Rényi relative entropies and sufficiency of quantum channels

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Dedicated to the memory of Dénes Petz

Classical Rényi relative entropies

For p, q probability measures over a finite set X, $0 < \alpha \neq 1$:

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}$$

In the limit $\alpha \to 1$: relative entropy

$$S(p\|q) = \sum_x p(x) \log(p(x)/q(x))$$

- introduced as the unique family of divergences satisfying a set of postulates
- fundamental quantities appearing in many information - theoretic tasks

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A. Rényi, *Proc. Symp. on Math., Stat. and Probability*, 1961.

Standard quantum Rényi relative entropies

For density matrices ρ, σ, $0 < \alpha \neq 1$,

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\, \rho^\alpha \sigma^{1-\alpha}$$

In the limit $\alpha \to 1$: quantum (Umegaki) relative entropy

$$S(\rho\|\sigma) = \mathrm{Tr}\, \rho(\log(\rho) - \log(\sigma))$$

- fundamental in quantum information theory
- obtained from Petz quasi-entropies[1,2]
- defined for normal states of a von Neumann algebra, using the relative modular operator

[1] D. Petz, *Rep. Math. Phys., 1986*

[2] D. Petz, Publ. RIMS, Kyoto Univ., 1985

Standard quantum Rényi relative entropies

It follows from the properties of quasi-entropies that: if $\alpha \in (0, 2]$

- strict positivity: $D_\alpha(\rho\|\sigma) \geq 0$ with equality iff $\rho = \sigma$;
- data processing inequality:

  $$D_\alpha(\rho\|\sigma) \geq D_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

  for any quantum channel Φ
- joint lower semicontinuity
- joint (quasi)-convexity: the map

  $$(\rho, \sigma) \mapsto \begin{cases} \exp\{(\alpha - 1)D_\alpha(\rho\|\sigma)\} & \alpha \neq 1 \\ D_1(\rho\|\alpha) & \alpha = 1 \end{cases}$$

  is jointly convex.

Standard quantum Rényi relative entropies

For $\alpha \in (0, 1]$, D_α have an operational significance:
as error exponents in quantum hypothesis testing

F. Hiai and D. Petz,
M. Hayashi
H. Nagaoka

They serve as parent quantities for other important information measures

Equality in DPI: sufficient quantum channels

Let the quantum states ρ, σ and a channel Φ be such that

$$S(\Phi(\rho)\|\Phi(\sigma)) = S(\rho\|\sigma) < \infty$$

This condition was introduced as a quantum extension of classical sufficient statistics:

A statistic T is sufficient with respect to a pair of probability distributons $\{p, q\}$ if

- the conditional expectation satisfies $E_p[\cdot|T] = E_q[\cdot|T]$
- an equivalent Kullback-Leibler characterization by the classical relative entropy:

$$S(p^T\|q^T) = S(p\|q) < \infty$$

D. Petz,

Sufficient quantum channels

Let ρ, σ be quantum states (normal states of a von Neumann algebra), σ faithful. Assume that $S(\rho\|\sigma) < \infty$

The following are equivalent.

- $S(\rho\|\sigma) = S(\Phi(\rho)\|\Phi(\sigma))$;
- There is a quantum channel Ψ such that

$$\Psi \circ \Phi(\rho) = \rho, \qquad \Psi \circ \Phi(\sigma) = \sigma$$

We say in this case that Φ is sufficient with respect to $\{\rho, \sigma\}$.

D. Petz,

D. Petz,

Sufficient quantum channels: divergences

A divergence D characterizes sufficiency if

$$D(\Phi(\rho)\|\Phi(\sigma)) = D(\rho\|\sigma) < \infty$$

implies that Φ is sufficient with respect to $\{\rho, \sigma\}$.

The following divergences characterize sufficiency:

- relative entropy
- D_α, with $\alpha \in (0,1)$[3,4]
- a class of f-divergences (in finite dimension)[5,6]

[3] D. Petz,
[4] AJ and D. Petz
[5] Hiai, Mosonyi, Petz, Bény
[6] F. Hiai, M. Mosonyi

Sufficient quantum channels: universal recovery channel

Assume that σ, $\Phi(\sigma)$ is faithful. The Petz recovery channel is defined as

$$\Phi_\sigma(Y) = \sigma^{1/2}\Phi^*(\Phi(\sigma)^{-1/2}Y\Phi(\sigma)^{-1/2})\sigma^{1/2}$$

Note that we always have $\Phi_\sigma \circ \Phi(\sigma) = \sigma$.

Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if

$$\Phi_\sigma \circ \Phi(\rho) = \rho.$$

Sufficient quantum channels: a conditional expectation

Note that by the last condition, ρ (and σ) must be invariant states of the channel $\Phi_\sigma \circ \Phi$.

There is a conditional expectation E, $\sigma \circ E = \sigma$, such that Φ is sufficient with respect to $\{\rho, \sigma\}$ if and only if $\rho \circ E = \rho$.

Structure ot the states: in finite dimensions (or on $B(\mathcal{H})$), there is a decomposition

$$\sigma = \bigoplus_n \lambda_n \sigma_n^L \otimes \sigma_n^R, \qquad \sigma_n^L, \sigma_n^R \text{ states, } \lambda_n \text{ probabilities}$$

such that Φ is sufficient with respect to $\{\rho, \sigma\}$ iff

$$\rho = \bigoplus_n \mu_n \rho_n^L \otimes \sigma_n^R, \qquad \rho_n^L \text{ states, } \mu_n \text{ probabilities}$$

Sufficient quantum channels: applications

Characterization of equality in various entropic inequalities:

- strong subadditivity: characterization of quantum Markov states[7]
- convexity of f-divergences, Minkowski-type inequalities
- monotonicity of quantum Fisher information, Holevo quantity, etc.

Approximate version[8,9]:

$$S(\rho\|\sigma) - S(\Phi(\rho)\|\Phi(\sigma)) \geq d(\rho\|\tilde{\Phi}_\sigma \circ \Phi(\rho))$$

d some "distence", $\tilde{\Phi}_\sigma$ a modification of Petz recovery channel

[7] Hayden, Josza, Petz, Winter

[8] O. Fawzi and R. Renner

[9] ...