JOURNAL OF MATHEMATICAL PHYSICS VOLUME 43, NUMBER 5 MAY 2002

Quantum information geometry and standard purification

Anna Jenčová[a)]
Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

(Received 30 November 2001; accepted for publication 14 January 2002)

We investigate relations between Uhlmann's parallelism, monotone Riemannian metrics and dual affine connections on the space of density matrices. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1467966]

I. INTRODUCTION

On the space of density operators, several differential geometrical structures were defined. In finite dimensional case, the class of monotone Riemannian metrics was specified by Petz in Ref. 1, using the set of real symmetric operator monotone functions. These metrics are the quantum generalizations of the classical Fisher metric, which is, together with the class of dual α-connections, the basic structure of the classical information geometry.

The quantum counterpart of the classical exponential and mixture connections was defined by Nagaoka in Ref. 2 and this definition was generalized to α-connections for $\alpha \in [-3,3]$ in Ref. 3.

In Ref. 4, the geometric phase of curves of pure states was extended to mixed states, using purifying lifts of curves of density matrices. Among such lifts, the one with the least Hilbert space length is used to define the phase. As it turns out, this defines a connection form a^{geo} in the space $\mathcal{W}$ of Hilbert–Schmidt operators, which is the $U(n)$–principal bundle over the space of non-normalized density operators $\mathcal{M}$, here n is the dimension of the underlying Hilbert space $\mathcal{H}$. Later, in Ref. 5 a class of such connection forms was introduced, containing the above form a^{geo}.

It is natural to ask what the relations between the above structures are. In Ref. 6 it was shown that the connection forms are naturally linked with some Riemannian or Hermitian metrics on the purifying space $\mathcal{W}$. Such metrics can be projected onto $\mathcal{M}$ and all the monotone metrics are obtained in this way. Further, in Ref. 7, a parallel transport, respecting the connection form a^{geo} was defined on $\mathcal{W}$, such that, if projected onto $\mathcal{M}$, it coincides with Nagaoka's (e)-connection. The aim of the present paper is to obtain all the affine α-connections and their duals in a similar way. For the sake of simplicity, we will assume that the underlying Hilbert space is finite dimensional and the density operators are invertible.

Section II below gives a brief description of purification and some formulas to be used later. In Sec. III, we define the class of connection forms and Riemannian metrics on $\mathcal{W}$ and relate them to the class of monotone metrics on $\mathcal{M}$ as in Ref. 6. In Secs. IV and V, we use the duality of two real subspaces in the tangent space to $\mathcal{W}$ to define two dual parallel transports, induced from the trivial affine connection on $\mathcal{B}=\mathcal{B}(\mathcal{H})$. We show that the projections of such parallel transports onto $\mathcal{M}$ coincide with the α-connection and its dual. Finally, we define a natural notion of duality of Riemannian metrics on $\mathcal{M}$.

II. THE PURIFICATION PROCEDURE

Let $\mathcal{H}$ be Hilbert space (not necessarily of finite dimension) and let $\mathcal{B}=\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators acting on $\mathcal{H}$. Let $\mathcal{K}$ be another Hilbert space, with $\dim(\mathcal{H}) \leq \dim(\mathcal{K})$ and let

$$\mathcal{A}=\mathcal{B}(\mathcal{H}\otimes\mathcal{K}).$$

[a)]Electronic mail: jenca@mat.savba.sk

Let us consider the *-representation $\phi:\mathcal{B}\to\mathcal{A}$, given by $b\mapsto b\otimes 1_{\mathcal{K}}$. Let ρ be a positive linear functional on $\mathcal{B}$. The vector $\xi\in\mathcal{H}\otimes\mathcal{K}$ is said to be a purification of ρ if for each $b\in\mathcal{B}$

$$\rho(b)=\mathrm{Tr}\rho b=\langle\xi,\phi(b)\xi\rangle=\mathrm{Tr}b\otimes 1 P_{\xi},$$

here the density operator was also denoted by ρ and P_{ξ} is the orthogonal projector onto the subspace spanned by ξ. It is clear that ξ is a purification of ρ if

$$\rho=\mathrm{Tr}_2 P_{\xi},$$

where Tr_2 is the partial trace over $\mathcal{K}$.

In the sequel, we will use the choice $\mathcal{K}=\mathcal{H}^*$. The Hilbert space $\mathcal{H}\otimes\mathcal{H}^*$ can be identified with the space of Hilbert–Schmidt operators over $\mathcal{H}$, with the Hilbert–Schmidt inner product

$$\langle w_1,w_2\rangle=\mathrm{Tr}w_1^*w_2.$$

This identification is easily seen if we use Dirac's notations, then the product vectors are identified with

$$\varphi\otimes\psi^*\mapsto|\varphi\rangle\langle\psi|,$$

where $\psi^*(\varphi)=\langle\psi,\varphi\rangle$. The operation $w\mapsto w^*$ on $\mathcal{W}$ corresponds to the anti-unitary operator J on $\mathcal{H}\otimes\mathcal{H}^*$, given by

$$J(\varphi\otimes\psi^*)=\psi\otimes\varphi^*.$$

The representation ϕ is equivalent to $b\mapsto L_b$, where $L_b(w)=bw$ is the left multiplication operator on $\mathcal{W}$. Then $w\in\mathcal{W}$ is a purification of ρ if

$$\rho(b)=\mathrm{Tr}b\rho=\langle w,L_bw\rangle=\mathrm{Tr}w^*bw=\mathrm{Tr}b\,\pi(w),$$

for each $b\in\mathcal{B}$, where $\pi(w)=ww^*$.

In a similar way, we may represent the algebra $\mathcal{B}^*=\mathcal{B}(\mathcal{H}^*)$ in $\mathcal{A}$ by

$$\phi':b\mapsto 1\otimes b\mapsto R_{b^*}.$$

Here, $b\in\mathcal{B}^*$ is the linear operator given by $b\varphi^*=(b\varphi)^*$ and $R_b(w)=wb$ is the right multiplication operator on $\mathcal{W}$. Then $\phi'(\mathcal{B}^*)$ is the commutant of $\phi(\mathcal{B})$. If σ is a positive linear functional on $\mathcal{B}^*$, the vector ξ purifies σ if

$$\sigma=\mathrm{Tr}_1P_{\xi}=\mathrm{Tr}_2P_{J\xi},$$

where Tr_1 is the partial trace over $\mathcal{H}$. This is equivalent to

$$\sigma=\pi'(w),\quad \pi'(w)=w^*w$$

where w is the Hilbert–Schmidt operator corresponding to the vector ξ.

Let $w\in\mathcal{W}$ be invertible, $\pi(w)=\rho$, $\pi'(w)=\sigma$ and let $w=\rho^{1/2}u=u\sigma^{1/2}$ be the polar decomposition of w. Then

$$\rho=\sum_i\lambda_i|\varphi_i\rangle\langle\varphi_i|,\quad \sigma=\sum_i\lambda_i|\psi_i\rangle\langle\psi_i|,$$

$\lambda_i>0$ for all i are the spectral decompositions of ρ and σ and $w=\Sigma_i\lambda_i^{1/2}|\varphi_i\rangle\langle\psi_i|$, where $u\psi_i=\varphi_i$.

Following Ref. 6, we will use a certain class of operators from $\mathcal{B}(\mathcal{W})$ to introduce a Riemannian structure in $\mathcal{W}$. Let us denote $L=L_\rho$, $R=R_\rho$, $\widetilde{L}=L_\sigma$ and $\widetilde{R}=R_\sigma$ and let Δ be the modular operator $\Delta=L\widetilde{R}^{-1}$. Then the spectral decomposition of Δ is

$$\Delta=\sum_i \frac{\lambda_i}{\lambda_j}(P_{\varphi_i}\otimes P_{\psi_j^*}),$$

so that if f is a smooth function and $x\in\mathcal{W}$, we have

$$f(\Delta)(x)=\sum_i f\left(\frac{\lambda_i}{\lambda_j}\right)|\varphi_i\rangle\langle\varphi_i|x\psi_j\rangle\langle\psi_j|.$$

Similar expressions are obtained for $f(LR^{-1})$, $f(\widetilde{L}\widetilde{R}^{-1})$, etc.

III. CONNECTION FORMS AND MONOTONE METRICS

Let $\dim(\mathcal{H})=n$. Let $\mathcal{W}_0\subset\mathcal{W}$ be the open subset of invertible operators, then it can be viewed as a real $2n^2$-dimensional differentiable manifold. Let T_w denote the tangent space at w, then T_w can be identified with $\mathcal{W}$ with the structure of a real linear space. Throughout the text, the capital letters X, Y, etc. denote the elements of T_w as differential operators, whereas small letters refer to their representations $x=X(w)$.

Let $\mathcal{M}\subset\mathcal{W}_0$ be the submanifold of (non-normalized) density matrices, i.e., the positive complex matrices. Then $\mathcal{W}_0$ can be realized as the principal bundle with base space $\mathcal{M}$ and structure group $U(n)$. The projection $\pi:\mathcal{W}_0\to\mathcal{M}, \pi(w)=ww^*$ is the canonical projection. Let $T_\rho(\mathcal{M})$ be the tangent space of $\mathcal{M}$ at ρ. The canonical projection induces a linear map $T_w\to T_{\pi(w)}(\mathcal{M})$ (also denoted by π)

$$\pi(x)=xw^*+wx^*.$$

The vertical subspace $V_w\subset T_w$ is given by the condition

$$x\in V_w\Leftrightarrow\pi(x)=0.$$

It is easy to see that

$$V_w=\{wa:a+a^*=0\}.$$

We will describe the class of connections defined in Refs. 5 and 6. If a connection is defined in $\mathcal{W}_0$, the tangent space T_w is decomposed as the direct sum of V_w and a horizontal subspace H_w. For each vector $x\in T_w$ there is a unique decomposition $x=x^h+x^v$ into its horizontal and vertical part. We introduce a Riemannian structure in $\mathcal{W}_0$ as follows. Let $(\cdot,\cdot)$ denote the real part of the Hilbert–Schmidt inner product, i.e.,

$$(x,y)=\Re\langle x,y\rangle=\tfrac{1}{2}\mathrm{Tr}(x^*y+y^*x).$$

Let $k:\mathbb{R}^+\to\mathbb{R}^+$ be a smooth function such that $k(1)=1$ and let Δ be the modular operator. Let us define

$$(x,y)_w=(x,k(\Delta)^{-1}(y)),\quad x,y\in T_w.$$

The horizontal subspace H_w is defined as the orthogonal complement of V_w with respect to $(\cdot,\cdot)_w$. A tangent vector x is then horizontal iff it has the form

$$x=k(\Delta)(gw)=k(L/R)(g)w,\quad g=g^*.$$

The corresponding connection form is given by

$$\omega = r(\tilde{L}/\tilde{R})(w^{-1}dw) - r(\tilde{R}/\tilde{L})(w^{-1}dw)^* \tag{1}$$

$$= w^{-1}dw - (w^{-1}dw)^* + w^{-1}(F(L/R)d\rho)(w^{-1})^*, \tag{2}$$

where r is a smooth positive function with $r(t)+r(t^{-1})=1$ and $F(t)=r(t)-r(t^{-1})$. The functions r and k are connected by

$$r(t) = \frac{tk(t^{-1})}{k(t)+tk(t^{-1})}.$$

The above Riemannian structure can be projected onto the base space $\mathcal{M}$ via the projection π: Let $\pi(w)=\rho$ and let $h \in T_\rho(\mathcal{M})$. Then there exists a unique vector $\hat{h} \in T_w$ such that $\hat{h}$ is horizontal and $\pi(\hat{h})=h$, it is called the horizontal lift of h to w. We have

$$\hat{h} = \tfrac{1}{2}R^{-1}(1-F(L/R))(h)w = r(R/L)(h)(w^*)^{-1}.$$

Let us define

$$\langle h,k\rangle_\rho = 4(\hat{h},\hat{k})_w \quad h,k \in T_\rho(\mathcal{M}). \tag{3}$$

It is easy to see that the right-hand side depends only on $ww^*=\rho$, so that this defines a Riemannian structure on $\mathcal{M}$. We are particularly interested in Riemannian metrics that are monotone with respect to completely positive, trace preserving maps. It was proved by Petz in Ref. 1, that such metrics are of the form

$$\langle h,k\rangle_\rho = \mathrm{Tr}\, hJ_\rho(k) \quad J_\rho = R^{-1}f(L/R)^{-1}, \tag{4}$$

where f is an operator monotone function which is symmetric, i.e., it satisfies $f(t)=tf(t^{-1})$. If we require that this metric coincides with the Fisher information metric on commutative submanifolds, we need an additional condition $f(1)=1$. The last requirement also gives rise to the multiplication factor 4 in (3).

If the induced Riemannian metric on $\mathcal{M}$ is of the form (4) for some smooth positive function f, then we have

$$k(t) = f(t)(1-F(t)), \quad f(t) = \tfrac{1}{2}(k(t)+tk(t^{-1})). \tag{5}$$

It is easy to see that given a monotone Riemannian metric, the connection that induces it is not unique. On the other hand, as the connection depends only from $k(t)/k(t^{-1})$, the induced Riemannian metric is again not unique. To get uniqueness, one more condition is needed. A natural requirement is that the inner product $(\cdot,\cdot)_w$ in T_w coincides with the real part of the Hilbert–Schmidt inner product, if restricted to horizontal vectors. It was proved in Ref. 6 that for each operator monotone function f there is a unique positive function k such that the above condition is satisfied. Let us put

$$p(t) = \frac{k(t)}{k(t^{-1})},$$

then $k(t)=\sqrt{p(t)}q(t)$, where the function $q(t)=q(t^{-1})$ is uniquely given by the condition $(x,k(\Delta)^{-1}y)=(x,y)$ for $x,y \in H_w$. Thus we have

Theorem 3.1: *For each monotone metric $\langle\cdot,\cdot\rangle_\rho$ on $\mathcal{M}$, there exists a unique smooth function $p:\mathbb{R}^+ \to \mathbb{R}^+$ satisfying $p(t^{-1})=(p(t))^{-1}$, such that* *(i)* *if*

$$r(t)=\frac{t}{p(t)+t},$$

then the corresponding connection form induces the given monotone metric;

(ii) *if*

$$k(t)=p(t)\frac{p(t)+t}{p(t)^2+t}, \tag{6}$$

then $p(t)=k(t)/k(t^{-1})$ *and the corresponding connection is the same as in* (i). *Moreover*, $(x,y)_w=(x,y)$ *for all horizontal vectors* x, y;

(iii) *the corresponding operator monotone function is of the form*

$$2\,f(t)=\frac{(p(t)+t)^2}{p(t)^2+t}.$$

In the sequel, we will always suppose that k and p are smooth functions related by Eq. (6).

Example 3.1: Let $p(t)=1$. Then $f(t)=(1+t)/2$ is the largest symmetric operator monotone function. It is related to the Bures metric, J_ρ is the symmetric logarithmic derivative (SLD). Here, $k(t)=1$ and $r(t)=t/(1+t)$. The horizontal subspace H_w consists of vectors of form $x=gw$, $g=g^*$. See Refs. 8 and 9.

Example 3.2: For the smallest symmetric operator monotone function $f(t)=2t/(1+t)$, we have $p(t)=t$, $k(t)=2t/(1+t)$ and $r(t)=\frac{1}{2}$. The vector $x\in H_w$ if and only if $x=2L/(L+R)(g)w$ for $g=g^*$. This is equivalent to $x=wh$ for some $h=h^*$. In this case, $J_\rho(h)=\frac{1}{2}(h\rho^{-1}+\rho^{-1}h)$ and the monotone metric is called the metric of the right logarithmic derivative (RLD). The connection form given by r is the canonical connection form on the bundle $GL(n)/U(n)$.[10]

Example 3.3: For $p(t)=\sqrt{t}$, we obtain the operator monotone function $f(t)=(1+\sqrt{t})^2/4$. The corresponding monotone metric is the smallest in a special class of monotone metrics, defined in Refs. 11 and 12. This class is important in the context of quantum information geometry, see Ref. 13. It contains also the monotone metric from the previous example. Here, $k(t)=(1+\sqrt{t})/2$ and $r(t)=\sqrt{t}/(1+\sqrt{t})$. The horizontal vectors for $w=\sqrt{ww^*}u$ are of the form $x=gu$, $g=g^*$.

IV. DUAL AFFINE CONNECTIONS

For the Riemannian manifold $\mathcal{M}$ with a monotone metric, Nagaoka in Ref. 2 defined the quantum version of the exponential and mixture connections of the classical information geometry. There, the mixture connection ((m)-connection) coincides with the trivial affine structure on $\mathcal{M}$ and the exponential (e)-connection is its dual with respect to the given Riemannian structure. The corresponding parallel transports of a vector $h\in T_{\rho_0}(\mathcal{M})$ along a curve $\rho\in\mathcal{M}$, with $\rho(0)=\rho_0$, $\rho(1)=\rho_1$, are given by

$$\Pi_\rho^{(m)}(h)=h,$$

$$J_{\rho_1}(\Pi_\rho^{(e)}(h))=J_{\rho_0}(h).$$

The aim of this section is to show that this structure on $\mathcal{M}$ appears naturally in the context of Sec. III.

Let us fix a monotone metric on $\mathcal{M}$ and let k be as in theorem 3.1. To define the two dual connections, it is convenient to use a different characterization of the horizontal subspace, given in Ref. 6. Let us consider the space

$$T_w=\{gw\ :\ g\in\mathcal{B}\}=\{wg\ :\ g\in\mathcal{B}\},$$

with the complex Hilbert space structure, $\langle x,y\rangle_w = \langle x,k(\Delta)^{-1}(y)\rangle$, ($\langle \cdot , \cdot \rangle$ is the Hilbert–Schmidt inner product.) Let S and F be the conjugate linear operators of the Tomita–Takesaki theory, given by $S(gw)=g^*w$, $F(wg)=wg^*$, $S=J\Delta^{1/2}=\Delta^{-1/2}J$ and $F=\Delta^{1/2}J=J\Delta^{-1/2}$, where J is the modular conjugation. Let S^k be the modification of S given by

$$S^k(k(\Delta)(gw))=k(\Delta)(g^*w).$$

Then H_w is the fix point set of S^k.

Lemma 4.1:

- *(i)* $S^k=Sp(\Delta)^{-1}=p(\Delta)S$;
- *(ii)* *The adjoint of* S^k *with respect to* $\langle \cdot , \cdot \rangle_w$ *is* F;
- *(iii)* *The polar decomposition of* S^k *is* $S^k=J^k(\Delta^k)^{1/2}$, *where*

$$J^k=Jp(\Delta)^{-1/2}=p(\Delta)^{1/2}J \quad \Delta^k=p(\Delta)^{-1}\Delta,$$

- *(iv)* *let* H_w^* *be the fix point set of the operator* F. *Then for* $x\in H_w$, $y\in H_w^*$ *we have* $\langle x,y\rangle_w \in \mathbb{R}$.

Proof: (i) Observe first, that $k(\Delta)(gw)=\sqrt{p(\Delta)}(q(L/R)(g)w)$ with $q(t)=q(t^{-1})$, so that $S^k=S^{\sqrt{p}}$. Obviously

$$S^{\sqrt{p}}=p(\Delta)^{1/2}Sp(\Delta)^{-1/2}=p(\Delta)S=Sp(\Delta)^{-1}.$$

The statements (ii) and (iii) are easy to prove. To prove (iv), let $x\in H_w$ and $y\in H_w^*$. Then

$$\begin{aligned}\langle x,y\rangle_w &= \langle S^k(x),y\rangle_w\\ &= \langle F(y),x\rangle_w = \langle y,x\rangle_w = \langle x,y\rangle_w^- \in \mathbb{R}.\end{aligned}$$

□

Let us introduce a linear operator $\mathcal{J}: H_w \to H_w^*$, such that for $x\in \mathcal{H}_w$, $\pi(x)=\pi(\mathcal{J}(x))$. Then $\mathcal{J}$ is well defined and invertible. Indeed, note that the real vector subspace $H_w^*=\{wg, g=g^*\}$ coincides with the horizontal subspace H_w from Example 3.2, so that both $\pi|_{H_w}$ and $\pi|_{H_w^*}$ are isomorphisms.

Further, let $\mathcal{L}_w^k$, $\mathcal{K}_w$ be two linear maps $T_w\to\mathcal{B}$, given by

$$\mathcal{L}_w^k(x)w=k(\Delta)^{-1}(x), \quad \mathcal{K}_w(x)=xw^*.$$

The operators $\mathcal{L}_w^k$ and $\mathcal{K}_w$ are invertible and $\mathcal{L}_w^k(H_w)=\mathcal{B}_h$, $\mathcal{K}_w(H_w^*)=\mathcal{B}_h$, where $\mathcal{B}_h$ is the subset of hermitian operators in $\mathcal{B}$. For $x,y\in T_w$, we have

$$\langle x,y\rangle_w = \langle \mathcal{L}_w^k(x),\mathcal{K}_w(y)\rangle. \tag{7}$$

Let $h\in T_\rho(\mathcal{M})$. The horizontal lift of h to w satisfies

$$\mathcal{L}_w^k(\hat{h})=\tfrac{1}{2}J_\rho(h), \tag{8}$$

$$y=\mathcal{J}(\hat{h}) \Leftrightarrow \mathcal{K}_w(y)=\tfrac{1}{2}h. \tag{9}$$

Proposition 4.1: Let us define the bilinear form $(\cdot,\cdot)_w^\sim$ *on* H_w *by*

$$(x,y)_w^\sim = \langle x,\mathcal{J}(y)\rangle_w,$$

then we have $(x,y)_w^\sim=(x,y)_w$. *Moreover, the operator* $\mathcal{J}$ *coincides with the restriction of*

$$\tfrac{1}{2}(1+\Delta^k)=\tfrac{1}{2}(1+p(\Delta)^{-1}\Delta),$$

to H_w.

Proof: Let $x,y\in H_w$ and let $\pi(x)=h$, $\pi(y)=k$. Then $\mathcal{K}_w(\mathcal{J}(y))=\frac{1}{2}k$ and $\mathcal{L}_w^k(x)=\frac{1}{2}J_\rho(h)$. It follows from (7) that

$$(x,y)_w^{\sim}=\langle x,\mathcal{J}(y)\rangle_w=\tfrac{1}{4}\langle J_\rho(h),k\rangle$$
$$=\tfrac{1}{4}\langle h,k\rangle_\rho=(x,y)_w.$$

Let now $x\in H_w$, then

$$F(1+\Delta^k)(x)=F(1+F)(x)=(1+\Delta^k)(x),$$

so that $\frac{1}{2}(1+\Delta^k)(x)\in H_w^*$. From

$$\langle x,\Delta^k(y)\rangle_w=\langle x,FS^k(y)\rangle_w=\langle S^k(y),S^k(x)\rangle_w=\langle y,x\rangle_w$$

we have that

$$\langle x,\tfrac{1}{2}(1+\Delta^k)(y)\rangle_w=(x,y)_w=(x,y)_w^{\sim}.$$

□

From the above Proposition, we see that we may extend $\mathcal{J}$ to a positive operator on T_w. The inner product $(\cdot,\cdot)_w^{\sim}$ on T_w, given by

$$(x,y)_w^{\sim}=\langle x,\mathcal{J}(y)\rangle_w,$$

corresponds to the complexification of the real Hilbert space H_w with the inner product $(\cdot,\cdot)_w$.

Example 4.1: Let $k=1$. Let $\rho(\theta)$, $\theta\in\Theta\subseteq\mathbb{R}^N$ be a smooth parametrization of the manifold $\mathcal{M}$. Let us denote

$$h_i(\theta)=\frac{\partial}{\partial\theta_i}\rho(\theta).$$

For simplicity, we will omit the indication of the point θ in the sequel.

In Ref. 14, Holevo introduced the symmetric logarithmic derivative L_i^S by

$$h_i=\tfrac{1}{2}(L_i^S\rho+\rho L_i^S),$$

and the right and left logarithmic derivatives

$$h_i=\rho L_i^R,\quad L_i^L=(L_i^R)^*.$$

It is easy to see that in our setting, the horizontal subspace H_w is spanned by the vectors

$$l_i^S=L_i^S w=2\hat{h}_i,\quad i=1,\ldots,N$$

and

$$\mathcal{J}(l_i^S)=l_i^L=L_i^L w,$$

so that H_w^* is the real linear span of $\{l_i^L,\ i=1,\ldots,N\}$, note that this is true for all k. We also have

$$l_i^R=L_i^R w=\Delta^{-1}(l_i^L).$$

Further, Holevo defined the sesquilinear forms in $a,b\in\mathcal{B}$

$$\langle a,b\rangle_\rho=\tfrac{1}{2}\mathrm{Tr}\,(a^*\rho b+a^*b\rho)=(aw,bw)^{\sim},$$

$$\langle a,b\rangle_\rho^+ = \mathrm{Tr}\, a^* b \rho = \langle aw, bw\rangle,$$

$$[a,b]_\rho = i\,\mathrm{Tr}\,(\rho a^* b - \rho b a^*) = \langle aw, i(1-\Delta)(bw)\rangle.$$

Another important notion is the commutation operator $\mathcal{D}_\rho$, given by

$$\langle a, \mathcal{D}_\rho(b)\rangle_\rho = [a,b]_\rho .$$

It is easy to see that the operator $aw \mapsto \mathcal{D}_\rho(a)w$ is equal to

$$i\mathcal{J}^{-1}(1-\Delta) = 2i\frac{1-\Delta}{1+\Delta} = -2iF(\Delta).$$

Example 4.2: Let $k(t)=2t/(1+t)$, then $p(t)=t$, so that $\Delta^k=1$. The horizontal subspace H_w coincides with H_w^* and the operator $\mathcal{J}$ is the identity. Clearly, $\hat{h}_i = \frac{1}{2} l_i^L$, $i=1,\ldots,N$. Let us denote by $\mathcal{J}^S$ the operator $\frac{1}{2}(1+\Delta)$ from the previous example, then $k(\Delta)=(\mathcal{J}^S)^{-1}\Delta$ and we get

$$(l_i^L, l_j^L)_w = \langle l_i^L, l_j^L\rangle_w = \langle l_i^L, \mathcal{J}^S(l_j^R)\rangle = (l_i^L, l_j^R)^{\sim},$$

where $(x,y)^{\sim} = \langle x, \mathcal{J}^S(y)\rangle$.

We can also describe the horizontal and vertical part of a vector in T_w, using the operators F, S^k, and $\mathcal{J}$ as follows. Let P be the orthogonal projector from T_w onto H_w with respect to $(\cdot,\cdot)_w$, then

$$P = \tfrac{1}{2}\mathcal{J}^{-1}(1+F).$$

Indeed, for each $x \in T_w$, we have $\frac{1}{2}(1+F)(x) \in H_w^*$ and $\pi(\frac{1}{2}(1+F)(x)) = \pi(x) = \pi(x^h)$. It follows that:

$$\tfrac{1}{2}\mathcal{J}^{-1}(1+F)(x) = x^h = Px.$$

From this, we have

$$x^v = x - x^h = \tfrac{1}{2}\mathcal{J}^{-1}(\Delta^k - F)(x) = \tfrac{1}{2}\mathcal{J}^{-1}F(S^k-1)(x).$$

We will now define dual affine connections in $\mathcal{W}_0$. The space $\mathcal{B}$ is a differentiable manifold and $\mathcal{W}_0$ is an open submanifold in $\mathcal{B}$. Let us consider the trivial affine connection in $\mathcal{B}$. The maps $\mathcal{L}_w^k : T_w \to T_w(\mathcal{B})$ and $\mathcal{K}_w : T_w \to T_w(\mathcal{B})$ induce affine connections in $\mathcal{W}_0$. Let Π^* and Π be the corresponding parallel transports and let γ be a smooth curve in $\mathcal{W}_0$, $\gamma(0)=w_0$, $\gamma(1)=w_1$. Then for $X \in T_w$,

$$\mathcal{K}_{w_1}(\Pi_\gamma(X)) = \mathcal{K}_{w_0}(X),$$

$$\mathcal{L}_{w_1}^k(\Pi_\gamma^*(X)) = \mathcal{L}_{w_0}^k(X).$$

The corresponding covariant derivatives ∇ and ∇^* are given by

$$\mathcal{K}_w(\nabla_X Y) = X\mathcal{K}_w(Y) = \mathcal{K}_w(XY) + yx^*,$$

$$\mathcal{L}_w^k(\nabla_X^* Y) = X\mathcal{L}_w^k(Y),$$

where X,Y are smooth vector fields on $\mathcal{W}_0$.

It is clear from (7) that $\langle X,Y\rangle_{w_0} = \langle \Pi_\gamma(X), \Pi_\gamma^*(Y)\rangle_{w_1}$ so that the connections ∇ and ∇^* are dual with respect to $\langle\cdot,\cdot\rangle_w$.

As we are interested in the projections onto $\mathcal{M}$, we restrict the parallel transports to horizontal vectors. From $\mathcal{L}_w^k(H_w)=\mathcal{B}_h$, we see that the restriction of Π^* to the bundle $\cup\{(w,H_w),\ w\in\mathcal{W}_0\}$ is induced from the trivial parallel transport restricted to $\cup\{(b,\mathcal{B}_h),\ b\in\mathcal{B}\}$. It means that the parallel transport of a horizontal vector stays horizontal. This is not necessarily true for Π. On the other hand, the above restriction of the trivial connection on $\mathcal{B}$ induces the restriction of Π to the bundle $\cup\{(w,H_w^*),\ w\in\mathcal{W}_0\}$. We may therefore define the parallel transport Π^h for $x\in H_w$ by

$$\Pi^h(x)=\mathcal{J}^{-1}(\Pi(\mathcal{J}(x))).$$

Then the parallel transports Π^* and Π^h are dual with respect to $(\cdot,\cdot)_w$. Indeed, if $x,y\in H_w$,

$$\begin{aligned}(\Pi_\gamma^*(x),\Pi_\gamma^h(y))_{w_1}&=\langle\Pi_\gamma^*(x),\Pi_\gamma(\mathcal{J}(y))\rangle_{w_1}\\&=\langle x,\mathcal{J}(y)\rangle_{w_0}=(x,y)_{w_0}.\end{aligned}$$

Note also that Π^h coincides with the horizontal part of Π.

Proposition 4.2: Let γ be a smooth curve in $\mathcal{W}_0$, $\gamma(0)=w_0$, $\gamma(1)=w_1$, where w_0 and w_1 are any elements of $\mathcal{W}_0$, satisfying $\pi(w_0)=\rho_0$ and $\pi(w_1)=\rho_1$. Let $h\in T_{\rho_0}(\mathcal{M})$ and let $\hat{h}$ be the horizontal lift to w_0. Then

$$\Pi_{\pi(\gamma)}^{(m)}(h)=\pi(\Pi_\gamma(\hat{h}))=\pi(\Pi_\gamma^h(\hat{h})),$$

$$\Pi_{\pi(\gamma)}^{(e)}(h)=\pi(\Pi_\gamma^*(\hat{h})).$$

Proof: Follows easily from (8) and (9). □

Let H,K be vector fields on $\mathcal{M}$ and let $\hat{H},\hat{K}$ be horizontal lifts. From the above Proposition, it also follows that $\pi(\nabla_{\hat{H}}^*\hat{K})=\nabla_H^{(e)}K$, so that $\nabla_{\hat{H}}^*\hat{K}$ is a horizontal lift of $\nabla_H^{(e)}K$. Similarly, $\nabla_{\hat{H}}^h\hat{K}$ is the horizontal lift of $\nabla_H^{(m)}K$.

V. THE α-CONNECTIONS

In Ref. 3, the α-connections on $\mathcal{M}$ were defined as generalizations of the (e) and (m)-connections for $\alpha\in[-3,3]$. There, the α-connection was given by the trivial affine structure on $g_\alpha(\mathcal{M})$, where

$$g_\alpha(t)=\begin{cases}\dfrac{2}{1-\alpha}t^{(1-\alpha)/2} & \alpha\neq 1\\ \log t & \alpha=1\end{cases}.$$

The connection was determined using the differentiation operators $L_\alpha[\rho]:T_\rho(\mathcal{M})\to T_{g_\alpha(\rho)}(\mathcal{M})$, given by

$$L_\alpha[\rho](H)=\frac{d}{dt}g_\alpha(\rho+tH)|_{t=0}.$$

The corresponding α-parallel transport from ρ_0 to ρ_1 along a curve ρ is

$$L_\alpha[\rho_1](\Pi_\rho^{(\alpha)}(X))=L_\alpha[\rho_0](X),\quad X\in T_{\rho_0}(\mathcal{M}).$$

The parallel transport $\Pi_\rho^{(\alpha)*}$, dual with respect to the given monotone metric is given by

$$L_\alpha^{-1}[\rho_1]J_{\rho_1}(\Pi_\rho^{(\alpha)*}(X))=L_\alpha^{-1}[\rho_0]J_{\rho_0}(X).$$

Again, we will show that the above affine connections are closely related to the class of connection forms from Sec. III. To this end, we define the projection π^α from $\mathcal{W}_0$ to $\mathcal{M}$ by

$$\pi^\alpha(w)=g_\alpha(ww^*).$$

Let $\rho=ww^*$. The induced linear map $\pi^\alpha:T_w\to T_{\pi^\alpha(w)}(\mathcal{M})$ is

$$\pi^\alpha(x)=L_\alpha[\rho](xw^*+wx^*).$$

The operator $L_\alpha[\rho]$ is invertible, therefore,

$$\pi^\alpha(x)=0\Leftrightarrow\pi(x)=0,$$

so that we have the same vertical subspace V_w. Hence we have the horizontal subspace H_w and the dual space H_w^* as in the previous section. Let $k\in T_{g_\alpha(\rho)}(\mathcal{M})$ and let us denote $\hat{k}^\alpha$ the horizontal lift of k with respect to π^α, i.e. a horizontal vector such that $\pi^\alpha(\hat{k}^\alpha)=k$. Let $h\in T_\rho(\mathcal{M})$. The horizontal lifts with respect to π and π^α are connected by

$$\hat{h}=\hat{k}^\alpha\Leftrightarrow k=L_\alpha[\rho](h). \tag{10}$$

In the sequel, we extend the differential operator $L_\alpha[\rho]$ to all of $\mathcal{B}$. We define the two maps $\mathcal{L}_w^\alpha,\mathcal{K}_w^\alpha:T_w\to\mathcal{B}$ by

$$\mathcal{L}_w^\alpha(X)=L_\alpha^{-1}[\rho](\mathcal{L}_w(X)),$$

$$\mathcal{K}_w^\alpha(X)=L_\alpha[\rho](\mathcal{K}_w(X)).$$

The operators $\mathcal{K}_w^\alpha$ and $\mathcal{L}_w^\alpha$ have similar properties as $\mathcal{K}_w$ and $\mathcal{L}_w$. We also have

$$\langle x,y\rangle_w=\langle\mathcal{L}_w(X),\mathcal{K}_w(Y)\rangle=\langle\mathcal{L}_w^\alpha(X),\mathcal{K}_w^\alpha(Y)\rangle. \tag{11}$$

The horizontal lift $\hat{h}^\alpha$ of a vector $h\in T_{\pi^\alpha(w)}(\mathcal{M})$ satisfies the conditions

$$\mathcal{L}_w^\alpha(\hat{h}^\alpha)=\tfrac{1}{2}L_\alpha^{-1}[\rho]J_\rho L_\alpha^{-1}[\rho](h)=:\tfrac{1}{2}K_\alpha(h), \tag{12}$$

$$y=\mathcal{J}(\hat{h}^\alpha)\Leftrightarrow\mathcal{K}_w^\alpha(y)=\tfrac{1}{2}h. \tag{13}$$

The induced inner product in $T_{\pi^\alpha(w)}(\mathcal{M})$ is then

$$4(\hat{h}^\alpha,\hat{k}^\alpha)_w=4\langle\hat{h}^\alpha,\mathcal{J}(\hat{k}^\alpha)\rangle_w$$
$$=4\langle\mathcal{L}_w^\alpha(\hat{h}^\alpha),\mathcal{K}_w^\alpha(\mathcal{J}(\hat{k}^\alpha))\rangle=\langle K_\alpha(h),k\rangle.$$

This coincides with the inner product, induced from $\langle\cdot,\cdot\rangle_\rho$ in the α-representation of the cotangent space, see Ref. 3.

As in the previous section, we use the trivial affine connection on $\mathcal{B}$ and the maps $\mathcal{K}_w^\alpha$ and $\mathcal{L}_w^\alpha$ to obtain the parallel transports Π^α and $\Pi^{\alpha*}$ on $\mathcal{W}_0$. From Eq. (11) it follows that the parallel transports are dual. The parallel transports Π^α and $\Pi^{\alpha*}$ can again be restricted to the bundles $\cup\{(w,H_w^*),\ w\in\mathcal{W}_0\}$ and $\cup\{(w,H_w),\ w\in\mathcal{W}_0\}$, respectively. We put

$$\Pi^{\alpha h}(x)=\mathcal{J}^{-1}\Pi^\alpha(\mathcal{J}(x)),$$

for $x\in H_w$. As before, this corresponds to the horizontal part of Π^α and the parallel transports $\Pi^{\alpha*}$ and $\Pi^{\alpha h}$ are dual with respect to $(\cdot,\cdot)_w$.

In the next Proposition, we show how these connections are projected onto $\mathcal{M}$.

Proposition 5.1: Let γ be a smooth curve in $\mathcal{W}_0$, $\gamma(0)=w_0$, $\gamma(1)=w_1$, where w_0 and w_1 are any elements of $\mathcal{W}_0$, satisfying $\pi(w_0)=\rho_0$ and $\pi(w_1)=\rho_1$. Let $h\in T_{\rho_0}(\mathcal{M})$. Then

$$\Pi^{(\alpha)}_{\pi(\gamma)}(h)=\pi(\Pi^{\alpha}_{\gamma}(\hat{h}))=\pi(\Pi^{\alpha h}_{\gamma}(\hat{h})),$$

$$\Pi^{(\alpha*)}_{\pi(\gamma)}(h)=\pi(\Pi^{\alpha*}_{\gamma}(\hat{h})).$$

Proof: The proof follows from (10), (12), (13) and the definition of the (α)- and (α^*)-parallel transports. □

From the above Proposition, it also follows that $\nabla^{\alpha h}_{\hat{H}}\hat{K}$ and $\nabla^{\alpha*}_{\hat{H}}\hat{K}$ are the π-horizontal lifts of $\nabla^{(\alpha)}_{H}K$ and $\nabla^{(\alpha*)}_{H}K$, respectively.

Let us now compute the Riemannian curvature of the α-connections.

Proposition 5.2: Let R^{α} and $R^{\alpha*}$ be the Riemannian curvature tensor of ∇^{α} and $\nabla^{\alpha*}$. Then

$$R^{\alpha}=R^{\alpha*}=0.$$

Proof: We have for smooth vector fields X,Y,Z on $\mathcal{W}$

$$\begin{aligned}\mathcal{L}^{\alpha}(R^{\alpha*}(X,Y)Z)&=\mathcal{L}^{\alpha}(\nabla^{\alpha}_{X}\nabla^{\alpha}_{Y}Z-\nabla^{\alpha}_{Y}\nabla^{\alpha}_{X}Z-\nabla^{\alpha}_{[X,Y]}Z)\\&=X\mathcal{L}^{\alpha}(\nabla^{\alpha}_{Y}Z)-Y\mathcal{L}^{\alpha}(\nabla^{\alpha}_{X}Z)-[X,Y]\mathcal{L}^{\alpha}(Z)\\&=(XY-YX-[X,Y])\mathcal{L}^{\alpha}(Z)=0.\end{aligned}$$

The statement for R^{α} can be proved similarly. It follows also from duality of the corresponding parallel transports.

□

VI. THE CURVATURE FORM

In this section, we compute the curvature form of the connection ω and show the relation between the curvature form and the torsion of the α-connections.

Let us choose a connection form ω from Sec. III 1. We will compute its curvature form. From the structure equation

$$d\omega=-\tfrac{1}{2}[\omega,\omega]+\Omega,$$

it follows that:

$$\omega[X,Y]=-2\Omega(X,Y), \tag{14}$$

for horizontal vector fields X,Y, see Ref. 10, pp. 81.

Let $w\in\mathcal{W}_0$ and let $\pi(w)=\rho$. Let us define the operator $C_{\rho}:\mathcal{B}\rightarrow\mathcal{B}$, given by

$$C_{\rho}(a)=R^{-1}r(R/L)(a)$$

Proposition 6.1: Let H,K be smooth vector fields on $\mathcal{M}$, such that $[H,K]=0$. Then

$$\begin{aligned}\Omega_w(\hat{H},\hat{K})=-\tfrac{1}{2}w^{-1}(HC_{\rho}(k)\\-KC_{\rho}(h)-[C_{\rho}(h),C_{\rho}(k)]_{\mathcal{B}})w,\end{aligned}$$

where $[a,b]_{\mathcal{B}}=ab-ba$ is the usual commutator on matrices.

Proof: From $\pi([\hat{H},\hat{K}])=[H,K]=0$ and Eq. (14), it follows that:

$$[\hat{H},\hat{K}]=[\hat{H},\hat{K}]^{v}=w\omega([\hat{H},\hat{K}])=-2w\Omega(\hat{H},\hat{K}).$$

Further, the horizontal lift of H is given by the condition

$$\hat{H}(w)=r(R/L)(H(\rho))(w^*)^{-1}=C_\rho(H(\rho))w,$$

so that

$$\hat{H}\hat{K}(w)=\hat{H}[C_\rho(k)]w+C_\rho(k)\hat{H}(w)=(HC_\rho(k)+C_\rho(k)C_\rho(h))w$$

It is now easy to finish the proof.

□

Example 6.1: Let $k=1$. Then $r(t)=t/(1+t)$ and $C_\rho=\frac{1}{2}J_\rho$. Let H,K be vector fields on $\mathcal{M}$, such that $[H,K]=0$. Let us denote $L_H^S=J_\rho(h)$. Then

$$-2w\Omega(\hat{H},\hat{K})w^{-1}=\tfrac{1}{2}(HL_K^S-KL_H^S)-\tfrac{1}{4}[L_H^S,L_K^S]_{\mathcal{B}}.$$

Let T^* be the torsion of the (e)-connection on $\mathcal{M}$, then

$$T^*(H,K)=J_\rho^{-1}(HL_K^S-KL_H^S)=\tfrac{1}{4}[[L_H^S,L_K^S]_{\mathcal{B}},\rho]_{\mathcal{B}},$$

see Refs. 2 and 6. We compute

$$\Omega(\hat{H},\hat{K})=\tfrac{1}{8}w^*J_\rho([L_H^S,L_K^S]_{\mathcal{B}})w.$$

We see that the curvature form $\Omega(H,K)=0$ if and only if the symmetric logarithmic derivatives commute. In this case also the torsion $T^*(H,K)$ vanishes.

Example 6.2: Let $\omega=\omega^{\mathrm{can}}$ be the connection form of the canonical connection (Example 3.2). Then $r=\frac{1}{2}$ and $C_\rho=\frac{1}{2}R^{-1}$. Let us denote $L_H^L=h\rho^{-1}=2C_\rho$. The curvature form is

$$\begin{aligned}\Omega^{\mathrm{can}}(H,K)&=\tfrac{1}{8}w^{-1}[L_K^L,L_H^L]_{\mathcal{B}}w\\&=\tfrac{1}{8}[w^{-1}k(w^*)^{-1},w^{-1}h(w^*)^{-1}]_{\mathcal{B}}.\end{aligned}$$

Again, the curvature form is zero iff the left (or, equivalently, the right) logarithmic derivatives commute. We have $T^*(H,K)=\mathfrak{R}[L_H^L,L_K^L]_{\mathcal{B}}$, so that the torsion also vanishes.

We see that in both examples, there is a close relation between the curvature form of ω and the exponential connection on $\mathcal{M}$: Vanishing of the curvature form implies vanishing of the torsion of the (e)-connection. As we shall see next, this is not always true.

Example 6.3: Let ω be the connection form from Example 3.3. Let $u\in U(n)$ and let us define $\sigma:\mathcal{M}\to\mathcal{W}_0$ by $\sigma(\rho)=\rho^{\frac{1}{2}}u$. Then σ is a global horizontal section. It follows that the curvature form of Ω vanishes. On the other hand, the torsion of the (e)-connection is not always zero, see Ref. 13.

Let now T^α and $T^{\alpha*}$ be the torsion of the connection $\nabla^{\alpha h}$ and $\nabla^{\alpha*}$ on $\mathcal{W}_0$.

Proposition 6.2: Let H,K be vector fields on $\mathcal{M}$ and let $T^{(\alpha*)}$ be the torsion of the connection $\nabla^{(\alpha*)}$ on $\mathcal{M}$. Then the horizontal part of $T^{\alpha*}(\hat{H},\hat{K})$ is the horizontal lift of $T^{(\alpha*)}(H,K)$ and

$$\omega(T^{\alpha*}(\hat{H},\hat{K}))=2\Omega(\hat{H},\hat{K}),$$

where Ω is the curvature form of ω.

Proof: The vector fields $\nabla_{\hat{H}}^{\alpha*}\hat{K}$ and $\nabla_{\hat{K}}^{\alpha*}\hat{H}$ are horizontal lifts of $\nabla_H^{(\alpha*)}K$ and $\nabla_K^{(\alpha*)}H$, so that

$$(T^{\alpha*}(\hat{H},\hat{K}))^h=\nabla_{\hat{H}}^{\alpha*}\hat{K}-\nabla_{\hat{H}}^{\alpha*}\hat{K}-[\hat{H},\hat{K}]^h.$$

But $[\hat{H},\hat{K}]^h$ is the horizontal lift of $[H,K]$, so that

$$\pi(T^{\alpha *}(\hat{H},\hat{K}))=T^{(\alpha *)}(H,K).$$

It also follows that

$$\omega(T^{\alpha *}(\hat{H},\hat{K}))=-\omega([\hat{H},\hat{K}])=2\Omega(\hat{H},\hat{K}).$$

□

Proposition 6.3: Let H, K be as above and let $T^{(\alpha)}$ be the torsion of the connection $\nabla^{(\alpha)}$ on $\mathcal{M}$. Then $T^{\alpha}(\hat{H},\hat{K})$ is vertical, with

$$\omega(T^{\alpha}(\hat{H},\hat{K}))=2\Omega(\hat{H},\hat{K}).$$

Proof: The proof is analogical to the previous proof, using the fact that the connection $\nabla^{(\alpha)}$ on $\mathcal{M}$ is torsion-free.

□

Proposition 6.4: *Let $x,y\in H_w^*$ and let $\overline{T}^{\alpha}$ be the torsion of the connection ∇^{α}. Then $\overline{T}^{\alpha}(X,Y)$ is vertical and*

$$\overline{T}^{\alpha}(X,Y)=2w\Omega^{\mathrm{can}}(X,Y)$$

Proof: Let the horizontal subspace be given by the cannonical connection, then $H_w=H_w^*$ and $\mathcal{J}$ is the identity map. It follows that $\nabla^{\alpha}=\nabla^{\alpha h}$. The vectors $X,Y\in H_w^*$ can be extended to vector fields on $\mathcal{W}_0$, such that $\pi(X)$ and $\pi(Y)$ are smooth vector fields on $\mathcal{M}$. The statement now follows from the previous Proposition.

□

Remark 6.1: If $\alpha=-1$, i.e., $\nabla^{\alpha}=\nabla$, it is easy to compute that

$$\mathcal{K}_w(\overline{T}(X,Y))=yx^*-xy^*,$$

and $\overline{T}(X,Y)$ is, therefore, vertical for all smooth vector fields X,Y.

VII. DUAL RIEMANNIAN METRICS

It follows from symmetry of the representation $\mathcal{B}\otimes 1$ and its commutant $1\otimes\mathcal{B}$ that we may equally well use the projection $\pi':\mathcal{W}_0\to\mathcal{M}$,

$$\pi':w\mapsto w^*w,$$

instead of π. In this case, the induced linear map $T_w\to T_{w^*w}(\mathcal{M})$ is

$$\pi'(x)=x^*w+w^*x.$$

It is easy to show that $\pi'(x)=0$ if xw^{-1} is anti-Hermitian, so that the vertical subspace is given by

$$V_w'=\{aw:a+a^*=0\}.$$

Let the horizontal subspace H_w' be the orthogonal complement of V_w' with respect to

$$(x,y)_w'=(x,k(\Delta^{-1})^{-1}(y)),$$

for a positive function k. It is quite clear that given a function k, we obtain the same Riemannian structure on $\mathcal{M}$ from $(\cdot,\cdot)_w'$ and π' as from $(\cdot,\cdot)_w$ and π. Moreover, $(\cdot,\cdot)_w$ coincides with $(\cdot,\cdot)$ on H_w if and only if the same is true for $(\cdot,\cdot)_w'$ and H_w'.

Let F^k be the modification of the Tomita–Takesaki operator F, given by

$$F^k(k(\Delta^{-1})(wg))=k(\Delta^{-1})(wg^*).$$

Lemma 7.1:

(i) $F^k=Fp(\Delta)=p(\Delta)^{-1}F$;
(ii) *The horizontal subspace* H'_w *is the fix point space of* F^k;
(iii) F^k *is the adjoint operator of* S^k *with respect to* $\langle\cdot,\cdot\rangle$;
(iv) *The polar decomposition of* S^k *with respect to* $\langle\cdot,\cdot\rangle$ *is* $S^k=J\Delta^{\sim k1/2}$, *where*

$$\Delta^{\sim k}=\Delta p(\Delta)^{-2}=\frac{\tilde{p}(\Delta)}{p(\Delta)}=\tilde{p}(\Delta)p(\Delta^{-1}),$$

(v) $F^{\sqrt{p}}=S^{\sqrt{\tilde{p}}}$, *where* $\tilde{p}=tp(t^{-1})=t/p(t)$

Proof: (i) is proved similarly as Lemma 4.1 (i), the statements (ii)–(iv) are easy to prove. To get (v), write

$$F^{\sqrt{p}}=Fp(\Delta)=J\Delta^{-1/2}p(\Delta)=J\Delta^{1/2}\tilde{p}(\Delta)^{-1}=S^{\sqrt{\tilde{p}}}.$$

□

Let the functions f and $\tilde{f}$ be obtained from p and $\tilde{p}$ as in Theorem 3.1 (iii). Then we will say that the corresponding Riemannian metrics on $\mathcal{M}$ are dual.

From the above Lemma, we see that the horizontal subspace H'_w, given by the function p, coincides with the subspace H_w, obtained from $\tilde{p}$. It means that a pair of dual Riemannian structures on $\mathcal{M}$ is obtained from the real part of Hilbert–Schmidt inner product on a horizontal subspace (H_w or H'_w), using the projections π and π'. Another way to do it is to use the same projection (π or π') on two dual subspaces H_w and H'_w.

Example 7.1: It is easy to see that if $k=1$, the dual subspace H'_w coincides with H^*_w from Sec. IV. Moreover, H_w and H^*_w are the SLD and RLD subspaces from Examples 3.1 and 3.2. It follows that SLD and RLD are dual Riemannian metrics. The metric from Example 3.3 is the unique self-dual Riemannian metric.

Let H_w and H'_w be two dual horizontal subspaces given by the function k, i.e., H_w and H'_w are the fix point space of S^k and F^k, respectively. Let $\mathcal{J}^k$ be the operator $H_w\to H'_w$, satisfying

$$\langle x,\mathcal{J}^k(y)\rangle=(x,y)_w,\quad x,y\in H_w.$$

Then

$$\mathcal{J}^k=\tfrac{1}{2}(1+\Delta^{\sim k})|_{H_w}.$$

Indeed, let $x,y\in H_w$, then

$$(1+\Delta^{\sim k})(x)=x+F^k(x)\in H'_w$$

and

$$\tfrac{1}{2}\langle x,(1+\Delta^{\sim k})(y)\rangle=\tfrac{1}{2}(\langle x,y\rangle+\langle x,F^kS^k(y)\rangle)=(x,y)=(x,y)_w.$$

Let now $h_1,h_2\in T_\rho(\mathcal{M})$ and let $\langle\cdot,\cdot\rangle_\rho$ be the inner product, given by the function $f=\frac{1}{2}(k(t)+tk(t^{-1}))$. Then it can be realized on H^*_w as

$$\begin{aligned}\langle h_1,h_2\rangle_\rho&=\langle l_1^L,\mathcal{J}^{-1}\mathcal{J}^k\mathcal{J}^{-1}(l_2^L)\rangle\\&=\langle l_1^L,2(1+\tilde{p}(\Delta))^{-1}[1+\tilde{p}(\Delta)p(\Delta^{-1})]\\&\quad\times(1+\tilde{p}(\Delta))^{-1}(L_2^L)\rangle,\end{aligned}$$

where $l_i^L = h_i(w^*)^{-1}$, $i=1,2$. Note that this corresponds to the decomposition of the function f

$$f(t)=\frac{1}{2}\,\frac{(1+\tilde{p}(t))^2}{1+\tilde{p}(t)p(t^{-1})},$$

from Theorem 3.1 (iii).

We conclude with two natural questions:

(1) From Theorem 3.1, we see that there is a one-to-one correspondence between the set of all operator monotone functions and a subset of

$$\{p:\mathbb{R}^+ \to \mathbb{R}^+,\ \ p \ \text{ is smooth,}\ \ p(t^{-1})=p(t)^{-1}\}.$$

How is this subset characterized?

(2) Let the Riemannian metric $\langle \cdot,\cdot \rangle_\rho$ be monotone. Does it follow that the dual Riemannian metric is monotone?

ACKNOWLEDGMENT

The research was supported by VEGA-Grant No. 1/7295/20.

[1] D. Petz, Linear Algebra Appl. **244**, 81 (1996).
[2] H. Nagaoka, in *Quantum Communication, Computing and Measurement*, edited by Hirota *et al.* (Plenum, New York, 1994).
[3] A. Jenčová, Rep. Math. Phys. **47**, 121 (2001).
[4] A. Uhlmann, Rep. Math. Phys. **24**, 229 (1986).
[5] J. Dittmann and G. Rudolph, J. Math. Phys. **33**, 4148 (1992).
[6] J. Dittmann and A. Uhlmann, J. Math. Phys. **40**, 3246 (1999).
[7] K. Matsumoto, Uhlmann's parallelism and Nagaoka's quantum information geometry, METR 97–09, 1997.
[8] A. Uhlmann, Rep. Math. Phys. **33**, 253 (1993).
[9] A. Uhlmann, Rep. Math. Phys. **36**, 461 (1995).
[10] S. Kobayashi and K. Nomizu *Foundation of Differential Geometry, Vol. I*, (Nauka, Moscow, 1981), (in Russian).
[11] H. Hasegawa, Rep. Math. Phys. **33**, 87 (1993).
[12] H. Hasegawa and D. Petz, *Non-commutative Extension of Information Geometry II, Quantum Communication, Computing and Measurement*, edited by Hirota *et al.* (Plenum, New York, 1997).
[13] A. Jenčová, Dualistic properties of the manifold of quantum states, Disordered and Complex Systems (2000).
[14] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (Nauka, Moscow, 1980), (in Russian).