On the double product integral $\Pi\Pi(1+\lambda(dA^\dagger\otimes dA - dA\otimes dA^\dagger))$

The following elementary identity lies behind the notion of double product integral in quantum stochastic calculus. Let $x_{j,k}, j=1,2,\cdots,M, k=1,2,\cdots,N$ be elements of an associative algebra such that x_{j_1,k_1} commutes with x_{j_2,k_2} whenever both $j^1\neq j^2$ and $k^1\neq k^2$. Then

$$\Pi_{j=1}^M(\Pi_{k=1}^N x_{j,k}) = \Pi_{k=1}^N(\Pi_{j=1}^M x_{j,k}).$$

For example, $x_{1,1}x_{1,2}x_{2,1}x_{2,2} = x_{1,1}x_{2,1}x_{1,2}x_{2,2}$.

For quantum independent increment processes such as the creation and annihilation processes $A^\dagger$ and A a continuous version of this identity allows the unambiguous definition of double product integrals as iterated single product integrals, that is, as the solutions of quantum stochastic differential equations. (But there is no sensible theory of triple product integrals in the quantum (noncommutative) context). Such product integrals have been used to construct quantum groups [1].

In this presentation we study the analytic theory of the double product integral

$$\Pi\Pi(1+\lambda(dA^\dagger\otimes dA - dA\otimes dA^\dagger))$$

as a unitary operator in the double Fock space $\mathcal{F}(\mathcal{L}^\in(\mathcal{R}^+))\otimes\mathcal{F}(\mathcal{L}^\in(\mathcal{R}^+))$, and relate it to the iterated product

$$\Pi_{a<s<t<b}(1+\lambda(dA^\dagger(s)dA(t) - dA(s)\otimes dA^\dagger(t)))$$

in the simple Fock space $\mathcal{F}(\mathcal{L}^\in(\mathcal{R}^+))$.

[1] R L Hudson and S Pulmannova, Double product integrals and Enriquez quantisation of Lie bialgebras II: The quantum Yang-Baxter equation, Letters in Mathematical Physics (2005) 72, 211-224.