A CHARACTERIZATION OF QUANTIC QUANTIFIERS IN SEVERAL LATTICES

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Definition 1. *A bounded lattice $L = (L, \vee, \wedge, 0, 1)$ is an ortholattice if there exists a unary operation $\perp : L \to L$ satisfying the conditions:*

(1) $a^{\perp\perp} = a$.
(2) $a, b \in L: \ (a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$.
(3) $a \vee a^{\perp} = 1$.

a, b being arbitrary elements of L.

If L is an ortholattice, we shall say L is an orthomodular lattice if it satisfies the following weak modularity property:

Given any $a, b \in L$ with $a \leq b$ we have: $b = a \vee (a^{\perp} \wedge b)$ (equivalently, $a = (a \vee b^{\perp}) \wedge b$).

Definition 2. *Let L be an arbitrary orthomodular lattice. By a quantic quantifier F on L we understand a function $F : L \to L$ satisfying the following conditions:*

(1) $F(0) = 0$.
(2) *For any $a \in L$, we have: $a \leq F(a)$.*
(3) *If $a, b \in L$ then $F(a \& F(b)) = F(a) \& F(b)$.*

Theorem 1. *Let L be an arbitrary orthomodular lattice and F a quantic quantifier on L. Then F and $F(L)$ satisfy:*

(1) *F(1)= 1.*
(2) *F is idempotent.*
(3) *F is a functor.*
(4) $F((F(a))^{\perp}) = F(a)^{\perp}$.
(5) *F is a projection in $S(L)$.*
(6) *For every $a \in L$ the set $[a, -) \cap F(L)$ has a least element. Namely, $F(a)$. In particular, $F(L)$ is reflective in L; i.e., the inclusion $I : F(L) \to L$ has a left adjoint, denoted by R.*
(7) *F(L), the set of the fixed points of L is a suborthomodular lattice of L.*

Definition 3. *An orthoposet $(L, \perp, 0, 1)$ is a bounded poset satisfying:*

(1) *For any $a \in L$ we have: $a^{\perp\perp} = a$.*
(2) *If $a \leq b$ in L then $b^{\perp} \leq a^{\perp}$.*
(3) *For any $a \in L$ $\quad a \wedge a^{\perp} = 0$.*
(4) *For any $a \in L$ $\quad a \wedge a^{\perp} = 0$.*
(5) *For any $a \in L$ $\quad a \vee a^{\perp} = 1$.*

We can formulate now the following:

Theorem 2. *Let $(L, 0, 1)$ be a bounded poset. Suppose we have two families of endofunctors $\{F(-, x) : L \to L\}_{x \in L}$, $\{H(x, -) : L \to L\}_{x \in L}$ satisfying the following conditions:*

(1) *For any $a \in L$, $F(-, a)$ is left adjoint to $H(a, -)$.*
(2) *If $x \leq y$ then $F(-, y) \circ F(-, x) = F(-, x)$.*
(3) *For any $x \in L$, if $H(x, 0) = x^{\perp}$ then $H(H(x, 0), 0) = x^{\perp\perp} \leq x$.*
(4) *For any $x, y \in L$, $F(F(y, x)^{\perp}, x) \leq y^{\perp}$.*
(5) *If $x \leq y$ then $x = F(x, y), y = H(x^{\perp}, y)$.*
(6) *For any $x \in L$ we have $F(1, x) \leq x$.*

then L is an orthomodular lattice. Conversely, any orthomodular lattice, has two families of endofunctors satisfying the last six conditions.

Theorem 3. *The category of orthomodular lattices, denoted by OML, is equivalent to the category of bounded posets $(P, 0, 1)$ with two families of endofunctors, parametrized by elements $a \in P$. Denoted by PosAdj , satisfying the conditions of theorem 2.*

Similar results can be proved for Heyting Algebras and right-sided, idempotent quantales.

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