$

If f is in $L_1([0,1])$ and a_n are its moments, then

$$\lim_{N\to\infty} \int_0^1 t^n L_N(t)[a] dt = a_n, n = 0, 1, \dots$$

If a_n are moments of f, then

$$\int_0^1 L_N(t)[a] dt = (N+1) \int_0^1 \int_0^1 \binom{N}{[Nt]} s^{[Nt]} (1-s)^{N-[Nt]} f(s) f(s)\, ds dt$$

$$= \int_0^1 \int_0^1 K_N(t,s) dt f(s) ds\, (N = 1, 2, , \dots, 0 \le s, t \le 1) \tag{4}$$

where

$$K_N(t,s) = (1+N) \binom{N}{[Nt]} s^{[Nt]} (1-s)^{N-[Nt]}.$$

It is easy to show that

$$\int_0^1 K_N(t,s) ds = \frac{N+1}{N} \sum_{m=0}^{N-1} \int_0^1 \binom{N}{m} s^m (1-s)^{N-m} ds \le M < \infty,$$

and

$$\int_0^1 K_N(t,s) dt = \frac{N+1}{N} \sum_{m=0}^{N-1} \int_0^1 \binom{N}{m} s^m (1-s)^{N-m} dt \le M < \infty,$$

for $N = 1, 2, \dots$

At least part of the following theorem is already known (see [Widder], pp.100-114, where the L_p case $(1 < p \le \infty)$ and (iii)-(iii)‘ are explicitly given)

Theorem 1. *Given a sequence $a_n, n = 0, 1, 2, \ldots,$ of complex numbers, there exists*
(i) a function $f \in L_1$ such that a_n are moments of f;
(ii) a function $f \in L_p, 1 < p \leq \infty$, such that the a_n are the moments of f;
(iii) a complex, regular Borel measure μ such that the a_n are the moments of μ;
(iv) a nonegative regular Borel measure μ such that the a_n are moments of μ:
if and only if the functions $L_k(t)\{a_n\}$
(i)' converge in the L_1-norm;
(ii)' are bounded in the L_p-norm;
(iii)' are bounded in L_1-norm.
(iv)' are nonnegative

2. Vector-valued moments

Let X be a quasi-complete, locally convex topological vector space. For each N, let $\Phi_N : C([0,1]) \to X$ be a linear mapping. The set of maps Φ_N is said to be weakly equi-compact if there is a weakly compact subset W of X such that

$$\{\Phi_N(\psi); \psi \in C([0,1]), ||\psi|| \leq 1, N = 1, 2, \ldots\} \subset W.$$

Let $\mathcal{B}([0,1])$ stand for the σ-algebra of all Borel sets in $[0,1]$.

Definition. Given a sequence $a_n, n = 0, 1, 2, \ldots$ of elements of X, we say that $a_n, n = 0, 1, 2, \ldots$ are moments of a regular measure $\mu : \mathcal{B}([0,1]) \to X$ if a_n are of the form defined by

$$a_n = \int_0^1 t^n \mu(dt).$$

Theorem 2. *Given a sequence $a_n, n = 0, 1, 2, \ldots$ of elements of X, there exists a regular measure $\mu : \mathcal{B}([0,1]) \to X$ such that the a_n are the moments of μ if and only if the set of maps $\Phi_N : C([0;1]) \to X, N = 1, 2, \ldots$ defined by*

$$\Phi_N(\psi) = \int_0^1 \psi(t) L_N(t) dt, \psi \in C([0,1]),$$

is weakly equi-compact.

Proof of Theorem 2. Suppose that such a measure exists. Then, for each $\psi \in C([0;1])$

$$\Phi_N(\psi) = \int_0^1 \psi(t) \int_0^1 K_N(t,s)\mu(ds)dt$$
$$= \int_0^1 (\int_0^1 K_N(t,s)\psi(t)dt)\mu(ds).$$

Let $R(\mu) = \{\mu(A); A \in \mathcal{B}([0,1])$, the range of μ, and let Q be the closed, absolutely convex hull of $R(\mu)$. Then $R(\mu)$ is relatively weakly compact in X (see [7]) and so by the Krein theorem (e.g. [7])), Q is weakly compact. Now, for all $\psi \in C([0,1]))$ with $||\psi|| \leq 1$, we have,

$$|\int_0^1 K_N(t,s)\psi(t)dt| \leq ||\psi|| \int_0^1 |K_N(t,s)|(t)dt \leq M$$

But for all measurable φ with $|\varphi(t)| \leq 1$ for all $t \in [0,1]$,

$$\int_0^1 \varphi(t)\mu(dt) \in Q.$$

Therefore, $\Phi_N(\psi)$ is in MQ for all N and all $\psi \in C([0,1])$ with $||\psi|| \leq 1$. That is, the set of Φ_N is weakly equi-compact.

Suppose now that the set of Φ_N is weakly equi-compact. Then there exists a weakly compact subset W of X such that $\{\Phi_N(\psi); \psi \in C([0,1]), ||\psi|| \leq 1, N = l, 2, \dots\} \subset W$. Take $x' \in X'$, the dual of X'. Then there exists a constant $\alpha_{x'}$ such that

$$| < \Phi_N(\psi), x' > | \leq \alpha_{x'},$$

for all N and all ψ with $||\psi|| \leq 1$. Therefore, for each N,

$$\sup_{||\psi|| \leq 1} | \int_0^1 \psi(t) < L_N(t), x' > dt| \leq \alpha_{x'};$$

that is

$$\int_0^1 | < L_N(t), x' > |dt \leq \alpha_{x'}$$

Therefore, part (iii) of Theorem 1 implies that there exists a scalar-valued measure $\mu_{x'}$ such that

$$< a_n, x' >= \int_0^1 t^n \mu_{x'}(dt) \tag{8}$$

and by (6) ,

$$\lim_N < \Phi_N(\psi), x' >= \int_0^1 \psi(t)\mu_{x'}(dt) \tag{9}$$

for all $\psi \in C([0,1])$. That is, for each fixed ψ, $\{< \Phi_N(\psi), x' >\}$ is convergent for all $x' \in X'$. Thus $\Phi_N(\psi)\}$ is weakly Cauchy and therefore weakly convergent since $\Phi_N(\psi); N = 1, 2, \dots\}$ is in the weakly compact set $||\psi||W$. Denote the weak limit by $\Phi(\psi)$. Then, for all ψ with $||\psi|| \leq 1$, $\Phi(\psi) \in W$. Since W is weakly ompact, Φ is weakly compact (i.e., it takes the unit ball of $C([0,1])$ to a relatively weakly compact set). So, by extensions [Deb, Groth., Lewis and Thomas] of a theorem of Bartle, Dunford and Schwartz ([6] Proposition 1), to (quasicomlete) locally convex spaces, there exists a regular measure $\mu : \mathcal{B}([0,1)) \to X$ such that

$$\Phi(\psi) = \int_0^1 \psi(t)\mu(dt)$$

for all $\psi \in C([0,1])$. Taking $\psi = t^n$ gives, for all $x' \in X'$,

$$< \Phi(t^n), x' >= \int_0^1 t^n < \mu(dt), x' >$$

But, by (8) and (9)

$$< \Phi(t^n), x' >= \int_0^1 t^n \mu_{x'}(dt) =< a_n, x' > .$$

Hence

$$a_n = \int_0^1 t^n \mu(dt).$$

q.e.d.

Recall that a function $f(t) : [a, b] \to X$ is called Pettis integrable if to every Borel set E in $[a, b]$ there corresponds an element $x_E \in X$ such that for all $x' \in X'$

$$x'(x_E) = \int_E x'[f(t)]dt$$

We then write

$$\int_E f(t)dt = x_E$$

In particular, if the space is reflexive, then $f(t)$ is Pettis integrable if and only if $x'[f(t)] \in L_1$ for all $x' \in X'$

Theorem 3. *Given a sequence $a_n, n = 0, 1, 2, \ldots,$ of elements of X and a Pettis integrable function f, the a_n are the moments of f if and only if*

$$\lim_N \int_0^1 \psi(t)(L_N(t) - f(t))dt = 0 \tag{7}$$

for all $\psi \in C([0, 1])$ with $||\psi|| \leq 1$.

Proof of Theorem 3. Suppose that the a_n are the moments of f. Let V be an absorbent neighbourhood of 0 in X. For all $\psi \in C([0, 1])$,

$$\int_0^1 \psi(t)(L_N(t) - f(t))dt =$$

$$= \int_0^1 \psi(t)(\int_0^1 K_N(t, s)f(s)ds - f(t))dt =$$

$$\int_0^1 f(s)(\int_0^1 K_N(t, s)\psi(t)dt - \psi(s)))ds$$

Also, there exists a constant $\delta > 0$ such that for all γ with $|\gamma| < \delta$,

$$\int_0^1 \gamma f(s)ds \in V.$$

But for each $\psi \in C([0, 1])$ with $||\psi|| \leq 1$, there exists an integer $N_0(\psi)$ such that that for all $N > N_0(\psi)$,

$$|\int_0^1 K_N(t, s)\psi(t)dt - \psi(s)| < \delta.$$

Hence, if ψ is in $C([0,1])$ with $||\psi|| \leq 1$,$N > N_0(\psi)$ implies

$$\int_0^1 \psi(t)(L_n(t) - f(t))dt \in V.$$

Conversely, define

$$\Phi_N(\psi) = \int_0^1 \psi(t)L_n(t)dt,\ N = 1, 2, \ldots,$$

$$\Phi(\psi) = \int_0^1 \psi(t)f(t),\ \psi \in C([0,1]),$$

and suppose that $\lim_N \Phi(\psi) = \Phi(\psi)$ for all $\psi \in C([0,1])$ with $||\psi|| \leq 1$. Then, for all $x' \in X'$ and all such ψ,

$$\lim_N < \Phi_N(\psi), x` >=< \Phi(\psi), x` > .$$

So, for every $x' \in X'$,

$$< a_n, x' >= \lim_N l_{N[Nt]} < a_n, x' >= \lim_N \int_0^1 t^n < L_N(t), x' > dt$$

$$=< \int_0^1 t^n f(t)dt, x' >$$

and hence

$$a_n = \int_0^1 t^n f(t)dt.$$

Theorem 4. *Given a sequence $a_n, n = 0, 1, 2, \ldots.$ of elements of X, there exists a regular measure $\mu : \mathcal{B}([0,1]) \to X$ of finite total variation such that the a_n are the moments of μ if and only if there exists a constant H such that*

$$\int_0^1 ||L_N(t)||dt \leq H,\ N = 1, 2, ...$$

Recall a definition of a Bochner integrable function. If $f : [a,b] \to X$ is simple, i.e., $f(s) = \sum_{i=1}^n \chi_{E_i}(s)x_i$, where χ_E denotes the indicator function of the set $E \subset [a,b]$, $x_i \in X$, then for any $E \in B([a,b])$

$$\int_E f d\lambda = \sum_{i=1}^n \lambda(E \cap E_i)x_i$$

where λ is a probability measure on $[a,b]$. Such functions are λ-measurable. Any function $f : [a,b] \to X$ which is the λ-almost everywhere limit of a sequence of simple functions is (called) λ-measurable.

A λ-measurable function $f : [a,b] \to X$ is called Bochner integrable if there exists a sequence of simple functions (f_n) such that

$$\lim \int_{[a,b]} ||f_n(s) - f(s)||d\lambda(s) = 0$$

In this case $\int_E f d\lambda$ is defined for each measurable set E in $[a,b]$ by

$$\int_E f d\lambda = \lim_n \int_E f_n d\lambda$$

A λ-measurable function $f : [a,b] \to X$ is Bochner integrable if and only if $\int_{[a,b]} ||f||d\lambda < \infty$

Theorem 5. *Given a sequence $a_n, n = 0, 1, 2, ..,$ of elements of X, there exists an X-valued Bochner integrable function f on $[0, 1]$ such that the a_n are the moments of f if and only if*

$$\lim_{N,J\to\infty} \int_0^1 ||L_N(t)(a) - L_J(t)(a)||dt = 0.$$

Remarks

If $f : [0.1] \to X$ is a Pettis or Bochner integrable function, then the members of the sequence a_n of elements in X are called moments of f if they are of the form defined by

$$a_n = \int_0^1 t^n f(t)dt, n = 1, 2, \ldots$$

REFERENCES

[1] BARGER, V., OLSSON, M.: CLASSICAL MECHANICS: A Modern Perspective, Second Edition, McGRAW-HILL, INC., New York 1995

[2] BARTLE, R., DUNFORD, N.,SCHWARTZ, J. C.: Weak compactness and vector measures. Canad. J. Math., 7, 1955, 289-305.

[3] DEBIEVE, C.: Intgration par rapport a une mesure vectorielle, Ann. de la Socit Scientif. de Bruxelles, 11,1973, 165–185.

[4] DUNFORD, N.-SCHWARTZ, J. T.: Linear Operators. Part I, Interscience, New York 1966.

[5] HAUSDORFF, F: Summations methoden und Momentenfolgen II, Math. Z., 9 (1921), 280–299.

[6] LEWIS, D. R.:Integration with respect to vector measures, Pac. J. Math., 33, 1970,157–165.

[7] TWEDDLE, I.: Weak compactness in locally convex spaces. Glasgow Math. J., 9, 1968, 123-127.

[8] WIDDER, D. V., The Laplace Transform. Princetom University Press 1946

[5] HAUSDORFF, F: Momentprobleme fuer ein endliches Interval, Math. Z., 16 (1923), 220–248.

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