Incompatibility in GPTs, generalized spectrahedra and tensor norms

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Why incompatibility

- one of fundamental nonclassical features of quantum mechanics, still not fully understood
- recognized as a valuable resource in quantum information theory
- necessary to violate any Bell inequality
- important task: characterize, detect and quantify incompatibility

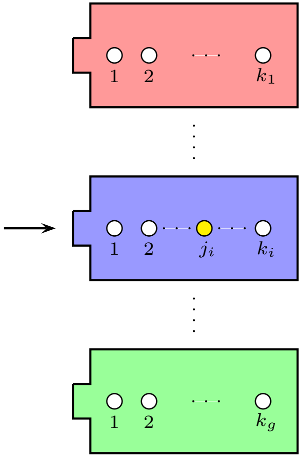
Why in GPTs

General probabilistic theories (GPTs):

- an operational framework for description of physical theories
- contains both classical and quantum theory
- incompatibility exists in any nonclassical GPT
- convenient tools (convexity, ordered vector spaces, tensor cones)
- relation of incompatibility to other problems in mathematics

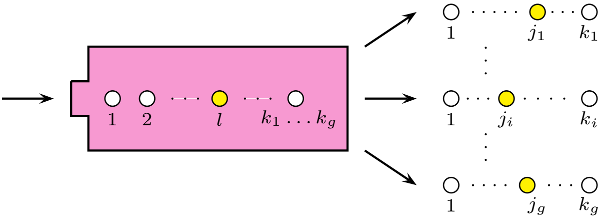
Measurements

A g-tuple of measurements with $\mathbf{k} = (k_1, \ldots, k_g)$ outcomes:



Compatible measurements

The measurements are compatible if all can be obtained from a single joint measurement:



Quantum measurements: POVMs and effects

A quantum measurement is represented by a POVM:

$$\{M_1, \ldots, M_k\}, \quad 0 \leq M_i \leq I, \quad \sum_i M_i = I$$

Outcome probabilities - Born rule: for any state ρ

$$\mathrm{Prob}(\text{outcome} = i|\rho) = \mathrm{Tr}\,[M_i \rho], \qquad i = 1, \ldots, k.$$

Binary measurements (two outcomes) are represented by effects:

$$0 \leq A \leq I, \quad \text{POVM} : \{A, I - A\}.$$

Characterization of compatible POVMs or effects?

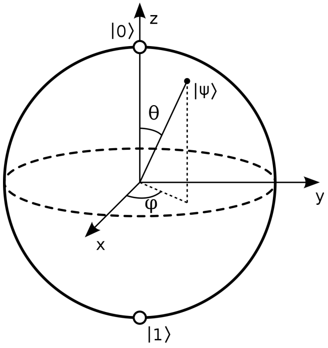
Qubit effects

Qubit effects are of the form

$$A = \frac{1}{2}(\alpha I + \boldsymbol{a} \cdot \boldsymbol{\sigma}), \qquad \|\boldsymbol{a}\|_2 \leq \alpha \leq 2 - \|\boldsymbol{a}\|_2$$

where

- $\boldsymbol{\sigma} = (\sigma_X, \sigma_Y, \sigma_Z)$ are the Pauli matrices
- $\boldsymbol{a} \in \mathbb{R}^3$, $\|\boldsymbol{a}\|_2 = \sqrt{\sum_i \boldsymbol{a}_i^2}$.



Compatible pairs of qubit effects

For qubit effects:

$$A = \frac{1}{2}(\alpha I + \boldsymbol{a} \cdot \boldsymbol{\sigma}), \qquad B = \frac{1}{2}(\beta I + \boldsymbol{b} \cdot \boldsymbol{\sigma})$$

- If A and B are compatible, then

$$\frac{1}{2}(\|\boldsymbol{a} + \boldsymbol{b}\|_2 + \|\boldsymbol{a} - \boldsymbol{b}\|_2) \leq 1.$$

- If A and B are unbiased (i.e. $\alpha = \beta = 1$), then also the converse holds.

Observation:

$$\frac{1}{2}(\|\boldsymbol{a} + \boldsymbol{b}\|_2 + \|\boldsymbol{a} - \boldsymbol{b}\|_2) \text{ is a tensor cross norm.}$$

Busch, Phys. Rev. D, 1986; Busch & Heinosaari, QIC, 2008

Tensor cross norms in Banach spaces

Let X and Y be Banach spaces. A cross norm is a norm $\|\cdot\|$ on $X \otimes Y$ such that for all $x \in X$, $y \in Y$, $\varphi \in X^*$, $\psi \in Y^*$

$$\|x \otimes y\| = \|x\|\|y\|, \quad \|\varphi \otimes \psi\| = \|\varphi\|\|\psi\|.$$

Injective cross norm:

$$\|z\|_\epsilon = \sup\{|(\varphi \otimes \psi)(z)|,\ \varphi \in X^*, \psi \in Y^*, \|\varphi\| = \|\psi\| = 1\}$$

Projective cross norm:

$$\|z\|_\pi = \inf\{\sum_i \|x_i\|\|y_i\|,\ z = \sum_i x_i \otimes y_i\}$$

For any cross norm we have

$$\|\cdot\|_\epsilon \leq \|\cdot\| \leq \|\cdot\|_\pi.$$

Pairs of qubit effects and cross norms

For e_1, e_2 basis elements in $\mathbb{R}^2$, $\boldsymbol{a}, \boldsymbol{b} \in \ell_2^3$, put

$$\varphi_{\boldsymbol{a},\boldsymbol{b}} := e_1 \otimes \boldsymbol{a} + e_2 \otimes \boldsymbol{b} \in \ell_\infty^2 \otimes \ell_2^3$$

The two cross norms are

$$\|\varphi_{\boldsymbol{a},\boldsymbol{b}}\|_\epsilon = \max\{\|\boldsymbol{a}\|_2, \|\boldsymbol{b}\|_2\}, \quad \|\varphi_{\boldsymbol{a},\boldsymbol{b}}\|_\pi = \frac{1}{2}(\|\boldsymbol{a}+\boldsymbol{b}\|_2 + \|\boldsymbol{a}-\boldsymbol{b}\|_2)$$

For $A = \frac{1}{2}(I + \boldsymbol{a}\cdot\boldsymbol{\sigma})$, $B = \frac{1}{2}(I + \boldsymbol{b}\cdot\boldsymbol{\sigma})$:

- A, B are (unbiased) effects $\iff \|\varphi_{\boldsymbol{a},\boldsymbol{b}}\|_\epsilon \leq 1$
- A, B are compatible effects $\iff \|\varphi_{\boldsymbol{a},\boldsymbol{b}}\|_\pi \leq 1$

The purpose of this talk

1. Compatibility of effects is characterized by a tensor cross norm for any (finite) number of any effects.

2. The cross norm is a compatibility degree.

3. We can bound the compatibility degree of g effects by the projective/injective ratio: (Aubrun et al., 2020)

$$\rho(\ell_\infty^g, X) := \max_{\varphi \in \ell_\infty^g \otimes X} \frac{\|\varphi\|_\pi}{\|\varphi\|_\epsilon}$$

4. For measurements with more outcomes, we have to find other characterizations.

We will work in the setting of general probabilistic theories (GPT).

General Probabilistic Theories

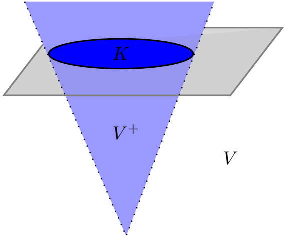
General probabilistic theories: definition

A GPT: $(V, V^+, \mathbb{1})$

- (V, V^+) an ordered vector space $(\dim(V) < \infty)$
- $\mathbb{1}$ unit effect: a strictly positive functional on (V, V^+).

The set of states: base of V^+ given by $\mathbb{1}$

$$K = \{\rho \in V^+,\ \langle \mathbb{1}, \rho\rangle = 1\}.$$



The base norm in V:

$$\|v\|_V := \inf\{\langle \mathbb{1}, v_+ + v_-\rangle :\ v = v_+ - v_-,\ v_\pm \in V^+\}.$$

Lami, arXiv:1803.02902

General probabilistic theories: the dual space

The dual ordered vector space: (A, A^+)

- $A = V^*$ dual vector space
- $A^+ = (V^+)^*$ the dual cone of positive functionals

$$(V^+)^* := \{f \in A,\ \langle f, v\rangle \geq 0,\ \forall v \in V^+\}$$

- $\mathbb{1} \in \mathrm{int}(A^+)$ - order unit in A

Order unit norm in A:

$$\|f\|_A := \inf\{\lambda > 0 :\ -\lambda \mathbb{1} \leq f \leq \lambda \mathbb{1}\}$$

the dual norm of the base norm $\|\cdot\|_V$.

Classical systems

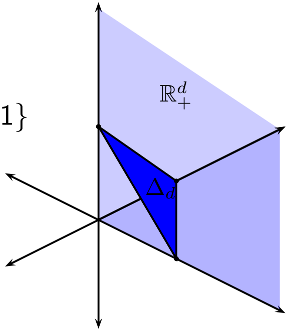
- $V \simeq A \simeq \mathbb{R}^d$
- $V^+ \simeq A^+ \simeq \mathbb{R}^d_+ = \{(x_1, \ldots, x_d),\ x_i \geq 0\}$ simplicial cone
- $\mathbb{1} = (1, 1, \ldots, 1)$

Classical state space: probability simplex

$$\Delta_d = \{(p_1, \ldots, p_d),\ p_i \geq 0,\ \sum_i p_1 = 1\}$$

The norms:

$$\|\cdot\|_V = \|\cdot\|_1, \qquad \|\cdot\|_A = \|\cdot\|_\infty.$$



Quantum systems

- $V \simeq A \simeq M_n^{sa}$ self-adjoint complex $n \times n$ matrices
- $V^+ \simeq A^+ \simeq M_n^+$ psd matrices
- $\mathbb{1} = I$ identity matrix

Quantum state space: set of density matrices

$$\mathcal{S}_n = \{\rho \in M_n^+,\ \mathrm{Tr}\,\rho = 1\}$$

The norms:

$$\|\cdot\|_V = \|\cdot\|_{\mathrm{Tr}}, \qquad \|\cdot\|_A = \|\cdot\| \text{ (operator norm).}$$

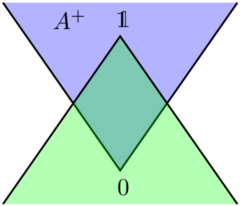
General probabilistic theories: measurements and effects

Effects: $f \in A$, $0 \leq f \leq \mathbb{1}$

- map states to probabilities:

$$K \ni \rho \mapsto \langle f, \rho \rangle \in [0, 1]$$

- binary measurements



Measurements with outcomes in $[k] = \{1, \ldots, k\}$:

- $f_1, \ldots, f_k$ effects, $\sum_i f_i = \mathbb{1}$
- Born rule: for any state $\rho \in K$

$$\mathrm{Prob}(\text{outcome} = i|\rho) = \langle f_i, \rho \rangle, \qquad i \in [k]$$

- map states to the probability simplex Δ_k

Compatibility in GPT

Let $f^{(1)}, \ldots, f^{(g)}$ be measurements in some GPT, with $\boldsymbol{k} = (k_1, \ldots, k_g)$ outcomes.

The measurements are compatible if all $f^{(i)}$ are marginals of a joint measurement h with outcomes in $[k_1] \times \cdots \times [k_g]$:

$$f_j^{(i)} = \sum_{n_1,\ldots,n_g} h_{n_1,\ldots,j,\ldots,n_g}, \quad i \in [g],\ j \in [k_i].$$

All measurements are compatible $\iff$ the GPT is classical

Plávala, Phys. Rev. A, 2016

Compatibility degree

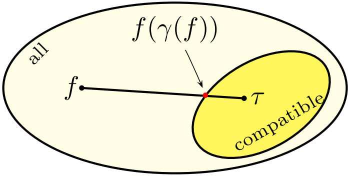
A trivial measurement with k outcomes: $\tau^k = (\frac{1}{k}\mathbb{1}, \ldots, \frac{1}{k}\mathbb{1})$:

For a tuple of measurements $f = (f^{(1)}, \ldots, f^{(g)})$ and $s \in [0,1]$:

$$f(s) := (f^{(1)}(s), \ldots, f^{(g)}(s)), \quad f^{(i)}(s) : sf^{(i)} + (1-s)\tau^{k_i}$$

Compatibility degree:

$$\gamma(f) = \sup\{s \in [0,1] \ : \ f(s) \text{ is compatible}\}$$



$$\gamma(\boldsymbol{k}; V, V^+) := \inf\{\gamma(f),\ f = (f^{(1)}, \ldots, f^{(g)}) \text{ measurements on } (V, V^+, \mathbb{1}) \text{ with } \boldsymbol{k} \text{ outcomes}\}$$

Busch et al., EPL 2013; Heinosaari et al., PRA 2015

Compatible effects in GPT and tensor norms

Compatible effects and cross norms

$f = (f_1, \ldots, f_g)$ a tuple of elements in $(A, \|\cdot\|_A)$:

$$\varphi^f := \sum_i e_i \otimes (2f_i - \mathbb{1}) \in \ell_\infty^g \otimes A$$

Interpretation of $\|\cdot\|_\epsilon$ and $\|\cdot\|_\pi$ cross norms in $\ell_\infty^g \otimes A$?

- $f_1, \ldots, f_g$ are effects if and only if

$$\|\varphi^f\|_\epsilon = \max_i \|2f_i - \mathbb{1}\|_A \leq 1$$

- what about the projective cross norm?

$$\|\varphi^f\|_\pi = \inf\{\sum_i \|h_i\|, \; \varphi^f = \sum_j z_j \otimes h_j, \; \|z_j\|_\infty = 1, \; h_j \in A\}$$

Compatible effects and cross norms

We introduce another norm in $\ell_\infty^g \otimes A$:

$$\|\varphi\|_c := \inf\{\|\sum_j h_j\|_A,\ \varphi = \sum_j z_j \otimes h_j,\ \|z_j\|_\infty = 1,\ h_j \in A^+\}.$$

Then $\|\cdot\|_c$ is a crossnorm, in particular

$$\|\cdot\|_\epsilon \leq \|\cdot\|_c \leq \|\cdot\|_\pi.$$

Characterization of compatibility

$f_1, \ldots, f_g$ are compatible effects if and only if $\|\varphi^f\|_c \leq 1$.

The norm $\|\cdot\|_c$ can be computed by a conic program.

Compatibility degree and cross norms

The compatibility degree:

$$\gamma(f) = \sup\{s \in [0,1] \ : \ s\|\varphi^f\|_c \leq 1\} = \|\varphi^f\|_c^{-1}$$

The smallest attainable compatibility degree:

$$\begin{aligned}
\gamma(g; V, V^+) &:= \inf\{\gamma(f) \ : \ f = (f_1, \ldots, f_g), \ f_i \text{ effects}\} \\
&= \min_{\varphi \in \ell_\infty^g \otimes A} \frac{\|\varphi\|_\epsilon}{\|\varphi\|_c} \\
&\geq \min_{\varphi \in \ell_\infty^g \otimes A} \frac{\|\varphi\|_\epsilon}{\|\varphi\|_\pi} = \rho(\ell_\infty^g, A)^{-1}
\end{aligned}$$

Lower bounds for compatibility degree

1. A first lower bound (tight if $\dim(V) \geq g+1$):

$$\gamma(g; V, V^+) \geq \rho(\ell_\infty^g, A)^{-1} \underset{\uparrow}{\geq} \max\{g^{-1}, \dim(V)^{-1}\}$$

(Aubrun et al., 2020)

2. A lower bound for all g:

$$\gamma(g; V, V^+) \geq \pi_1(V)^{-1}$$

$\pi_1(V)$ is the 1-summing constant: the least $c > 0$ such that:

$$\sum_{i=1}^{g} \|z_i\|_V \leq c \sup_{\|f\|_A \leq 1} \sum_{i=1}^{g} |\langle f, z_i \rangle|, \quad \forall g \in \mathbb{N},\ z_1, \ldots, z_g \in V$$

Aubrun et al., CMP, 2020

Centrally symmetric GPTs

$(V, V^+, \mathbb{1})$ is centrally symmetric if:

K is isomorphic to the unit ball of some norm $\|\cdot\|$ in $\mathbb{R}^d$.

Then

- $V \simeq \mathbb{R}\mathbb{1} \times \mathbb{R}^d$
- $V^+ \simeq \{(\lambda, x),\ \|x\| \leq \lambda\}$
- $\mathbb{1} = (1, 0)$

The dual space: for $\|\cdot\|_*$ the dual norm to $\|\cdot\|$

- $A \simeq \mathbb{R}\mathbb{1} \times \mathbb{R}^d$
- $A^+ \simeq \{(\alpha, \bar{f}),\ \|\bar{f}\|_* \leq \alpha\}$

Important example: qubits

Lami et al., CMP, 2018

The tensor norms in centrally symmetric GPTs

For $\varphi \in \ell_\infty^g \otimes A \simeq \ell_\infty^g \times (\ell_\infty^g \otimes \mathbb{R}^d)$:

$$\varphi = (y, \bar{\varphi}) \quad \text{for some } y \in \ell_\infty^g,\ \bar{\varphi} \in \ell_\infty^g \otimes \mathbb{R}^d$$

- the injective norm:

$$\max\{\|y\|_\infty, \|\bar{\varphi}\|_\epsilon\} \leq \|\varphi\|_\epsilon \leq \|y\|_\infty + \|\bar{\varphi}\|_\epsilon$$

- the norm $\|\cdot\|_c$:

$$\max\{\|y\|_\infty, \|\bar{\varphi}\|_\pi\} \leq \|\varphi\|_c \leq \|y\|_\infty + \|\bar{\varphi}\|_\pi$$

- if $y = 0$ then

$$\|\varphi\|_\epsilon = \|\bar{\varphi}\|_\epsilon, \qquad \|\varphi\|_c = \|\bar{\varphi}\|_\pi$$

Compatibility conditions in centrally symmetric GPTs

Let $f = (f_1, \ldots, f_g)$, $f_i = \frac{1}{2}(\alpha_i, \bar{f}_i)$, $i \in [g]$.

- f_i is an effect if and only if $\|\bar{f}_i\|_* \leq \alpha_i \leq 2 - \|\bar{f}_i\|_*$
- f_i is unbiased if $\alpha_i = 1$

The corresponding tensor $\varphi^f \in \ell_\infty^g \otimes A$:

$$\varphi^f = (y_f, \bar{\varphi}^f), \quad y_f = \sum_i (\alpha_i - 1) e_i, \quad \bar{\varphi}^f = \sum_i e_i \otimes \bar{f}_i$$

Compatibility conditions in centrally symmetric GPTs

Compatible effects in centrally symmetric GPTs:

- If $f_1, \ldots, f_g$ are compatible, then

$$\|\bar{\varphi}^f\|_\pi \leq 1.$$

- If all f_i are unbiased then also the converse holds.

Compatibility degree centrally symmetric GPTs

Minimal compatibility degree is attained at a g-tuple of unbiased effects:

$$\gamma(g; V, V^+) = \min_{\bar{\varphi} \in \ell_\infty^g \otimes \mathbb{R}^d} \frac{\|\bar{\varphi}\|_\epsilon}{\|\bar{\varphi}\|_\pi} = \rho(\ell_\infty^g, (\mathbb{R}^d, \|\cdot\|_*))^{-1}$$

From bounds on the π/ϵ-ratio: (Aubrun et al., 2020)

$$2^{-1/2} \geq \gamma(g; V, V^+) \geq \max\{g^{-1}, d^{-1}\}$$

Minimal compatibility degree for $g \to \infty$:

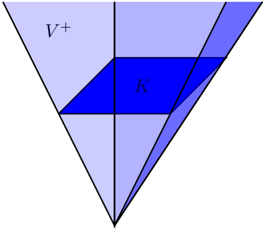
$$\gamma(g; V, V^+) \downarrow \pi_1(\mathbb{R}^d, \|\cdot\|)^{-1}$$

Centrally symmetric GPTs: examples

Cubic GPTs:

- $\|\cdot\| = \|\cdot\|_\infty$
- the state space:

$$K = [-1, 1]^d$$



- attains the lower bound for compatibility degree:

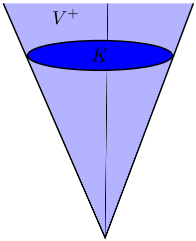
$$\gamma(g; V, V^+) = \max\{g^{-1}, d^{-1}\}$$

Centrally symmetric GPTs: examples

Spherical GPTs:

- $\|\cdot\| = \|\cdot\|_2$
- the state space:

  Euclidean ball



- compatibility degree:

$$\gamma(g; V, V^+) = g^{-1/2}, \qquad \text{if } g \leq d$$

$$d^{-1/2} \geq \gamma(g; V, V^+) \geq \max\{g^{-1/2}, \frac{\Gamma(\frac{d}{2})}{\sqrt{\pi}\Gamma(\frac{d+1}{2})}\}, \qquad \text{if } g \geq d$$

Compatibility of qubit effects

Qubits (spherical GPT with $d = 3$):

- $g = 2, 3$:
$$\gamma(g; M_2^{sa}, M_2^{+}) = g^{-1/2}$$
- $g \geq 4$:
$$3^{-1/2} > \gamma(g; M_2^{sa}, M_2^{+}) \downarrow 1/2$$

Is the lower bound $1/2$ attained for some $g \geq 4$?

Compatibility of quantum effects: matrix convex sets

Matrix convex sets

Matrix convex set in g variables: g-tuples of self-adjoint matrices

$$\mathcal{C} = \cup_n \mathcal{C}_n, \qquad \mathcal{C}_n \subseteq (M_n^{sa})^g$$

- closed under directs sums:

$$(X_i)_{i=1}^g \in \mathcal{C}_k,\ (Y_i)_{i=1}^g \in \mathcal{C}_n \implies (X_i \oplus Y_i)_{i=1}^g \in \mathcal{C}_{n+k}$$

- closed under application of ucp maps: for $\phi : M_k \to M_m$,

$$(X_i)_{i=1}^g \in \mathcal{C}_k \implies (\phi(X_i))_{i=1}^g \in \mathcal{C}_m.$$

Related to optimization (relaxation of linear matrix inequalities), operator algebras (dilation of operators, operator systems)

Examples of matrix convex sets

Main examples: given by a g-tuple $(A_i)_{i=1}^g \in (B(\mathcal{H})^{sa})^g$

- Matrix range:

$$\mathcal{W}(A) = \cup_n \mathcal{W}_n(A)$$
$$\mathcal{W}_n(A) := \{(\phi(A_i))_{i=1}^g,\ \phi : B(\mathcal{H}) \to M_n \text{ ucp map}\}$$

- Free spectrahedron:

$$\mathcal{D}_A = \cup_n \mathcal{D}_A(n)$$
$$\mathcal{D}_A(n) := \{(X_i)_{i=1}^g,\ \sum_{i=1}^g A_i \otimes X_i \leq I\}$$

At level 1: $\mathcal{D}_A(1) = \{x \in \mathbb{R}^g,\ \sum_i x_i A_i \leq I\}$ - spectrahedron

Duality of matrix convex sets

The matrix dual:

for a matrix convex set $\mathcal{C}$ (in g variables): $\mathcal{C}^\bullet = \cup_n \mathcal{C}_n^\bullet$,

$$\mathcal{C}_n^\bullet := \{(X_i)_{i=1}^g \in (M_n^{sa})^g,\ \sum_{i=1}^g C_i \otimes X_i \leq I, \forall (C_i) \in \mathcal{C}\}$$

Duality of free spectrahedron and matrix range:

$$\mathcal{W}(A)^\bullet = \mathcal{D}_A, \qquad \mathcal{D}_A^\bullet = \mathcal{W}(A).$$

Minimal and maximal matrix convex sets

Let $K \subseteq \mathbb{R}^g$ be a convex set.

- A matrix convex set over K: $\mathcal{C} = \cup_n \mathcal{C}_n$, $\mathcal{C}_1 = K$.
- Minimal and maximal matrix convex set over K:

$$\mathcal{C}_1^{min} = \mathcal{C}_1^{max} = K, \quad \mathcal{C}^{min}(K) \subseteq \mathcal{C} \subseteq \mathcal{C}^{max}(K)$$

  for all $\mathcal{C}$ with $\mathcal{C}_1 = K$.
- Duality: with K° the polar of K.

$$\mathcal{C}^{max}(K)^\bullet = \mathcal{C}^{min}(K^\circ), \quad \mathcal{C}^{min}(K)^\bullet = \mathcal{C}^{max}(K^\circ)$$

- If K is a polytope: there are some $(A_i)^g, (B_i)^g$ such that

$$\mathcal{C}^{min}(K) = \mathcal{W}(A), \qquad \mathcal{C}^{max}(K) = \mathcal{D}_B$$

Matrix convex sets and tensor norms

Let $\varphi = \sum_i e_i \otimes \tilde{f}_i \in \ell_\infty^g \otimes M_n^{sa}$

Unit ball of the injective norm:

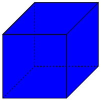
$$\|\varphi\|_\epsilon \leq 1 \iff (\tilde{f}_i)_{i=1}^g \in \mathcal{C}_n^{max}(B_{g,\infty}).$$

Unit ball of $\|\cdot\|_c$:

$$\|\varphi\|_c \leq 1 \iff (\tilde{f}_i)_{i=1}^g \in \mathcal{C}_n^{min}(B_{g,\infty}).$$

$B_{g,\infty} = [-1,1]^g$ is the unit ball in ℓ_∞^g

The matrix cube and the matrix diamond



$B_{g,\infty}$ - hypercube

$B_{g,\infty}^{\circ} = B_{g,1}$ - cross polytope

The matrix cube:

$$\mathcal{C}_n^{max}(B_{g,\infty}) = \{(X_i) \in (M_n^{sa})^g,\ \|X_i\| \leq 1,\ i \in [g]\} =: \mathcal{D}_{\square,g}(n)$$

The matrix diamond:

$$\mathcal{C}_n^{max}(B_{g,1}) = \{(X_i) \in (M_n^{sa})^g,\ \sum_i \varepsilon_i X_i \leq I, \varepsilon \in \{-1,1\}^g\}$$
$$=: \mathcal{D}_{\Diamond,g}(n)$$

Equivalent characterizations of compatible quantum effects

Let $f_1, \ldots, f_g$ be quantum effects, $\tilde{f}_i = 2f_i - I$.

$f_1, \ldots, f_g$ are compatible iff any of the following holds:

(i) $(\tilde{f}_i) \in \mathcal{C}_n^{min}(B_{g,\infty}) = \mathcal{D}^{\bullet}_{\Diamond,g}(n)$

(ii) **free spectrahedra inclusion:** $\mathcal{D}_{\Diamond,g} \subseteq \mathcal{D}_{\tilde{f}}$

(iii) $\tilde{f}_i = \phi(A_i),\ i \in [g],$ for a **ucp map** $\phi : M_{2^g} \to M_n$

(iv) $(\tilde{f}_i)$ dilates to a g-tuple of **commuting contractions**

$$A_i = I \otimes \cdots \otimes I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I \otimes \cdots \otimes I \in M_{2^g}^{sa}, \quad i \in [g].$$

Bluhm & Nechita, JMP, 2018

The compatibility degree and inclusion constant

Minimal compatibility degree:

$$\gamma(g; M_n^{sa}, M_n^+) = \sup\{s > 0,\ s\mathcal{D}_{\square,g}(n) \subseteq \mathcal{D}^{\bullet}_{\lozenge,g}(n)\}$$

Lower bounds:

1. For all n and g,

$$\gamma(g; M_n^{sa}, M_n^+) \geq g^{-1/2}.$$

   Equality holds for $n \geq 2^{\frac{g-1}{2}}$, attained by anti-commuting s.a. unitaries.

2. Dimension-dependent bound:

$$\gamma(g; M_n^{sa}, M_n^+) \geq \frac{1}{2n}.$$

Bluhm & Nechita, JMP, 2018

Compatible measurements and positive maps

Tensor cones and positive maps

(E, E^+), (F, F^+) - ordered vector spaces

Tensor cone: a positive cone $C \subset E \otimes F$:

$$E^+ \otimes_{min} F^+ \subseteq C \subseteq E^+ \otimes_{max} F^+$$

- minimal tensor product (separable elements are positive)

$$E^+ \otimes_{min} F^+ := \{\sum_i v_i \otimes w_i, v_i \in E^+, w_i \in F^+\}$$

- maximal tensor product (separable effects are positive)

$$E^+ \otimes_{max} F^+ := ((E^+)^* \otimes_{min} (F^+)^*)^*$$

Tensor cones and positive maps

Let $\phi : (E, E^+) \to (F, F^+)$ be a positive map, (L, L^+) an OVS:

- ϕ is completely positive with respect to the max or min cone:

$$(\phi \otimes id_L)(E^+ \otimes_{max/min} L^+) \subseteq F^+ \otimes_{max/min} L^+$$

- ϕ is entanglement breaking if:

$$(\phi \otimes id_L)(E^+ \otimes_{max} L^+) \subseteq F^+ \otimes_{min} L^+, \ \forall (L, L^+)$$

$$\Longleftrightarrow$$

$$(\phi \otimes id_L)(E^+ \otimes_{max} (F^+)^*) \subseteq F^+ \otimes_{min} (F^+)^*$$

Compatible measurements in a GPT

Let $(V, V^+, \mathbb{1})$ be a GPT
$f^{(1)}, \ldots, f^{(g)}$ measurements with $\boldsymbol{k} = (k_1, \ldots, k_g)$ outcomes

We define an affine map: $\boldsymbol{f} : K \to \mathcal{P}_{\boldsymbol{k}} := \Delta_{k_1} \times \cdots \times \Delta_{k_g}$

$$\rho \mapsto (f^{(1)}(\rho), \ldots, f^{(g)}(\rho))$$

Compatibility:

$$\begin{tikzcd} K \arrow[r, "\boldsymbol{f}"] \arrow[dr, "h"'] & \mathcal{P}_{\boldsymbol{k}} \\ & \Delta_{\boldsymbol{k}} = \bigotimes_i \Delta_{k_i} \arrow[u, "J"'] \end{tikzcd}$$

A GPT with state space $\mathcal{P}_{\boldsymbol{k}}$

We construct a subspace $E_{\boldsymbol{k}} \subseteq \mathbb{R}^{\boldsymbol{k}} := \bigotimes_i \mathbb{R}^{k_i}$, with $J : \mathbb{R}^{\boldsymbol{k}} \to E_{\boldsymbol{k}}$ orthogonal projection, such that

- $(E_{\boldsymbol{k}}, J(\mathbb{R}^{\boldsymbol{k}}_+), 1_{\boldsymbol{k}} = (1, \ldots, 1))$ is a GPT with state space

$$K = J(\Delta_{\boldsymbol{k}}) \simeq \mathcal{P}_{\boldsymbol{k}}$$

- $J|_{\Delta_{\boldsymbol{k}}}$ corresponds to taking marginals
- The dual space is $(E_{\boldsymbol{k}}, E_{\boldsymbol{k}} \cap \mathbb{R}^{\boldsymbol{k}}_+)$

The affine map $\boldsymbol{f} : K \to \mathcal{P}_{\boldsymbol{k}}$ extends to a positive map

$$\boldsymbol{f} : (V, V^+) \to (E_{\boldsymbol{k}}, J(\mathbb{R}^{\boldsymbol{k}}_+)), \quad \boldsymbol{f}^* : (E_{\boldsymbol{k}}, E_{\boldsymbol{k}} \cap \mathbb{R}^{\boldsymbol{k}}_+) \to (A, A^+)$$

A GPT with state space $\mathcal{P}_{\boldsymbol{k}}$

We choose a suitable basis in $E_{\boldsymbol{k}}$:

$$v := \{1_{\mathbf{k}}, v_j^{(i)}, j \in [k_i - 1], i \in [g]\}$$

The adjoint map $\boldsymbol{f}^* : E_{\boldsymbol{k}} \to A$ is determined by

$$\boldsymbol{f}^*(1_{\boldsymbol{k}}) = \mathbb{1}, \quad \boldsymbol{f}^*(v_j^{(i)}) = \tilde{f}_j^{(i)} := 2f_j^{(i)} - 2/k_i \mathbb{1},\ j \in [k_i - 1], i \in [g]$$

Compatibility, ETB maps and extendable maps

Compatibility: for the extended (positive) maps

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[dr, "h"'] & E_{k} \\
 & \mathbb{R}^{k} \arrow[u, "J"']
\end{tikzcd}
\quad \overset{\text{duality}}{\Longleftrightarrow} \quad
\begin{tikzcd}
E_{k} \arrow[r, "f^*"] \arrow[dr, hook] & A \\
 & \mathbb{R}^{k} \arrow[u, "h^*"']
\end{tikzcd}
$$

Equivalent compatibility conditions:

- f^* extends to a positive map $(\mathbb{R}^{k}, \mathbb{R}^{k}_{+}) \to (A, A^{+})$.
- f (equivalently f^*) is entanglement breaking.

The quantum case

Let $f^{(1)}, \dots, f^{(g)}$ be quantum measurements,

$$\boldsymbol{f}^* : (E_{\boldsymbol{k}}, E_{\boldsymbol{k}} \cap \mathbb{R}^{\boldsymbol{k}}) \to (M_n^{sa}, M_n^+).$$

- $E_{\boldsymbol{k}}$ is an operator system, generated by diagonal matrices

$$A_j^{(i)}(\boldsymbol{k}) = \mathrm{diag}(v_j^{(i)}), \qquad j \in [k_i - 1], i \in [g].$$

- $\boldsymbol{f}^*$ has a positive extension iff it is completely positive iff $\boldsymbol{f}^*$ has an ucp extension $\phi : M_{\boldsymbol{k}} \to M_n$
- this means that

$$(\boldsymbol{f}^*(v_j^{(i)})) = (\tilde{f}_j^{(i)}) \in \mathcal{W}(A(\boldsymbol{k})), \quad A(\boldsymbol{k}) = (A_j^{(i)}(\boldsymbol{k}))$$

a matrix range.

The matrix jewel

The matrix jewel: a free spectrahedron in $\sum_i (k_i - 1)$ variables

$$\mathcal{D}_{\blacklozenge,\mathbf{k}} = \mathcal{D}_{A(\boldsymbol{k})} = \mathcal{C}^{max}(\mathcal{D}_{\blacklozenge,\boldsymbol{k}}(1)), \qquad \mathcal{D}_{\blacklozenge,\boldsymbol{k}}(1) \simeq \mathcal{P}_{\boldsymbol{k}}^{\circ}$$

Put $\tilde{f} = \{\tilde{f}_j^{(i)} := 2f_j^{(i)} - 2/k_i \mathbb{1}, j \in [k_i - 1],\ i \in [g]\}$.

Equivalent compatibility conditions:

- $(\tilde{f}_j^{(i)}) \in \mathcal{D}^{\bullet}_{\blacklozenge,\mathbf{k}}(n) = \mathcal{W}_n(A(\boldsymbol{k}))$
- **Free spectrahedra inclusion:** $\mathcal{D}_{\blacklozenge,\mathbf{k}} \subseteq \mathcal{D}_{\tilde{f}}$.

Bluhm & Nechita, SIAM J. Appl. Algebra Geometry, 2020

Compatible measurements and generalized spectrahedra

Generalized spectrahedra

(E, E^+), (L, L^+), a tensor cone $C \subseteq E \otimes L$, $a_1, \ldots, a_N \in E$.

A generalized spectrahedron: a positive cone in L^N,

$$\mathcal{D}_a(L, C) := \{(v_1, \ldots, v_N) \in L^N, \ \sum_i a_i \otimes v_i \in C\}$$

Assume that $\{a_1, \ldots, a_N\}$ is a basis of E.

- $\mathcal{D}_a(\mathbb{R}, C) \simeq C$.
- Let (F, F^+), $C' \subseteq F \otimes L$ a tensor cone, $\phi : E \to F$. Then

  $$\phi \otimes id_L : (E \otimes L, C) \to (F \otimes L, C')$$

  is positive if and only if $\mathcal{D}_a(L, C) \subseteq \mathcal{D}_{\phi(a)}(L, C')$.

The GPT jewel and compatibility

Let $E_{\mathbf{k}} \subseteq \mathbb{R}^{\mathbf{k}}$ be as before, with basis

$$v := \{1_{\mathbf{k}}, v_j^{(i)}, j \in [k_i - 1], i \in [g]\}$$

The GPT jewel: for (L, L^+),

$$\mathcal{D}_{GPT,\blacklozenge}(\mathbf{k}; L, L^+) := \mathcal{D}_v(L, E_{\mathbf{k}}^+ \otimes_{max} L^+)$$

$f^{(1)}, \ldots, f^{(g)}$ are **compatible measurements** if and only if

$$\mathcal{D}_{GPT,\blacklozenge}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_{\tilde{f}}(V, A^+ \otimes_{min} V^+),$$

with $\tilde{f} = \{\mathbb{1},\ \tilde{f}_j^{(i)}, j \in [k_i - 1], i \in [g]\}$.

The GPT diamond

For binary measurements (effects), $\boldsymbol{k} = (2, \ldots, 2)$:

The GPT diamond:

$$\mathcal{D}_{GPT,\lozenge}(g; L, L^+) := \mathcal{D}_{GPT,\blacklozenge}((2, \ldots, 2); L, L^+)$$
$$= \{(z_0, \ldots, z_g),\ z_0 + \sum_i \varepsilon_i z_i \in L^+,\ \forall \varepsilon \in \{\pm 1\}^g\}$$

$$\mathcal{D}_{GPT,\lozenge}(g; V, V^+) =$$
$$= \{(z_0, \ldots, z_g),\ z_0 \in V^+, \|\sum_i e_i \otimes z_i\|_{c^*} \leq \mathbb{1}(z_0)\}$$

where $\|\cdot\|_{c^*}$ is the cross norm in $\ell_1^g \otimes V$ dual to $\|\cdot\|_c$.

Relation to the matrix jewel

For quantum theory:

$$\mathcal{D}_{GPT,\blacklozenge}(\mathbf{k}; M_n^{sa}, M_n^+)$$
$$= \{(Z_0, Z_0^{1/2} X_j^{(i)} Z_0^{1/2}), \quad Z_0 \in M_n^+,\ (X_j^{(i)}) \in \mathcal{D}_{\blacklozenge,\mathbf{k}}(n)\}$$

At dimension 1: $\mathcal{D}_{GPT,\blacklozenge}(\mathbf{k}; \mathbb{R}, \mathbb{R}_+) = \mathbb{R}_+(1, \mathcal{D}_{\blacklozenge,\boldsymbol{k}}(1))$

The GPT jewel and compatibility degree

For $(z_0, z_j^{(i)}) \in \mathcal{D}_{GPT,\blacklozenge}(\mathbf{k}; V, V^+)$ and $s > 0$, we define

$$s \cdot (z_0, z_j^{(i)}) := (z_0, s z_j^{(i)}).$$

The compatibility degree as inclusion constant:

$$\gamma(\mathbf{k}; V, V^+) =$$
$$\sup\{s > 0,\ s \cdot \mathcal{D}_{GPT,\blacklozenge}(\mathbf{k}; V, V^+) \subseteq \mathcal{D}_{\tilde{f}}(V; A^+ \otimes_{min} V^+)$$
$$\forall f_j^{(i)} \in A,\ \text{such that } \mathcal{D}_{GPT,\blacklozenge}(\mathbf{k}; \mathbb{R}, \mathbb{R}_+) \subseteq \mathcal{D}_{\tilde{f}}(\mathbb{R}, A^+))\}$$

Lower bounds on compatibility degree

Let $k_m = \max\{k_i\}$, $\bar{k} = \sum_i (k_i - 1)$. From the relation

$$\frac{k_m}{2}(k_m - 1)^{-1}\mathcal{D}_{\lozenge,\bar{k}}(1) \subseteq \mathcal{D}_{\blacklozenge,\mathbf{k}}(1) \subseteq \frac{k_m}{2}(k_m - 1)\mathcal{D}_{\lozenge,\bar{k}}(1)$$

From the lower bound for effects:

$$\begin{aligned}\gamma(\mathbf{k}; V, V^+) &\geq (k_m - 1)^{-2}\gamma(\bar{k}; V, V^+) \\ &\geq (k_m - 1)^{-2}\rho(\ell_\infty^{\bar{k}}, A)^{-1}\end{aligned}$$

In conclusion...

Incompatibility of measurements in GPT is related to

- tensor norms in Banach spaces
- inclusion problems for matrix convex sets
- properties of positive maps on OVS
- inclusion of generalized spectrahedra

Further results

Compatibility region: different coefficients in the mixture with noise

$$
\begin{aligned}
\Gamma(f) = \{ & \boldsymbol{s} \in [0,1]^g, \\
& (s_i f^{(i)} + (1 - s_i)\tau^{k_i}) \text{ are compatible}\} \\
\Gamma(\boldsymbol{k}; V, V^+) = \cap \{ & \Gamma(f), \\
& (f^{(i)}) \text{ are measurements with } \boldsymbol{k} \text{ outcomes}\}
\end{aligned}
$$

Incompatibility witnesses: obtained from

- spectrahedra
- duality of maps
- duality of norms $\|\cdot\|_c$, $\|\cdot\|_{c^*}$