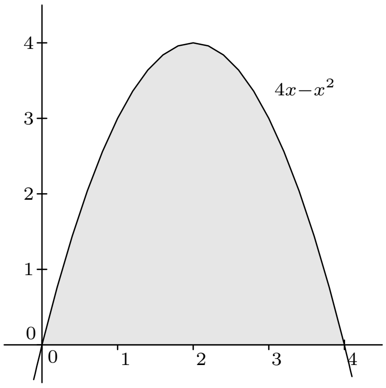
Určte obsah ohraničený krivkami:

$y = 4x - x^2$, o_x

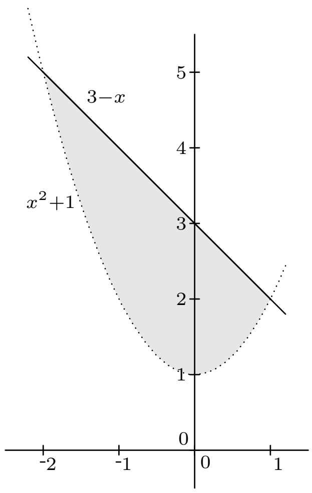


Prienik kriviek $y = 4x - x^2$ a $o_x \equiv y = 0$ je riešením kvadratickej rovnice $4x - x^2 = 0 = x(4-x)$. Hranice sú 0 a 4

$\int_0^4 |4x - x^2 - 0| \, \mathrm{d}x = \int_0^4 (4x - x^2) \, \mathrm{d}x = \left[2x^2 - \frac{x^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$

$$\boxed{\frac{32}{3}}$$

$y = x^2 + 1$, $x + y = 3$



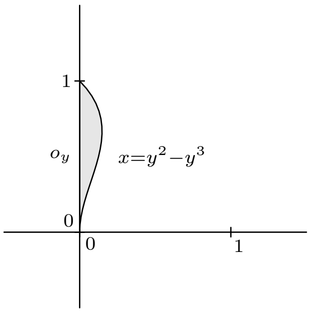
Prienik: $y = x^2 + 1 = 3 - x$, t.j. $x^2 + x - 2 = 0$, $x_1 = -2, x_2 = 1$. Na intervale $\langle -2, 1\rangle$ je $3 - x \geq x^2 + 1$.

$\int_{-2}^1 (3 - x - (x^2 + 1)) \, \mathrm{d}x = \int_{-2}^1 (2 - x - x^2) \, \mathrm{d}x = \left[2x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-2}^1 = (2 - \frac{1}{2} - \frac{1}{3}) - (-4 - 2 + \frac{8}{3}) =$

$= \frac{12-3-2+24+6-16}{6} = \frac{21}{6} = \frac{7}{2}$

$$\boxed{\frac{7}{2}}$$

o_y, $x = y^2 - y^3$

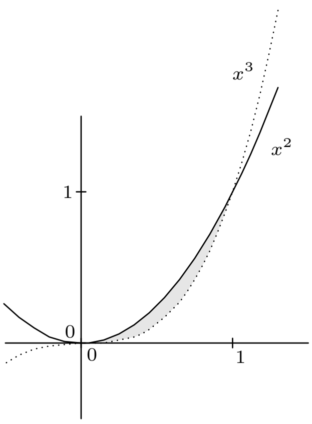


Vymeníme úlohu osí a budeme tentokrát integrovať podľa y, akoby to bolo x. Počítame prienik

$o_y \equiv x = 0$ a $x = y^2 - y^3$, $0 = y^2 - y^3 = y^2(1-y)$, $y_1 = 0, y_2 = 1$. Na intervale $\langle 0, 1\rangle$ je funkcia $y^2 - y^3 \geq 0$, preto pre obsah platí: $\int_0^1 (y^2 - y^3)\,\mathrm{d}y = \left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$$\boxed{\frac{1}{12}}$$

$y = x^2$, $y = x^3$

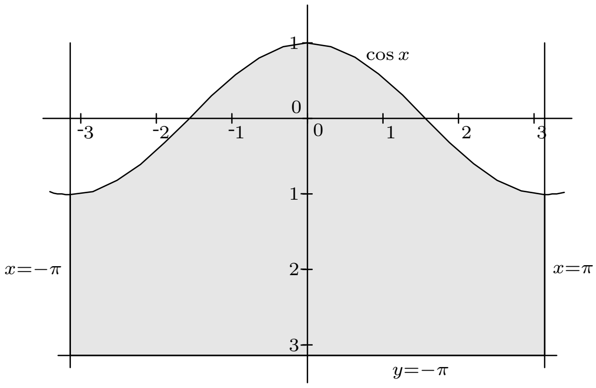


Prienik: $x^2 = x^3$, $x^2(1-x) = 0$, $x_1 = 0$, $x_2 = 1$, na intervale $\langle 0, 1\rangle$ je funkcia $x^2 \geq x^3$, preto sa obsah rovná:

$\int_0^1 (x^2 - x^3)\,\mathrm{d}x = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

$$\boxed{\frac{1}{12}}$$

$y = \cos x$, $y = -\pi$, $x = -\pi$, $x = \pi$

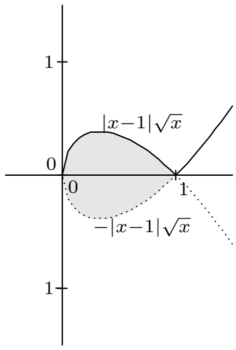


Ohraničenie je dané priamkami $x = -\pi$ a $x = \pi$.

$\int_{-\pi}^{\pi} (\pi + \cos x)\,\mathrm{d}x = [\pi x + \sin x]_{-\pi}^{\pi} = \pi^2 - (-\pi^2) = 2\pi^2$

$$\boxed{2\pi^2}$$

$y^2 = x(x-1)^2$

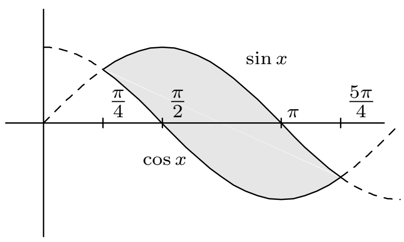


"Funkcia" je definovaná len pre nezáporné hodnoty x a skladá sa z dvoch vetiev $y_1 = |x-1|\sqrt{x}$ a $y_2 = -|x-1|\sqrt{x}$. Tieto majú prienik v dvoch bodoch 0 a 1, preto plocha sa rovná:

$\int_0^1 2|x-1|\sqrt{x}\,\mathrm{d}x = 2\int_0^1 (1-x)\sqrt{x}\,\mathrm{d}x = 2\int_0^1 (x^{\frac{1}{2}} - x^{\frac{3}{2}})\,\mathrm{d}x = \left[2 \cdot \frac{2}{3}x^{\frac{3}{2}} - 2 \cdot \frac{2}{5}x^{\frac{5}{2}}\right]_0^1 = \frac{4}{3} - \frac{4}{5} = \frac{8}{15}$

$$\boxed{\frac{8}{15}}$$

$y = \cos x$, $y = \sin x$, $x = \frac{5\pi}{4}$, $x = \frac{\pi}{4}$

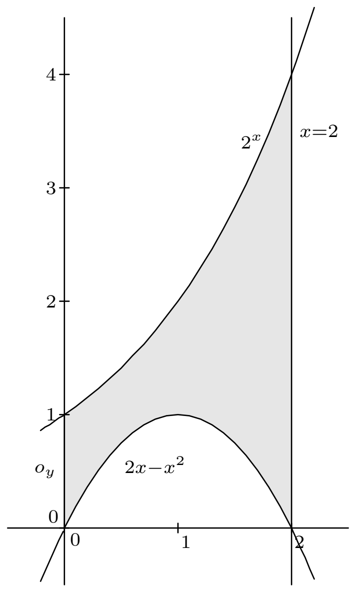


Prienikom funkcií $\sin x$, $\cos x$ sú body $\frac{\pi}{4} + k\pi$, $k \in Z$, čiže body $\frac{\pi}{4}$ a $\frac{5\pi}{4}$ sú body prieniku. Na intervale $\langle \frac{\pi}{4}, \frac{5\pi}{4} \rangle$ platí $\sin x \geq \cos x$.

$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (\sin x - \cos x)\,\mathrm{d}x = [-\cos x - \sin x]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = -(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) = 2\sqrt{2}$

$$\boxed{2\sqrt{2}}$$

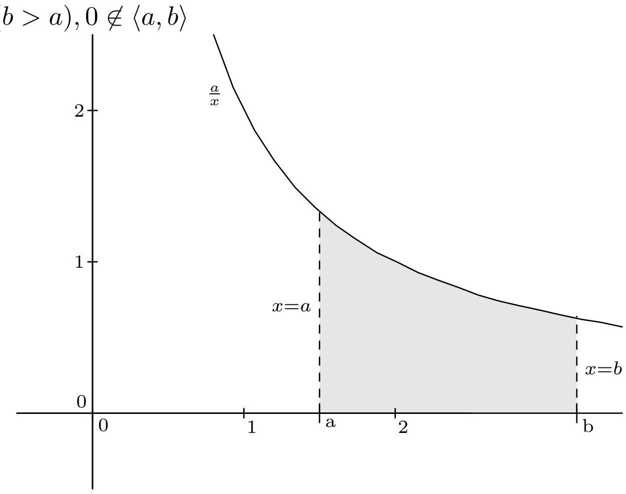
$y = 2^x,\ y = 2x - x^2,\ x = 2,\ o_y$



$$\int_0^2 (2^x - 2x + x^2)\,\mathrm{d}x = \left[\frac{1}{\ln 2}2^x - x^2 + \frac{x^3}{3}\right]_0^2 = \frac{3}{\ln 2} - 4 + \frac{8}{3} = \frac{3}{\ln 2} - \frac{4}{3}$$

$$\boxed{\frac{3}{\ln 2} - \frac{4}{3}}$$

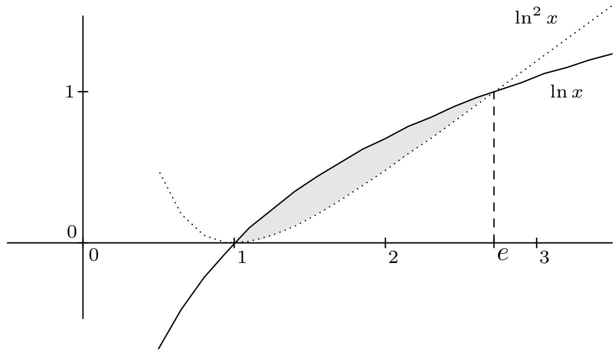
$xy = a,\ x = a,\ x = b,\ o_x,\ (b > a), 0 \notin \langle a, b\rangle$



$$\int_a^b \frac{a}{x}\,\mathrm{d}x = a\left[\ln|x|\right]_a^b = a(\ln|b| - \ln|a|) = a\ln\left|\frac{b}{a}\right|$$

$$\boxed{a\ln\left|\frac{b}{a}\right|}$$

$y = \ln x$, $y = \ln^2 x$



Prienik kriviek: $\ln x = \ln^2 x$, $\ln x(1 - \ln x) = 0$, $x_1 = 1$, $x_2 = e$. Na intervale $\langle 1, e\rangle$ platí: $\ln x \geq \ln^2 x$. Obsah sa rovná:

$\int_1^e (\ln x - \ln^2 x)\,\mathrm{d}x = \int_1^e \ln x\,\mathrm{d}x - \int_1^e \ln^2 x\,\mathrm{d}x = 1 - (e-2) = 3 - e$

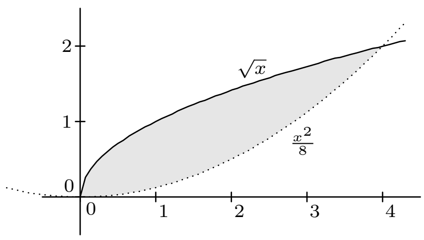
$\int_1^e \ln x\,\mathrm{d}x = \begin{vmatrix} \ln x & 1 \\ \frac{1}{x} & x \end{vmatrix} = [x \ln x]_1^e - \int_1^e 1\,\mathrm{d}x = e \ln e - 1 \ln 1 - [x]_1^e = e - (e-1) = 1$

$\int_1^e \ln^2 x\,\mathrm{d}x = \begin{vmatrix} \ln^2 x & 1 \\ 2\frac{\ln x}{x} & x \end{vmatrix} = \left[x \ln^2 x\right]_1^e - 2\int_1^e \ln x\,\mathrm{d}x = e - 2$

$$\boxed{3 - e}$$

Určte objem rotačných telies ohraničených krivkami
rotácia okolo osi x

$y = \sqrt{x}$, $y = \frac{x^2}{8}$

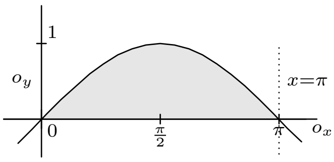


Prienik kriviek: $\sqrt{x} = \frac{x^2}{8}$, $8\sqrt{x} = x^2$, $64x = x^4$, $x(64 - x^3) = 0, x_1 = 0, x_2 = 4$, $\sqrt{x}$ dominuje na intervale $\langle 0, 4\rangle$.

$\pi \int_0^4 ((\sqrt{x})^2 - (\frac{x^2}{8})^2)\,\mathrm{d}x = \pi \int_0^4 (x - \frac{1}{64}x^4)\,\mathrm{d}x = \pi \left[\frac{x^2}{2} - \frac{1}{320}x^5\right]_0^4 = \pi(8 - \frac{16}{5}) = \frac{24\pi}{5}$

$$\boxed{\frac{24\pi}{5}}$$

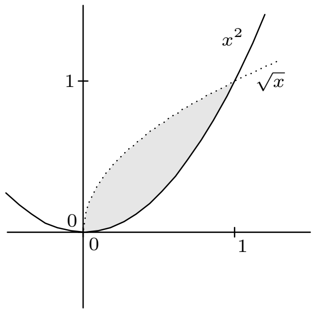
$y = \sin x$, o_x, $x = 0$, $x = \pi$



$\pi \int_0^\pi \sin^2 x\,\mathrm{d}x = \pi \int_0^\pi \frac{1-\cos 2x}{2}\mathrm{d}x = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^\pi = \frac{\pi^2}{2}$

$$\boxed{\frac{\pi^2}{2}}$$

$y = x^2$, $x = y^2$

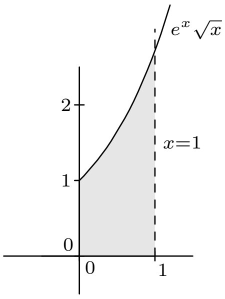


Prienik kriviek: $y^2 = x = x^4$, $x(x^3 - 1) = 0$, $x_1 = 0, x_2 = 1$. $x = y^2$ je vlastne dvojicou funkcií, $y = \sqrt{x}$, $y = -\sqrt{x}$. Neprázdny prienik s funkciou $y = x^2$ má len prvá z nich. $\sqrt{x}$ dominuje na $\langle 0, 1\rangle$.

$\pi \int_0^1 (x - x^4)\,\mathrm{d}x = \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi(\frac{1}{2} - \frac{1}{10}) = \frac{3\pi}{10}$

$$\boxed{\frac{3\pi}{10}}$$

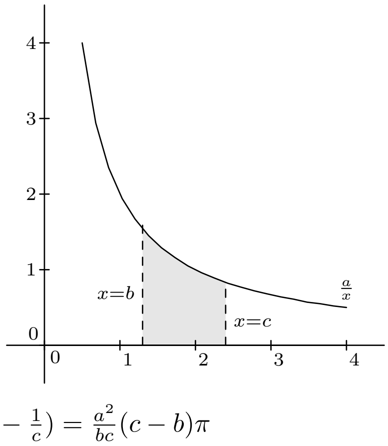
$y = e^x\sqrt{x}$, $x = 1$, $y = 0$



$\pi \int_0^1 xe^{2x}\mathrm{d}x = \begin{vmatrix} x & e^{2x} \\ 1 & \frac{1}{2}e^{2x} \end{vmatrix} = \frac{\pi}{2}\left[xe^{2x}\right]_0^1 - \frac{\pi}{2}\int_0^1 e^{2x}\mathrm{d}x = \frac{\pi}{2}e^2 - \frac{\pi}{4}\left[e^{2x}\right]_0^1 = \frac{\pi}{4}(1 + e^2)$

$$\boxed{\frac{\pi}{4}(1 + e^2)}$$

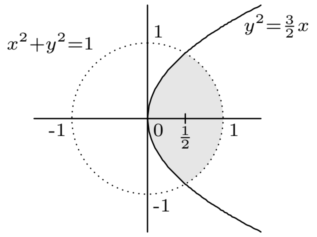
$xy = a$, o_x, $x = b$, $x = c$, $(0 < b < c)$



$\pi \int_b^c \frac{a^2}{x^2}\mathrm{d}x = \pi a^2 \left[-\frac{1}{x}\right]_b^c = a^2\pi(\frac{1}{b} - \frac{1}{c}) = \frac{a^2}{bc}(c - b)\pi$

$$\boxed{\frac{a^2}{bc}(c - b)\pi}$$

$x^2 + y^2 = 1$, $y^2 = \frac{3}{2}x$

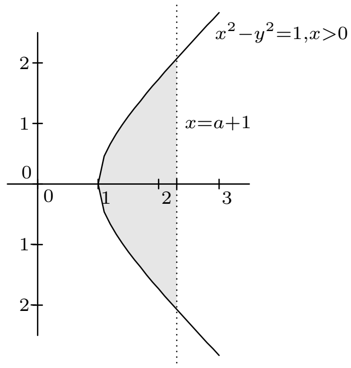


Prienik: $x^2 + y^2 = 1 = x^2 + \frac{3x}{2}$, $x = \frac{1}{2}$. Po vizualizácii dostávame, že na intervale $\langle 0, \frac{1}{2}\rangle$ je oblasť ohraničená ramenami $y_1 = \sqrt{\frac{3x}{2}}$ a $y_2 = -\sqrt{\frac{3x}{2}}$ a na $\langle \frac{1}{2}, 1\rangle$, zase $y_3 = \sqrt{1-x^2}$ a $y_4 = -\sqrt{1-x^2}$. Keďže sú tieto vetvy symetrické podľa osi rotácie, tak uvažujem iba jednu z nich a pre objem platí:

$\pi \int_0^{\frac{1}{2}} \frac{3x}{2} \mathrm{d}x + \pi \int_{\frac{1}{2}}^{1} (1-x^2)\,\mathrm{d}x = = \frac{\pi}{2}\left[\frac{3x^2}{2}\right]_0^{\frac{1}{2}} + \pi\left[x - \frac{x^3}{3}\right]_{\frac{1}{2}}^{1} = \pi(\frac{3}{16} + \frac{2}{3} - \frac{1}{2} + \frac{1}{24}) = \frac{19}{48}\pi$

$\boxed{\frac{19}{48}\pi}$

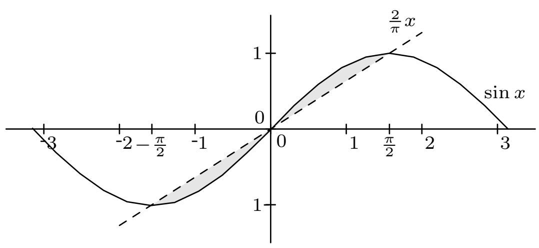
$x^2 - y^2 = 1$, $x > 0$ a $x = a + 1, a > 0$



$\pi \int_1^{1+a} (x^2 - 1)\mathrm{d}x = \pi\left[\frac{x^3}{3} - x\right]_1^{1+a} = \pi(\frac{(1+a)^3-1}{3} - a) = \frac{\pi}{3}a^2(a+3)$

$\boxed{\frac{\pi}{3}a^2(a+3)}$

$y = \sin x$, $y = \frac{2}{\pi}x$



Prienik: $x_1 = -\frac{\pi}{2}$, $x_2 = \frac{\pi}{2}$. Útvary sú symetrické podľa počiatku(nepárne funkcie), preto stačí počítať objem iba na intervale $\langle 0, \frac{\pi}{2}\rangle$ a výsledný objem bude dvojnásobkom. Funkcia $\sin x$ dominuje na $\langle 0, \frac{\pi}{2}\rangle$.

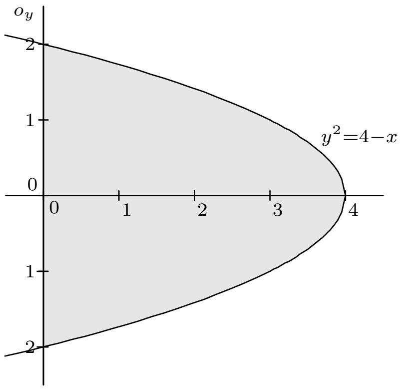
$2\pi \int_0^{\frac{\pi}{2}} (\sin^2 x - \frac{4}{\pi^2}x^2)\,\mathrm{d}x = \pi \int_0^{\frac{\pi}{2}} (1 - \cos 2x)\,dx - \frac{8}{\pi}\int_0^{\frac{\pi}{2}} x^2\,\mathrm{d}x = \pi\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}} - \frac{8}{\pi}\left[\frac{x^3}{3}\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{2} - \frac{8}{\pi}\frac{\pi^3}{24} =$

$= \pi^2(\frac{1}{2} - \frac{1}{3}) = \frac{\pi^2}{6}$

$\boxed{\frac{\pi^2}{6}}$

Určte objem rotačných telies ohraničených krivkami
rotácia okolo osi y

$x = 0,\ y^2 + x - 4 = 0$



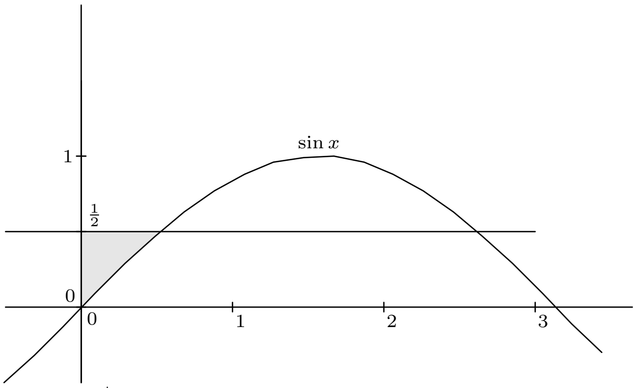
Horná a dolná polovica. Objem je teda dvojnásobkom.

$2 \cdot 2\pi \int_0^4 x\sqrt{4-x}\,\mathrm{d}x = \begin{vmatrix} 4-x=t & 4 \mapsto 0 \\ -\mathrm{d}x = \mathrm{d}t & 0 \mapsto 4 \end{vmatrix} = -4\pi \int_4^0 (4-t)\sqrt{t}\,\mathrm{d}t = \ 4\pi \int_0^4 (4\sqrt{t} - t\sqrt{t})\,\mathrm{d}t =$

$= 16\pi \left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^4 - 4\pi \left[\frac{2}{5}t^{\frac{5}{2}}\right]_0^4 = \frac{32}{3}\pi \cdot 8 - \frac{8}{5}\pi \cdot 32 = \frac{512}{15}\pi$

$\boxed{\frac{512}{15}\pi}$

$y = \sin x,\ x = 0,\ y = \frac{1}{2}$

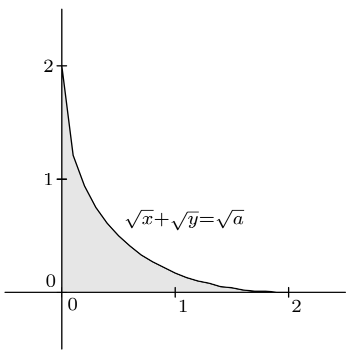


$2\pi \int_0^{\frac{\pi}{6}} x \sin x\,\mathrm{d}x = \begin{vmatrix} x & \sin x \\ 1 & -\cos x \end{vmatrix} = 2\pi\,[-x\cos x]_0^{\frac{\pi}{6}} + 2\pi \int_0^{\frac{\pi}{6}} \cos x\,dx = -2\pi \frac{\pi}{6}\frac{\sqrt{3}}{2} + 2\pi\,[\sin x]_0^{\frac{\pi}{6}} = \pi - \frac{\sqrt{3}\pi^2}{6} =$

$= \frac{1}{6}\pi(6 - \pi\sqrt{3})$

$\boxed{\frac{1}{6}\pi(6 - \pi\sqrt{3})}$

$\sqrt{x} + \sqrt{y} = \sqrt{a}$

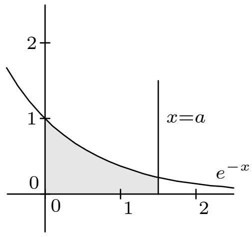


$2\pi\int_0^a x(\sqrt{a}-\sqrt{x})^2\mathrm{d}x = 2\pi\int_0^a (ax-2\sqrt{a}x^{\frac{3}{2}}+x^2)\,\mathrm{d}x = 2\pi\left[a\frac{x^2}{2}-2\sqrt{a}\frac{2}{5}x^{\frac{5}{2}}+\frac{x^3}{3}\right]_0^a =$

$= 2\pi(\frac{a^3}{2}-\frac{4}{5}a^3+\frac{a^3}{3}) = 2a^3\pi\frac{15-24+10}{30} = \frac{\pi a^3}{15}$

$$\boxed{\frac{\pi a^3}{15}}$$

$y=e^{-x}$, $x=0$, $x=a$, $y=0$, $(a>0)$

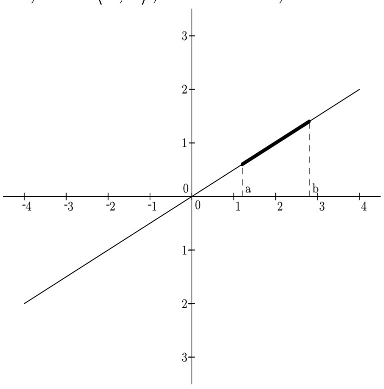


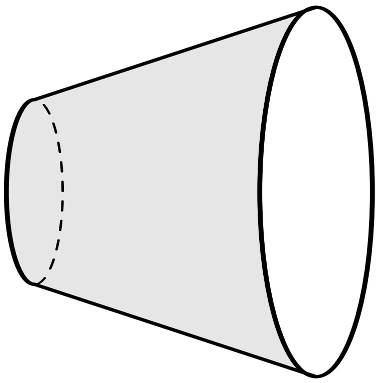
$2\pi\int_0^a xe^{-x}\mathrm{d}x = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = 2\pi\left[-xe^{-x}\right]_0^a + 2\pi\int_0^a e^{-x}\mathrm{d}x = -2\pi\frac{a}{e^a} - 2\pi\left[e^{-x}\right]_0^a = 2\pi(1-\frac{a+1}{e^a})$

$$\boxed{2\pi(1-\frac{a+1}{e^a})}$$

Určte povrchy rotačných telies, rotácia okolo osi x

$y=kx$, $x\in\langle a,b\rangle$, $0<a<b$, $k>0$

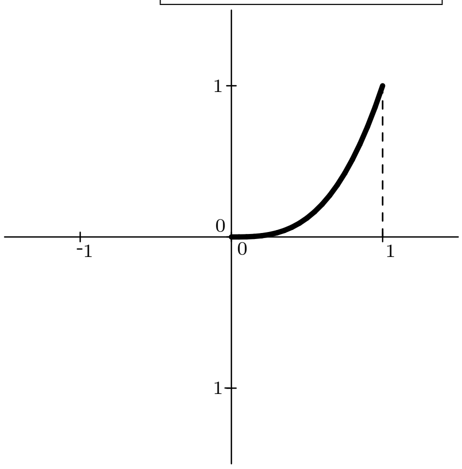


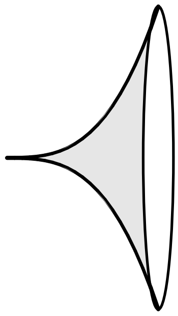


$2\pi\int_a^b kx\sqrt{1+k^2}\,\mathrm{d}x = 2\pi k\sqrt{1+k^2}\int_a^b x\,\mathrm{d}x = 2\pi k\sqrt{1+k^2}[\frac{x^2}{2}]_a^b = 2\pi\frac{k\sqrt{1+k^2}}{2}(b^2-a^2) =$

$= \pi k\sqrt{1+k^2}(b^2-a^2)$

$$\boxed{\pi k\sqrt{1+k^2}(b^2-a^2)}$$



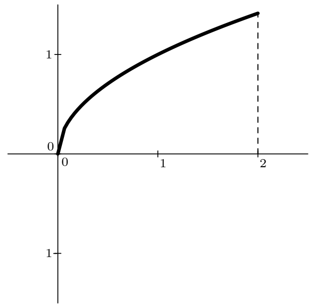


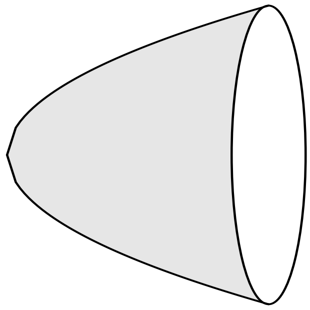
$y=x^3$, $x\in\langle 0,1\rangle$

$2\pi\int_0^1 x^3\sqrt{1+9x^4}\,\mathrm{d}x = \begin{vmatrix} t=1+9x^4 & 0\mapsto 1, 1\mapsto 10 \\ \mathrm{d}t=36x^3\mathrm{d}x & \mathrm{d}x=\frac{\mathrm{d}t}{36x^3} \end{vmatrix} = \frac{2\pi}{36}\int_1^{10}\sqrt{t}\,\mathrm{d}t = \frac{\pi}{18}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^{10} = \frac{\pi}{27}(10\sqrt{10}-1)$

$$\boxed{\tfrac{\pi}{27}(10\sqrt{10}-1)}$$

$y=\sqrt{x}$, $x\in\langle 0,2\rangle$



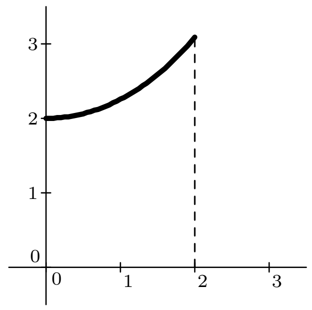


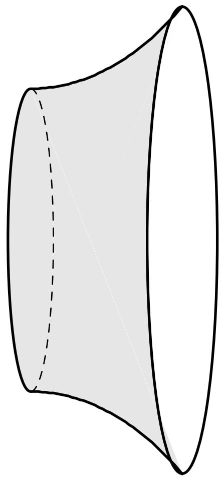
$$2\pi\int_0^2\sqrt{x}\sqrt{1+\left(\tfrac{1}{2\sqrt{x}}\right)^2}\,\mathrm{d}x=2\pi\int_0^2\sqrt{x}\sqrt{\tfrac{4x+1}{4x}}\,\mathrm{d}x=\pi\int_0^2\sqrt{1+4x}\,\mathrm{d}x=\begin{vmatrix} t=1+4x & 0\mapsto 1,2\mapsto 3\\ \mathrm{d}t=4\,\mathrm{d}x & \mathrm{d}x=\frac{\mathrm{d}t}{4}\end{vmatrix}=$$

$$=\tfrac{\pi}{4}\int_1^3\sqrt{t}\,\mathrm{d}t=\tfrac{\pi}{4}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^3=\tfrac{\pi}{6}(3\sqrt{3}-1)$$

$$\boxed{\tfrac{\pi}{6}(3\sqrt{3}-1)}$$

$y=2\cosh\left(\frac{x}{2}\right)$, $x\in\langle 0,2\rangle$



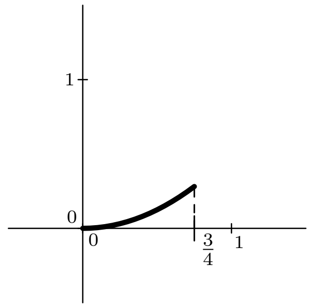


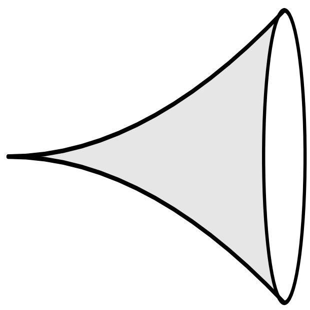
$$2\pi\int_0^2 2\cosh\tfrac{x}{2}\sqrt{1+\left(\tfrac{2\sinh\frac{x}{2}}{2}\right)^2}\mathrm{d}x=4\pi\int_0^2\cosh\tfrac{x}{2}\sqrt{1+\sinh^2\tfrac{x}{2}}\,\mathrm{d}x=4\pi\int_0^2\cosh^2\tfrac{x}{2}\mathrm{d}x=$$

$$=4\pi\int_0^2\tfrac{1+\cosh 2\frac{x}{2}}{2}\,\mathrm{d}x=2\pi\int_0^2(1+\cosh x)\,\mathrm{d}x=2\pi\left[x+\sinh x\right]_0^2=2\pi(2+\sinh 2)=\pi(4+e^2-e^{-2})$$

$$\boxed{\pi(4+e^2-e^{-2})}$$

$y=\frac{x^2}{2}$, $x\in\langle 0,\frac{3}{4}\rangle$





$$2\pi \int_0^{\frac{3}{4}} \frac{x^2}{2}\sqrt{1+x^2}\,\mathrm{d}x = \pi \int_0^{\frac{3}{4}} x^2\sqrt{1+x^2}\,\mathrm{d}x = \left| \begin{array}{ll} x = \sinh t & t = \ln(x+\sqrt{1+x^2}) \\ \mathrm{d}x = \cosh t\,\mathrm{d}t & 0 \mapsto 0, \frac{3}{4} \mapsto \ln 2 \end{array} \right| =$$

$$= \pi \int_0^{\ln 2} \sinh^2 t \cosh^2 t\,\mathrm{d}t = \frac{\pi}{4}\int_0^{\ln 2} \sinh^2 2t\,\mathrm{d}t = \frac{\pi}{4}\int_0^{\ln 2} \frac{\cosh 4t - 1}{2}\,\mathrm{d}t = \frac{\pi}{8}\left[\frac{\sinh 4t}{4} - t\right]_0^{\ln 2} =$$

$$= \frac{\pi}{8}\left(\frac{e^{4\ln 2} - e^{-4\ln 2}}{8} - \ln 2\right) = \frac{\pi}{8}\left(\frac{16-\frac{1}{16}}{8} - \ln 2\right) = \frac{\pi}{8}\left(2 - \frac{1}{128} - \ln 2\right) = \frac{\pi}{4} - \frac{\pi}{1024} - \frac{\pi \ln 2}{8} = \frac{\pi}{1024}(255 - 128\ln 2)$$

$$\boxed{\frac{\pi}{1024}(255 - 128\ln 2)}$$