Notes on elliptic curves and Weierstrass form transformations

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The problem: Find three positive integer numbers, x, y, z such that

$$\frac{x}{y+z} + \frac{y}{x+z} + \frac{z}{x+y} = 4.$$

After reading some internet articles, e.g., [1] I have stumbled across that the problem is equivalent to finding certain rational solutions of elliptic curve

$$y^2 = x^3 + 109x^2 + 224x$$

leading to the original solutions with the same sign.

Now, I would like to take a detailed look, how this transformation has been performed.

First we reduce the equation to polynomial form

$$F(x,y,z) = x^3 + y^3 + z^3 - 3x^2(y+z) - 3y^2(x+z) - 3z^3(x+y) - 5xyz = 0.$$

Now we find the inflex/saddle point on the curve.

$\frac{\partial F}{\partial x} = 3x^2 - 3y^2 - 3z^2 - 6xy - 6xz - 5yz;$

$\frac{\partial F}{\partial y} = 3y^2 - 3x^2 - 3z^2 - 6xy - 5xz - 6yz;$

$\frac{\partial F}{\partial z} = 3z^2 - 3x^2 - 3y^2 - 5xy - 6xz - 6yz;$

After computing Hessian

$$H = \begin{vmatrix} 6x-6y-6z & -6x-6y-5z & -6x-5y-6z \\ -6x-6y-5z & -6x+6y-6z & -5x-6y-6z \\ -6x-5y-6z & -5x-6y-6z & -6x-6y+6z \end{vmatrix} =$$

$$138(x^3+y^3+z^3) - 726(x^2(y+z) + y^2(x+z) + z^2(x+y) - 898xyz$$

We see that point $[x, -x, 0]$ is the solution of both F, G and

$$H(x,-x,0) = \begin{pmatrix} 12x & 0 & -x \\ 0 & -12x & x \\ -x & x & 0 \end{pmatrix}$$

with eigenvalues equal to $-\sqrt{146}x$, $\sqrt{146}x$ and 0. Thus the point $[x, -x, 0]$ is saddle point, for any $x \neq 0$.

Next we consider $x = 1$, i.e. point $Q \equiv [1, -1, 0]$.

The tangent to F at this point is equal to

$$\frac{\partial F}{\partial x}(1,-1,0)\cdot(x-1)+\frac{\partial F}{\partial y}(1,-1,0)(y+1)+\frac{\partial F}{\partial z}(1,-1,0)\cdot z=0,$$

i.e.

$$6(x-1)+6(y+1)+(-1)z=0=6x+6y-z.$$

Next step is to find transformation of the variables sending Q to the point $[0,1,0]$ such that z_1 is transformed along tangent, i.e. $z_1=6x+6y-z$. $x_1=x+y$ and $y_1=-y$ is one such transformation. Thus old variables are $x=x_1+y_1$, $y=-y_1$ and $z=z_1-6x-6y=z_1-6(x_1+y_1)-6(-y_1)=6x_1-z_1$. Another way to see this is to compute inverse of the transformation. Namely,

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 6 & 6 & -1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 6 & 6 & -1 \end{pmatrix}^{-1}\cdot\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}=\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 6 & 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$$

Substituting the variables into F we get that

$$F(x_1,y_1,z_1)=91x_1^3-69x_1^2z_1+x_1y_1z_1+15x_1z_1^2-z_1^3+y_1^2z_1.$$

We need to multiply the equation by certain value in order to find easy transformation that subsequently transforms to x_2^3 and $-y_2^2$. Such multiple is 91^2 to get

$$91^3x^3-69\cdot 91^2x_1^2z_1+91^2x_1y_1z_1+91^2\cdot 15x_1z_1^2-91^2z_1^3+91^2z_1y_1^2.$$

We immediately see that $x_2=91x_1$, $y_2=91y_1$ and $z_2=-z_1$ (to get opposite sign). Thus $x_1=\frac{1}{91}x_2$, $y_1=\frac{1}{91}y_2$ and $z_1=-z_2$.

$$91^2F(x_2,y_2,z_2)=x_2^3+69x_2^2z_2+1365x_2z_2^2+8281z_2^3-(y_2^2z_2+x_2y_2z_2).$$

We eliminate $x_2y_2z_2$ term by completing to square (" + half of the coefficient"), i.e. transformation $x_3=x_2$, $y_3=y_2+x_2/2$, $z_3=z_2$. Or $x_2=x_3$, $y_2=y_3-\frac{x_3}{2}$, $z_3=y_2$. We get:

$$91^2F(x_3,y_3,z_3)=x_3^2+\frac{277}{4}x_3^2z_3+1365x_3z_3^2+8281z_3^3-y_3^2z_3.$$

Finally, we eliminate $x_3^2z_3$ term by completing to cube (+ "third of the coefficient"), i.e. $x_4=x_3+\frac{277}{12}z_3$, $y_4=y_3$, $z_4=z_3$, i.e. $x_3=x_4-\frac{277}{12}y_4$, $y_4=y_3$ and $z_4=z_3$. We get:

$$91^2F(x_4,y_4,z_4)=x_4^3-\frac{11209}{48}x_4z_4^2+\frac{1185157}{864}z_4^3-y_4^2z_2.$$

We have the following transformation of variables:

$$\begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix} = \begin{pmatrix} -\frac{95}{2} & -\frac{95}{2} & \frac{277}{12} \\ \frac{91}{2} & -\frac{91}{2} & 0 \\ -6 & -6 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{182} & \frac{1}{91} & -\frac{277}{2184} \\ \frac{1}{182} & -\frac{1}{91} & -\frac{277}{2184} \\ \frac{6}{91} & 0 & -\frac{95}{182} \end{pmatrix} \begin{pmatrix} x_4 \\ y_4 \\ z_4 \end{pmatrix}$$

After setting $z_4 = 1$, we've got equivalent Weierstrass form

$$y^2 = x^3 - \frac{11209}{48}x + \frac{1185157}{864}.$$

To obtain this transformation in `sagemath` program we could use the following commands:

```
R.<x,y,z> = QQ[];
F = x^3 + y^3 + z^3 - 3*x^2*(y+z) -3*y^2*(x+y) - 3*z^2*(x+y)-5*x*y*z;
WeierstrassForm(F)
```

The outcome is

```
(-11209/48, 1185157/864)
```

Next we are trying to find another coefficient to transform this form to the so called minimal form, i.e. $y^2+Axy+By = y^3+Cx+D$. The used transformation is the following $x = u^2x'+r$, $y = su^2x'+u^3y'+t$, with certain values of (u, r, s, t). After substitution we get:

$$\begin{aligned} u^6x^3 + (3r - s^2)u^4x^2 + (-2st + 3r^2 - \frac{11209}{48})u^2x+ \\ + (r^3 - \frac{11209}{48}r + \frac{1185157}{864}) = u^6y^2 + 2su^5xy + 2tu^3y - t^2. \end{aligned} \tag{1}$$

After choosing $u = 1$, $s = \frac{1}{2}$, $r = 1/12$ (to zero coefficient by u^4x^2), we get

$$y^2 + xy + 2ty = x^3 - (\frac{467}{2} + t)x + \frac{5409}{4} - t^2$$

Substituting $t = \frac{1}{2}$ we get the form:

$$y^2 + xy + y = x^3 - 234x + 1352.$$

It is the minimal form obtained by `sagemath` program:

```
H = EllipticCurve(-11209/48, 1185157/864);
H.minimal_model()
```

With result

```
Elliptic Curve defined by y^2 + x*y + y = x^3 - 234*x + 1352 over Rational Field
```

Another possibility.

Equation $r^3 - \frac{11209}{48}r + \frac{1185157}{864} = 0$ has a rational solution $r = \frac{109}{12}$.
Substituting $r = 109/12$ in (1) we get

$$u^6x^3 - s^2u^4x^2 + \frac{109}{4}u^4x^2 - 2stu^2x + 14u^2x - t^2 = u^6y^2 + 2su^5xy + 2tu^3y.$$

Dividing equation by u^6 we get

$$x^3 - \frac{s^2}{u^2}x^2 + \frac{109}{4u^2}x^2 - \frac{2st}{u^4}x + \frac{14}{u^4}x - \frac{t^2}{u^6} = y^2 + 2\frac{s}{u}xy + \frac{2t}{u^3}y.$$

In order to get rid of xy and y terms, we have $s = t = 0$.

$$x^3 + \frac{109}{4u^2}x^2 + \frac{14}{u^4}x = y^2.$$

In order to get integer coefficients we put $u = \frac{1}{2}$ and obtain the final form:

$$y^2 = x^3 + 109x^2 + 224x. \tag{2}$$

Lastly we turn into finding the solution to the original equation.

Checking small negative values of x reveals that $(x, y) = (-4, 28)$ is the point on the curve (2). Such point is transformed to $(x_4, y_4, z_4) = (\frac{1}{4}x + \frac{109}{12}, 1/8y, 1) = (97/12, 7/2, 1)$. Finally, transforming to original coordinates we get that

$$(-4/91, -11/91, 1/91)$$

is the solution. Since the original equation is homogeneous also $(4, 11, -1)$ is the solution (we have chosen "the most positive" one).

Point $2P = (\frac{676}{49}, \frac{55796}{3433}) \mapsto (5165, -8784, 9499)$.
Point $3P = (\frac{-731025}{11881}, \frac{527529870}{1295029}) \mapsto (-375326521, 679733219, 883659076)$
Point $4P = (\frac{102131236}{9534155449}, -\frac{1445821255910884}{930943540506707})$
$x = 6696085890501216$,
$y = 6334630576523495$,
$z = -6531563383962071$
Point $5P = (-\frac{97524655611626884}{1922978696871961}, -\frac{31388186331916061221894348}{84326040682894618812541})$
$x = 5048384306267455380784631$
$y = -2798662276711559924688956$
$z = 5824662475191962424632819$
Point $6P = (x_6, y_6)$, where
$x_6 = \frac{2527858405259639371198721}{1322534768408511595560 0}$,
$y_6 = -\frac{34376465376083164578497028229439456 9}{15209349758988684590003854422960 00}$.
The solution
$x = -39986625862443873723249364624438370 9$,
$y = 28766304889722455433744691834440542 9$,
$z = 43402140409109114078200023459161832 0$.
Point $7P = (x_7, y_7)$, where
$x_7 = -\frac{16190434804910928688794461871168 4}{48593717397294424684495546259881}$,
$y_7 = -\frac{69972020958031305848824257026342684737522178306 12}{33874288772722379300463151452548536334563256322 1}$
The solution:

$x = 33869282463293272597638491845101850314062113248O4,$
$y = 1637627722378544613543242758851617912968156867151,$
$z = -6782669709301339235789161616483503982063541013 81.$

Point $8P = (x_8, y_8)$, where

$x_8 = \frac{647805352455262562383962846512445288 4145956}{1245636875525200464539184202153614229609},$

$y_8 = \frac{16659913412846099183252330384510918009297731314186105439712688964}{439629862403720607099271058167630124229993124898178055 53227}.$

The solution:

$x = 21107606492313258550470889745604686675326161643975201426221 04465,$

$y = -2054217703980198940765993621567260834791816664149006217306067776,$

$z = 3432583032546353432111754845885724305752899389276569722015637 91.$

Point $9P = (x_9, y_9)$, where

$x_9 = -\frac{66202368404229585264842409883878874707453676645038225}{13514400292716288512070907945002943352692578000406921},$

$y_9 = \frac{588008351573080833073767517273471813300856728502967303518717487133079887 00611210}{15710686685979784345563647072918962688380869454300313221967543904202804073 46469}.$

Which corresponds to solution (the desired one with positive coefficients):

$x = 368751317941299998271978115652254748254929799689719709962831374716372246340555 79,$

$y = 1544768021087461664419513150199198374856643256695654317000266348982532020352779 99,$

$z = 4373612677928697257861252602371390152816537558161613618621437993378423467772036.$

Note 1: Computing $2P$, where $P \equiv [-4, 28]$:

$y^2 = x^3 + 109x^2 + 224x \mapsto T(x, y) = x^3 + 109x^2 + 224x - y^2.$

We are computing the tangent at the point P, thus we need partial derivations of T.

$\frac{\partial T}{\partial x} = 3x^2 + 218x + 224;$

$\frac{\partial T}{\partial y} = -2y.$

$\frac{\partial T}{\partial x}(-4, 28) = -600$, $\frac{\partial T}{\partial y}(-4, 28) = -56$, thus the tanget line at P is $-600(x+4) - 56(y - 28) = 0$, which by dividing $-\gcd(600, 56, 832) = -8$ is equivalent to:

$$75x + 7y + 104 = 0.$$

Thus $y = \frac{-104-75x}{7}$, we get equation:

$$x^3 - \frac{284}{49}x^2 - \frac{4624}{49}x - \frac{10816}{49} = 0.$$

We know that -4 is double root of this equation, thus $(-4)+(-4)+x_3 = \frac{284}{49}$. Thus $x_3 = 676/49$.

$y_3 = \frac{-104-75x}{7} = -\frac{55796}{343}$. In order to get $2P$ we need to flip obtained point around x axis.

$$2P = \left[\frac{676}{49}, \frac{55796}{343}\right].$$

Note 2: Computing $3P = P + 2P$, where $P \equiv [-4, 28]$.

We need to find intersection of the line going through P and $2P$ with the elliptic curve.

The line is $y = y_P + \frac{y_{2P}-y_P}{x_{2P}-x_P}(x - x_P)$.

$y = \frac{5774}{763}x + \frac{44460}{763}$. After substitution into T we get:

$$x^3 + \frac{30117345}{582169}x^2 - \frac{383018224}{582169}x - \frac{1976691600}{582169} = 0.$$

Again x_P and x_{2P} are roots of this equation thus the third root is $x_3 = -\frac{30117345}{582169} - x_P - x_{2P} = -\frac{731025}{11881}$. $y_3 = -\frac{527529870}{1295029}$. Thus (after flip over axis x)

$$P + 2P = \left(-\frac{731025}{11881}, \frac{527529870}{1295029}\right).$$

Note 3: Commands in `sagemath`:

```
// (a1, a2, a3, a4,a5) -> y^2 + a1*xy +a3*y = x^3 + a2*x^2 + a4*x + a5
E = EllipticCurve([0, 109, 0, 224, 0]);
E
P = E(-4,28);
P
2*P
3*P
P + 2*P
```

The output is

```
Elliptic Curve defined by y^2 = x^3 + 109*x^2 + 224*x over Rational Field
(-4 : 28 : 1)
(676/49 : 55796/343 : 1)
(-731025/11881 : 527529870/1295029 : 1)
(-731025/11881 : 527529870/1295029 : 1)
```

References

[1] https://www.quora.com/How-do-you-find-the-integer-solutions-to-frac-x-y+z-+-frac-y-z+x-+-frac-z-x+y-4

[2] http://www.sagemath.org/