Spectral Order of Operators

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Let x and y be self-adjoint operators acting on a Hilbert space with the spectral families $(E^x_\lambda)_{\lambda\in\mathbb{R}}$ and $(E^y_\lambda)_{\lambda\in\mathbb{R}}$, respectively. We say that x is less than y in the spectral order (in symbols $x \le_S y$) if $E^x_\lambda \ge E^y_\lambda$ for all $\lambda \in \mathbb{R}$. The interest in the spectral order lies in the fact that it organizes the self-adjoint part of a von Neumann algebra M into a boundedly complete lattice [8] . In particular, the positive part, $\mathcal{E}(M)$, of the unit ball of a von Neumann algebra M, called the effect algebra, is a complete lattice with the order $\le_S$ which contains the projection lattice $P(M)$ of M (equipped with the standard order) as a sublattice. In this paper we study the effect algebra and its relation to the projection lattice from the point of view of the spectral order. In the first part of the paper we deal with spectral orthomorphisms, i.e. with the maps between the effect algebras which preserve the spectral order and orthogonality of elements. Although they need not be linear, we show that, under mild condition of preserving the multiples of the units, spectral orthomorphisms extend to Jordan $*$-homomorphisms between the underlying von Neumann algebras. Moreover, we prove that any spectral orthomorphism fulfilling the previous conditions which preserves suprema of two projections preserves automatically suprema of all countable families of elements in the whole effect algebra. This generalizes results on morphisms of von Neumann lattices in [1, 2, 5, 7]. In the second part we study the spectral order in the context of Jordan algebras. As a main result we show that the projection lattice $P(M)$ of a JBW algebra M is modular if, and only if, the map $r : \mathcal{E}(M) \to P(M)$ assigning to each element in the effect algebra its range projection is a complete spectral lattice homomorphism. This generalizes results in [3, 4]. Besides, other results on spectral lattices and their relevance to quantum theory are presented.

References

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