From Gleason measure to Shannon information

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Let $(\Omega, \mathcal{F}, P)$ be the unit interval with Lebesgue measure, and let $\mathfrak{M}$ be the class of all finite fields of events. We say that $I\colon \mathfrak{M} \to \mathbb{R}$ is an *information* if

$$I(\mathcal{A}) + I(\mathcal{B}) = I(\sigma(\mathcal{A} \cup \mathcal{B}))$$

for any independent $\mathcal{A}, \mathcal{B} \in \mathfrak{M}$. It turns out that for a suitable continuous in some sense information, the Shannon function

$$H_{\mathcal{A}} = -\sum_{i=1}^{n} \log P(A_i) \mathbf{1}_{A_i}$$

for $\mathcal{A}$ generated by finite partition $\{A_1, \ldots, A_n\}$, where $\mathbf{1}_A$ is the indicator of the set A, is crucial. Namely, the general form of such I is given by

$$I(\mathcal{A}) = \int_{\Omega} H_{\mathcal{A}}\, dm + \beta \mathbb{D}^2 H_{\mathcal{A}},$$

with some $\beta \in \mathbb{R}$ and real measure m; $\mathbb{D}^2$ being the variance of a random variable.

Any information I is an analogue of a Gleason measure, namely, I can be treated as a function defined on projections in $L^2(\Omega, \mathcal{F}, P)$ such that

$$I(P \vee Q) = I(P) + I(Q),$$

for $P \perp Q$.