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A HELLY TYPE THEOREM IN THE SETTING OF BANACH SPACES

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Abstract. There is given an extension of the classical Helly and Helly-Bray theorems to functions with values in Banach spaces. An application is given to a moment problem. From this, a representation theorem for compact mappings of the space of continuous functions into a Banach space is derived.

1. Introduction

Helly's theorem had been of some importance a long time above all in the probability theory in connection with a problem of moments of distributions. Recall that in this connection, real-valued nondecreasing functions f on the interval $[a, b]$ of the real line are considered and that the following facts are true.

- (1) (*First Helly's theorem*) Given a uniformly bounded sequence (f_n) of real-valued nondecreasing functions, there exists a subsequence (f_{n_k}) of (f_n) converging to a real-valued nondecreasing function f on $[a, b]$.
- (2) (*Second Helly's theorem*) Given a sequence (f_n) of real-valued nondecreasing functions on $[a, b]$, converging to a real-valued nondecreasing function f, then, for every continuous function g on $[a, b]$, we have
$$\lim_{n\to\infty} \int_a^b g(t)\, df_n(t) = \int_a^b g(t)\, df(t).$$

We extend these theorems to normed space valued functions satisfying some compactness conditions. Then an application to a certain moment problem in Banach spaces is given. From this, as an application, we obtain a well-known representation theorem for compact mappings of the space of continuous functions into a Banach space. It can be shown that moment problem theorem is equivalent to a representation theorem.

2. Helly and Helly-Bray theorems

Let X be a Banach space and g a function on an interval $[a, b]$ with values in X. We denote by X' the topological dual of X. We recall now some definitions and facts (see [5]),(see also [2]).

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Definition 1. We will say that g has *bounded semi-variation* if the set

$$V_g = \{\sum_{j=1}^{k} \alpha_j (g(t_j) - g(t_{j-1})) \mid k \in \mathbb{N}^*, a = t_0 < t_1 < \ldots < t_k = b, |\alpha_j| \leq 1\}$$

is a bounded set in X.

Proposition 1. *A function g has bounded semi-variation if and only if elements of the set $\{x' \circ g \mid x' \in X', \|x'\| \leq 1\}$ have uniformly bounded variations.*

Proof. a) Let us suppose that g has bounded semi-variation and let $\|x'\| \leq 1$. For any division $a = t_0 < t_1 < \ldots < t_k = b$ of interval $[a, b]$ and for any j, there exists α_j such that $|\alpha_j| = 1$ and $|x'(g(t_j - g(t_{j-1}))| = \alpha_j x'(g(t_j - g(t_{j-1}))$. It follows that

$$\begin{aligned}\sum_{j=1}^{k} |x'(g(t_j - g(t_{j-1}))| &= x' \left(\sum_{j=1}^{k} \alpha_j (g(t_j) - g(t_{j-1})) \right) \\ &\leq \|x'\| K \leq K\end{aligned}$$

where K is a bound for the norm of elements of V_g. b) We now suppose that the elements of the set $\{x' \circ g \mid x' \in X', \|x'\| \leq 1\}$ have uniformly bounded variations. We have

$$\left\| \sum_{j=1}^{k} \alpha_j \left(g(t_j) - g(t_{j-1}) \right) \right\| = \sup_{\|x'\| \leq 1} \left| x' \left(\sum_{j=1}^{k} \alpha_j (g(t_j) - g(t_{j-1})) \right) \right|$$

and

$$\begin{aligned}\left| x' \left(\sum_{j=1}^{k} \alpha_j \left(g(t_j) - g(t_{j-1}) \right) \right) \right| &= \left| \sum_{j=1}^{k} \alpha_j x'(g(t_j) - g(t_{j-1})) \right| \\ &\leq \sum_{j=1}^{k} |x'(g(t_j) - g(t_{j-1}))|\end{aligned}$$

and the last sum is bounded by a constant independent of x'. □

The following definitions are rather classical. We recall them for sake of completeness.

Definition 2. We will say that a real-valued function f defined on interval $[a, b]$ has a *Riemann-Stieltjes integral with respect to* g if there exists an element $A \in X$ such that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any division $\{t_0, t_1, \ldots, t_k\}, a = t_0 < t_1 < \ldots < t_k = b$ of the interval $[a, b]$ such that $|t_j - t_{j-1}| < \delta$, and for any choice of s_j with $t_{j-1} < s_j < t_j$ we have

$$\left\| A - \sum_{j=1}^{k} f(s_j)(g(t_j) - g(t_{j-1})) \right\| \leq \varepsilon.$$

In that case, the element A will be unique and denoted by $\int f \, dg$. We will denote by $t(D)$ the maximum of $|t_j - t_{j-1}|$.

Definition 3. We will call a repartition of points in the interval $[a, b]$ a division $\{t_0, t_1, \ldots, t_k\}$ with

$$a = t_0 < t_1 < \ldots < t_k = b$$

and a choice of points (s_j) such that $t_{j-1} < s_j < t_j$ for $0 \leq j \leq k$. If D is a repartition of points in the interval $[a, b]$, then we define

$$s_D = \sum_{j=1}^{k} f(s_j)(g(t_j) - g(t_{j-1})).$$

Proposition 2. *If f is a continuous real-valued function defined on $[a, b]$ and if $g : [a, b] \to X$ has bounded semi-variation, then the Riemann-Stieltjes integral $\int f\, dg$ exists.*

Proof. (see [5]) By continuity of f, for fixed $\varepsilon > 0$, there exists $\delta > 0$ such that $|t - t'| < \delta \Rightarrow |f(t) - f(t')| < \varepsilon$.

Let D and D' be two repartitions of points in the interval $[a, b]$ such that D is a refinement of D' and $t(D) < \delta$. If D is defined by points t_j, s_j and D' by points t'_r, s'_r, for each j, the interval $[t_{j-1}, t_j]$ is included in some interval $[t'_{r-1}, t'_r]$. In that case we let $s''_j = s'_r$. We have

$$s_D = \sum_{j=1}^{k} f(s_j)(g(t_j) - g(t_{j-1}))$$

and

$$s_{D'} = \sum_{r=1}^{k'} f(s'_r)(g(t'_r) - g(t'_{r-1})) = \sum_{j=1}^{k} f(s''_j)(g(t_j) - g(t_{j-1})).$$

It follows that

$$\|s_D - s_{D'}\| = \|\sum_{j=1}^{k}(f(s_j) - f(s''_j))(g(t_j) - g(t_{j-1}))\| \leq \varepsilon K$$

which shows that the net $(s_D)_{D\in\mathcal{D}}$ is Cauchy in X. Its limit obviously satisfies the condition of the Riemann-Stieltjes integral. □

Definition 4. We will say that g has compact semi-variation if g has bounded semi-variation and if the set V_g is included in a compact set W of X.

It is clear that, when g has compact semi-variation, the compact set W may be supposed to be absolutely convex.

Moreover, for any continuous function $f : [a, b] \to \mathbb{R}$, the net $(s_D)_{D\in\mathcal{D}}$ defined in the proof of the Proposition 2 is such that $s_D \in \|f\| V_g$. It follows that $\int f\, dg \in \|f\|\, W$ if $V_g \subseteq W$.

Proposition 3. *If $V_g \subseteq W$, then $g([a, b]) \subseteq W + g(a)$.*

Proof. Clear. □

Definition 5. A sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly of bounded (resp. compact) semi-variation if there exists a bounded (resp. compact) set containing V_{g_n} for any n.

The following proposition is proved in the same way as Proposition 1.

Proposition 4. *Elements of the sequence $(g_n)_{n\in\mathbb{N}}$ are uniformly of bounded semi-variation if and only if the elements of the set $\{x' \circ g_n \mid \|x'\| \leq 1, n \in \mathbb{N}\}$ have uniformly bounded variations.*

Theorem 6. *Let X be a Banach space and $(g_n)_{n\in\mathbb{N}}$ a sequence of functions from $[a, b]$ into X such that*
a) the sequence $(g_n(a))_{n\in\mathbb{N}}$ is relatively compact,
b) $(g_n)_{n\in\mathbb{N}}$ is uniformly of compact semi-variation.
Then there exists a subsequence of the sequence $(g_n)_{n\in\mathbb{N}}$ converging to a function $g : [a, b] \to X$ of compact semi-variation with the same compact subset as the sequence $(g_n)_{n\in\mathbb{N}}$.

Proof. As each function of the sequence $(g_n)_{n\in\mathbb{N}}$ is of compact semivariation, by Proposition 3, these functions have separable range. So, we may suppose, without lost of generality, that the space X is separable. In that case, we can choose a total sequence $(x'_k)_{k\in\mathbb{N}}$ in the dual space X'. By the classical Helly's first theorem (see [8],[9]), and using the classical diagonal process, we can pick a subsequence $(g_{n_i})_{i\in\mathbb{N}}$ of the sequence $(g_n)_{n\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, the sequence $(x'_k \circ g_{n_i}(t))_{i\in\mathbb{N}}$ converges to some $g_{x'_k}(t)$ for every $t \in [a, b]$.

For each $t \in [a, b]$, the sequence $(g_{n_i}(t))_{i\in\mathbb{N}}$ is relatively compact, so it has a subsequence converging to some element. Let us denote such an element by g_t.

As the sequence $(x'_k \circ g_{n_i}(t))_{i\in\mathbb{N}}$ converges to $g_{x'_k}(t)$, we have $g_{x'_k}(t) = x'_k(g_t)$. Since the sequence $(x'_k)_{k\in\mathbb{N}}$ is total, the sequence $(g_{n_i}(t))_{i\in\mathbb{N}}$ converges to g_t which we shall write $g(t)$. This defines the limit function g. Let us denote by W a compact set containing V_{g_n} for each n. As $\sum_{j=1}^{r} \alpha_j(g(t_j) - g(t_{j-1}))$ is the limit of $\left(\sum_{j=1}^{r} \alpha_j(g_{n_i}(t_j) - g_{n_i}(t_{j-1}))\right)_{i\in\mathbb{N}}$, we have $V_g \subseteq W$ which shows that g is of compact semivariation with the same compact subset as the sequence $(g_n)_{n\in\mathbb{N}}$. □

Theorem 7 (Helly-Bray Theorem)**.** *If the sequence $(g_n)_{n\in\mathbb{N}}$ is uniformly of compact semi-variation, then there exists a subsequence (g_{n_k}) of (g_n) converging on a given dense subset D of $[a, b]$ to a function g of compact semi-variation, such that for any continuous function $f : [a, b] \to \mathbb{R}$, we have*

$$\lim_{n\to\infty} \int f \, dg_{n_k} = \int f \, dg.$$

Proof. Without lost of generality, we may suppose that $g_n(a) = 0$. From theorem 6, there exists a subsequence (g_{n_k}) of (g_n) converging on D to a function g of compact semi-variation. By Proposition 3, $\int f\ dg_{n_k}$ and $\int f\ dg$ exist and are elements of $\|f\|W$ for some compact set W. It follows that the sequence $\left(\int f\ dg_{n_k}\right)_{k\in\mathbb{N}}$ has a converging subsequence. As

$$\lim_{k\to\infty} x'(\int f\ dg_{n_k}) = x'(\int f\ dg)$$

by the classical Helly-Bray theorem, the subsequence of

$$\left(\int f\ dg_{n_k}\right)_{k\in\mathbb{N}}$$

has to converge to $\int f\ dg$.

□

3. Application to Moment Problem

Definition 8. A sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach space X is a *moment sequence* if there exists a function $f\ : [0,1] \to X$, with bounded semivariation, such that for all k, we have :

$$a_k = \int_0^1 t^k\ df(t).$$

Definition 9. We say that a sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach space X is completely compact if there exists a compact set W in X such that

$$\sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k \in W \quad n = 0, 1, \ldots$$

where $|\alpha_k^n| \leq 1$ are complex numbers.

In that case, it is obvious that there exists a constant M such that

$$||\sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k|| \leq M, \text{ for } n \in \mathbf{N}.$$

Theorem 10. *(Moment theorem)*
A sequence $(a_k)_{k\in\mathbb{N}}$ in a Banach space X is the moment sequence of a function f of a compact semivariation if and only if the sequence $(a_k)_{k\in\mathbb{N}}$ is completely compact.

Proof. Let us suppose that f is of compact semivariation, and denote by W the compact set associated to f. If the sequence (a_k) is the moment sequence of f, then the sequence (a_k) is completely compact. In fact, if we define $x^{(n)}(t) =$

$\sum_{k=0}^{n} C_n^k \alpha_k^n t^k (1-t)^{n-k}$ for a fixed family of complex numbers (α_k^n) such that $|\alpha_k^n| \leq 1$, then, by the remark after definition 4, we have

$$| \ x^n(t)| \ = | \ \sum_{k=0}^{n} C_n^k \alpha_k^n t^k (1-t)^{n-k}| \ \leq 1$$

and

$$\sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k = \int_0^1 x^n(t) df(t) \in W.$$

Conversely, for $n = 1, 2, ...,$ let us define a function $f_n \ : [0,1] \to X$ by the formulas :

$$f_n(t) = \sum_{j=0}^{m-1} \binom{n}{j} \Delta^{n-j} a_j \qquad \text{for } t \in \left(\frac{m-1}{n}, \frac{m}{n} \right)$$

$$f_n(\frac{m}{n}) = \sum_{j=0}^{m} \binom{n}{j} \Delta^{n-j} a_j, \ f_n(0) = 0, \ f_n(1) = a_0.$$

The functions f_n are uniformly of compact semivariation. This is an easy consequence of the fact that the sequence $(a_k)_{k\in\mathbb{N}}$ is completely compact. If we define the operator Λ on the space of polynomials by

$$\Lambda(\sum_{j=0}^{n} c_j t^j) = \sum_{j=0}^{n} c_j a_j$$

it is clear that the Bernstein polynomials

$$B_{k,n}(t) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{j}{n} \right)^k t^j (1-t)^{n-j}$$

verify

$$\Lambda(B_{n,k}) = \int_0^1 t^k \ df_n(t).$$

Using Theorem 6, we can choose a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of (f_n) and a function f of compact semivariation such that $(f_{n_i}(t))_{i\in\mathbf{N}}$ converges to $f(t)$ at each continuity point $t \in]0,1[$ of f and for $t = 0$ and $t = 1$. By Theorem 7, for every k, we have :

$$\lim_{i\to\infty} \int_0^1 t^k \ df_{n_i}(t) = \int_0^1 t^k \ df(t), \qquad \text{for } \sigma(X, X').$$

We now show that $\lim_{n\to\infty} \Lambda(B_{k,n}) = a_k$ for the norm topology of X, and the conclusion will follow. By classical algebraic computations (see [10]), it is easy to show that $a_0 = \Lambda(B_{0,n})$ and that

$$a_k = \sum_{j=k}^{n} \frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} \binom{n}{j} \Delta^{n-j} a_j.$$

Consequently, we obtain :

$$a_k - \Lambda(B_{k,n}) = \sum_{j=k}^{n} \left(\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \right) \binom{n}{j} \Delta^{n-j} a_j$$
$$- \sum_{j=0}^{k-1} \binom{n}{j} \left(\frac{j}{n}\right)^k \Delta^{n-j} a_j.$$

Let $y = \frac{j}{n}$, and observe that

$$\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k = \prod_{i=0}^{k-1} \frac{ny-i}{n-i} - y^k.$$

It follows that, given $\varepsilon > 0$, there exists n_0 such that

$$\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \leq \varepsilon$$

for $n \geq n_0$ and

$$\left\| \sum_{j=k}^{n} \left(\frac{j(j-1)\dots(j-k+1)}{n(n-1)\dots(n-k+1)} - \left(\frac{j}{n}\right)^k \right) \binom{n}{j} \Delta^{n-j} a_j \right\| \leq \varepsilon$$

for $n \geq n_0$ It is also clear that

$$\left\| \sum_{j=0}^{k-1} \binom{n}{j} \left(\frac{j}{n}\right)^k \Delta^{n-j} a_j \right\| \leq \left(\frac{k}{n}\right)^k ||a_0||.$$

Now, it is easy to conclude that if n is large enough we have

$$||a_k - \Lambda(B_{k,n})|| \leq 2\varepsilon ||a_0||,$$

which proves the theorem. □

We obtained our moment theorem only with the help of an "Helly's theorem", not a representation theorem of a compact operator, see, e. g., ([11]). We conclude this paper by deducing a representation theorem for compact operator on $C[0,1]$ into X from our moment problem theorem.

Theorem 11 (Representation theorem)**.** *Every compact linear operator L on $C([\alpha,\beta],\mathbb{R})$, with values in a Banach space X, is representable in the form*

$$L(g) = \int_{\alpha}^{\beta} g(t)\, df(t),$$

where f is a function of compact semivariation from $[\alpha,\beta]$ into X.

Proof. It is clear that it suffices to prove the result for the interval $[0,1]$. The sequence $(a_k)_{k\in\mathbb{N}}$ defined by the formula $a_k = L(t^k)$ is completely compact. In fact, we have

$$\sum_{k=0}^{n} C_n^k \alpha_k^n \Delta^{n-k} a_k = \sum_{j=0}^{n} \binom{n}{j} \alpha_k^n L(t^k(1-t)^{n-k})$$
$$= L(\sum_{j=0}^{n} \binom{n}{j} \alpha_k^n t^k (1-t)^{n-k}) \in W$$

W being a compact set. By Theorem 7, there exists a f of compact semivariation such that

$$L(t^k) = \int_0^1 t^k \, df(t)$$

and, by Weierstrass theorem, this equality extends to every continuous function. We recall that a compact mapping from a Banach space into a normed vector space is automatically continuous. □

4. Example

The following example shows that we may not omit in Helly theorem the first condition concerning (weak) compactness. Take $X = \ell^1$, $[a,b] = [0,1]$. Let $\mathbf{e}_n$ denote the n-th unit vector and define $g_n(t) = t\mathbf{e}_n$. Clearly $g_n(0) = 0$ for all n. Also the variation of each g_n is 1. For each n the set V_{g_n} is contained in the unit ball of ℓ^1 which is not (weakly) compact. Now clearly no subsequence of $g_n(t)$ converges if $t > 0$. Define the linear function $lin_{[0,1]}$ as $lin_{[0,1]}(t) = t$ for $t \in [0,1]$. Then $g_n(t) = lin_{[0,1]}(t)\mathbf{e}_n = t\mathbf{e}_n$ for $t \in [0,1]$, so $g_n = lin_{[0,1]}\mathbf{e}_n : [0,1] \to \ell^1$.

We have

$$\|g_n(t)\|_{\ell^1} = t, n = 1, 2, \dots,$$

for every $t \in [0,1]$.

Using the following well-known result ([7], p. 282) stating that a bounded subset C of ℓ^1 is strongly relatively compact if and only if

$$\lim_{n\to\infty} \sup_{\xi=(\xi_i)\subset C} \sum_{i=n}^{\infty} |\xi_i| = 0,$$

it is easy to see that the set $C = \{g_n(t) \mid t \in [0,1], n = 1, 2, \dots\}$ is not strongly relatively compact. and

$$g_n(t) = g_{ni}(t), i = 1, 2, \dots$$

But

$$g_{11}(t) = (t, 0, 0, \dots), g_{22}(t) = (0, t, 0, \dots), g_{33}(t) = (0, 0, t, \dots), \dots$$

Further

$$\sum_{i=n}^{\infty} g_{ni}(t) = t$$

$$\sum_{i=n+k}^{\infty} g_{(n+k),i}(t) = t$$

Hence

$$\lim_{n\to\infty} \sup_n \sum_{i=n}^{\infty} |g_{ni}(t)| = t.$$

Take now $t > 0$.

As the n-th component is the only non zero component of $g_n(t)$ which is equal to t, we have

$$\lim_{n\to\infty} \sup_n \sum_{i=n}^{\infty} |g_{ni}(t)| = t.$$

For $t > 0$ the required condition is not satisfied. (see [7]).

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