A NOTE ON EFFECT ALGEBRAS AND DIMENSION THEORY OF AF C*-ALGEBRAS

ANNA JENČOVÁ and SYLVIA PULMANNOVÁ

Mathematical Institute, Slovak Academy of Sciences,
SK-814 73 Bratislava, Slovakia
(e-mails: pulmann@mat.savba.sk, JENCA@mat.savba.sk)

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We continue in the investigation of the relations between effect algebras and AF C*-algebras started by Pulmannová, 1999. In particular, in analogy with the notion of a dimension group, we introduce the notion of a dimension effect algebra as an effect algebra obtained as the direct limit of finite effect algebras with RDP. We also give an intrinsic characterization of dimension effect algebras. It turns out that every dimension effect algebra is a unit interval in a dimension group with a unit. We prove that there is a categorical equivalence between the category of countable dimension effect algebras and unital AF C*-algebras.

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1. Introduction

In the traditional approach, the mathematical description of a quantum mechanical system is based on a Hilbert space. Experimentally verifiable propositions about a physical system are modeled by projections on the corresponding Hilbert space.

Recently, to describe unsharp propositions and measurements, new abstract partial algebraic structures were introduced, called effect algebras (alternatively, D-posets). In the Hilbert space approach, the effect algebra is realized by self-adjoint operators lying between zero and identity, which are called the Hilbert space effects. The transition from projections to effects corresponds to the transition from projection-valued (PV) to positive operator-valued (POV) observables. The latter play an important role in the theory of quantum measurements [6].

For the systems with infinitely many degrees of freedom, the C*-algebraic approach is more appropriate [18]. For example, the physical system arising as the thermodynamic limit of finite spin-like systems is described by AF C*-algebras, i.e. the norm closure of an ascending sequence of finite-dimensional C*-algebras.

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Among general effect algebras, a special role is played by MV-algebras, introduced as an algebraic description of many-valued logic [7, 8]. In [22], a correspondence was shown between (countable) MV-algebras and the subclass of unital AF C*-algebras, whose Murray–von Neumann order is a lattice. This result was extended in [24] to all unital AF C*-algebras, and a certain subclass of effect algebras with the Riesz decomposition property, extending the class of MV-algebras.

In the present paper, we give an intrinsic characterization of the latter subclass of effect algebras. We also show that they can be obtained as direct limits of sequences of finite effect algebras with the Riesz decomposition property. In analogy with the notion of dimension groups, we introduce the notion of dimension effect algebras. It turns out that dimension effect algebras are exactly the unit intervals in unital dimension groups. We prove a categorical equivalence between countable dimension effect algebras and unital AF C*-algebras. A similar result was stated in [24, Theorem 8] without proof.

2. Preliminaries on K_0-theory of AF C*-algebras

The purpose of dimension theory of C*-algebras is to measure the “dimensions” of projections in the algebra. The dimension of a projection in a matrix algebra is a nonnegative integer, while in a finite von Neumann factor one obtains a nonnegative real number. In a C*-algebra A, in general, the dimension function must be given by values in a pre-ordered abelian group (a so-called K_0 group), rather than in real numbers. For basics of K_0-theory for C*-algebras see e.g. [4, 17, 27, 28].

Given a C*-algebra A, two projections $p, q \in A$ are equivalent, written $p \sim g$, if there is a partial isometry u such that $u^*u = p$ and $uu^* = q$. Let $\mathrm{Proj}(A)$ denote the set of all equivalence classes of projections of A. A partial binary operation $+$ can be defined on $\mathrm{Proj}(A)$ in the following way: for two equivalence classes $[p]$ and $[q]$ their ”sum” $[p] + [q]$ exists iff there are representatives $p' \in [p]$ and $q' \in [q]$ with $p'q' = 0$, in which case $[p] + [q] = [p' + q']$.

Recall that in the K_0-theory of C*-algebras, the definition of $K_0(A)$ for a C*-algebra A requires simultaneous consideration of all matrix algebras over A. Denote by $\mathcal{M}_n(A)$ the set of all $n \times n$ matrices with entries in A. Let $\mathcal{M}_\infty$ denote the algebraic direct limit of $\mathcal{M}_n(A)$ under the embedding $a \mapsto \mathrm{diag}(a, 0)$. Then $\mathcal{M}_\infty(A)$ can be thought as the algebra of all infinite matrices with only finitely many nonzero entries. Let $V(A) := \mathrm{Proj}(\mathcal{M}_\infty(A))$. If A is separable, then $V(A)$ is countable. There is a binary operation on $V(A)$: If $[e], [f] \in V(A)$, choose $e' \in [e], f' \in [f]$ with $e' \perp f'$. Notice that this is always possible by “moving down” the diagonal. Then define $[e] + [f] = [e' + f']$. This operation is well defined and makes $V(A)$ into an abelian semigroup with the additive zero $[0]$.

If A is a unital C*-algebra, then $K_0(A)$ is defined as the Grothendieck group for $V(A)$. For nonunital C*-algebras the construction of K_0 is more refined. The embedding of $V(A)$ into $K_0(A)$ is injective iff $V(A)$ has cancelation, i.e. if $[e] + [g] = [e] + [h]$ implies $[g] = [h]$ for $[e], [g], [h] \in V(A)$. $K_0(A)$ can be pre-ordered taking the image of $V(A)$ in $K_0(A)$ as $K_0(A)^+$. We can also define the *scale* $D(A)$ as the image of $\mathrm{Proj}(A)$ in $K_0(A)$. If A is a unital C*-algebra, then $D(A) = [0, 1_A]$. If $\phi : A \to B$ is a homomorphism of C*-algebras, then ϕ extends to a semigroup homomorphism $\phi^* : V(A) \to V(B)$. Moreover, $V : A \to V(A)$ is a covariant functor from the category of C*-algebras to the category of abelian semigroups, which preserves direct products and inductive limits.

It turns out that to every abelian group G there is a C*-algebra A with $K_0(A) = G$. But this correspondence is not one-to-one; the C*-algebra A is not uniquely defined by its $K_0(A)$. The situation is better in the class of approximately finite C*-algebras.

A C*-algebra A is called approximately finite-dimensional (AF) if A is the direct limit of an increasing sequence of finite-dimensional C*-algebras. We will be concerned with unital AF C*-algebras that arise as direct limits of sequences of unital finite-dimensional C*-algebras with the same unit. We refer to [17] for the definition of the functor K_0 from the category of AF C*-algebras with C*-algebra homomorphisms to the category of countable partially ordered abelian groups with order-preserving group homomorphisms.

It was proved in [12] that any AF C*-algebra is uniquely determined by its K_0-group. Such groups are just the ordered direct limits of sequences of simplicial groups (i.e. groups order-isomorphic to Z^r, $r \in \mathbb{N}$, with the standard positive cone) called the (countable) *dimension groups* [11, 16]. In [11], it was proved that the dimension groups can be characterized as unperforated directed interpolation groups. We note that such groups coincide with the Riesz groups [15].

We note that some authors allow AF C*-algebras to be nonseparable by defining them as direct limits of arbitrary directed sets of finite-dimensional C*-algebras. Their K_0 groups are then the general dimension groups, obtained as direct limits of directed systems of simplicial groups.

We follow the classical definition of AF C*-algebras, although the more general case is immediate.

3. Interpolation groups and their unit intervals

By a *partially ordered group* we shall mean an abelian group G together with a subset G^+ such that $G^+ + G^+ \subseteq G^+$, $G^+ \cap (-G^+) = \{0\}$. The set G^+ is called the *positive cone* of G and we write $a \le b$ iff $b - a \in G^+$. G is said to have the *Riesz interpolation property* (RIP), or to be an *interpolation group*, if given a_i, b_j $(1 \le i \le m, 1 \le j \le n)$ with $a_i \le b_j$ for all i, j, there must be a $c \in G$ with $a_i \le c \le b_j$ for all i, j. The Riesz interpolation property is equivalent to the *Riesz decomposition property* (RDP): given a_i, b_j in G^+ $(1 \le i \le m, 1 \le j \le n)$ with $\sum a_i = \sum b_j$, there exist $c_{ij} \in G^+$ with $a_i = \sum_j c_{ij}$, $b_j = \sum_i c_{ij}$. Equivalently, RDP can be expressed as follows: given $a, b_i, i \le n$ in G^+ with $a \le \sum_{i \le n} b_i$, there exist $a_i, i \le n$, with $a_i \le b_i, i \le n$, and $a = \sum_{i \le n} a_i$. In order to verify these properties, it is only necessary to consider the case $m = n = 2$ (see [15, 16]).

An element u of a partially ordered group G is an *order unit* if for all $a \in G$, $a \le nu$ for some $n \in \mathbb{N}$. A partially ordered group G is said to be *directed* if

$G = G^+ - G^+$. If G has an order unit u, then it is directed; indeed, if $g \leq nu$, then we may write $g = nu - (nu - g)$.

An abelian partially ordered group G is called *unperforated* if given $n \in \mathbb{N}$ and $a \in G$, $na \in G^+$ implies $a \in G^+$. Any unperforated abelian group must be torsion free. G is *archimedean* if whenever $x, y \in G$ such that $nx \leq y$ for all positive integers n, then $x \leq 0$. According to [16, Proposition 1.24], if G is an archimedean directed abelian group, then G is unperforated. Also, by [16, Proposition 1.22], all lattice ordered abelian groups are unperforated.

Given partially ordered groups G and H, we say that a homomorphism $\phi : G \to H$ is *positive* if $\phi(G^+) \subseteq H^+$, and that an isomorphism $\phi : G \to H$ is an *order isomorphism* if $\phi(G^+) = H^+$. If G and H have order units u and v respectively, then a positive homomorphism $\phi : G \to H$ such that $\phi(u) = v$ is called *unital*. Fixing an order unit $u \in G^+$, a positive homomorphism $f : G \to \mathbb{R}$ with $f(u) = 1$ is called a *u-state* on G.

We say that a partially ordered group G is a *Riesz group* if it is directed, unperforated and has the interpolation property. Riesz groups were first studied in [15].

Let G be a partially ordered group. We say that $u \in G^+$ is a *generating unit* if every $g \in G^+$ can be expressed in the form of a finite sum of (not necessarily different) elements of the interval $[0, u]$. It is straightforward to show that a generating unit is an order unit. Conversely, if G is an interpolation group, then any order unit is generating. Indeed, if $0 \leq g \leq nu$, then the Riesz decomposition property implies that $g = \sum_{i \leq n} g_i$ with $g_i \leq u$, $i \leq n$.

For any $v \in G^+$, where G is a partially ordered group, the interval $G^+[0, v] = \{a \in G : 0 \leq a \leq v\}$ can be endowed with a partial algebraic structure in the following way: we define a unary operation $'$ by putting $a' = v - a$, and a partial binary operation $\oplus$ defined iff $a + b \leq v$, and $a \oplus b = a + b$. This partial algebraic structure is a special case of a more general structure, so-called effect algebra. A general axiomatic definition is the following [13]. Notice that an alternative structure of a so-called D-poset was introduced in [21].

DEFINITION 3.1. An *effect algebra* is an algebraic system $(E; 0, 1, \oplus)$, where $\oplus$ is a partial binary operation and 0 and 1 are constants, such that the following axioms are satisfied for every $a, b, c \in E$:

(i) if $a \oplus b$ is defined then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity);
(ii) if $a \oplus b$ and $(a \oplus b) \oplus c$ is defined, then $a \oplus (b \oplus c)$ is defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);
(iii) for every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$;
(iv) $a \oplus 1$ is defined iff $a = 0$.

In an effect algebra E, we define $a \leq b$ if there is $c \in E$ with $a \oplus c = b$. It turns out that $\leq$ is a partial order such that $0 \leq a \leq 1$ for all $a \in E$. Moreover, if $a \oplus c_1 = a \oplus c_2$, then $c_1 = c_2$, and we define $c = b \ominus a$ iff $a \oplus c = b$. In particular, $1 - a = a'$ is called the *orthosupplement* of a.

A very important example of effect algebras is the interval $[0, I]$ in the group of self-adjoint operators on a Hilbert space, which are called the Hilbert space effects. This example gave a motivation for introducing the abstract definition.

Effect algebras include as special subclasses algebraic structures that have been used so far as quantum logics (i.e. mathematical models of quantum propositions): boolean algebras, MV-algebras, orthomodular lattices and posets, orthoalgebras [10]. In particular, MV-algebras were originally introduced as an algebraic basis for many-valued logics [7, 8]. Recall that an MV-algebra is an algebraic structure $(M; 0, 1,', \dot{+})$, where $0, 1$ are constants, $' : M \to M$ is a unary operation and $\dot{+}$ is a binary operation which is associative and commutative, 0 is its neutral element, $1\dot{+}a = 1$, $a\dot{+}a' = 1$, $0' = 1$ and $a\dot{+}(a\dot{+}b')' = b\dot{+}(b\dot{+}a')'$. With $a \vee b = a\dot{+}(a\dot{+}b')'$ as the join, and $(a' \vee b')'$ as the meet, M becomes a distributive lattice with respect to the partial order $a \leq b$ iff $a \vee b = b$. If we restrict the operation $\dot{+}$ to those pairs (a, b) for which $a \leq b'$, M becomes a lattice ordered effect algebra (a so-called *MV-effect algebra*) with the same ordering as the original MV-algebra and with $a' = 1 \ominus a$.

We say that an effect algebra E has the *Riesz decomposition property* (RDP) if one of the following equivalent properties is satisfied:

(R1) $a \leq b_1 \oplus b_2 \oplus \cdots \oplus b_n$ implies $a = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ with $a_i \leq b_i$, $i \leq n$;

(R2) $\oplus_{i \leq m} a_i = \oplus_{j \leq n} b_j$, $m, n \in \mathbb{N}$, implies $a_i = \oplus_j c_{ij}$, $i \leq m$, and $b_j = \oplus_i c_{ij}$, $j \leq n$, where $(c_{ij})_{ij}$ are orthogonal elements in E.

Similarly, as for partially ordered groups, it suffices to prove the above properties for $m, n = 2$.

Let $(P, \leq)$ be a partially ordered set. We say that P has the *interpolation property* if $a_1, a_2, b_1, b_2 \in P$ and $a_i \leq b_j$ for all $i, j = 1, 2$, there is an element $x \in P$ with $a_i \leq x \leq b_j, i, j = 1, 2$. Every effect algebra with RDP has the interpolation property (see [20] for a direct proof; it also follows from the fact that any such algebra is an interval of an interpolation group, as will be shown below). But, differently from the groups, the interpolation property for effect algebras is weaker than the Riesz decomposition property. (There are lattice ordered effect algebras that do not satisfy the RDP, see e.g. the ”diamond” [10]).

A characterization of MV-algebras among effect algebras was formulated in [9]. By an easy reformulation of it, we find the following ([10]). Every MV-effect algebra satisfies RDP. Conversely, every lattice ordered effect algebra with RDP can be organized into an MV-algebra by putting $a\dot{+}b = a \oplus a' \wedge b$. So MV-algebras are in one-to-one correspondence with lattice ordered effect algebras with RDP.

DEFINITION 3.2 ([2]). An effect algebra E is called an *interval effect algebra* if it is isomorphic to an effect algebra of the form $G^+[0, e]$, where G is a partially ordered abelian group and $e \in G^+$.

DEFINITION 3.3. Let E be an effect algebra. A map $\phi : E \to K$, where K is any abelian group, is called a K-valued measure if $\phi(a \oplus b) = \phi(a) + \phi(b)$ whenever $a \oplus b$ is defined in E.

THEOREM 3.4 ([2]). *Let E be an interval effect algebra. Then there exists a unique (up to isomorphism) partially ordered directed abelian group G with positive cone G^+ and an element $u \in G^+$ such that the following conditions are satisfied:*
(i) *E is isomorphic to the interval effect algebra $G^+[0, u]$,*
(ii) *u is a generating unit,*
(iii) *every K-valued measure $\phi : E \to K$ can be extended uniquely to a group homomorphism $\phi^* : G \to K$.*

The group G in the preceding theorem is called the *universal group* for E. We will denote it by G_E. The element u is a generating unit in G_E.

There are effect algebras that are not interval effect algebras. E.g., there exist orthomodular lattices with no group valued measures [23]. On the other hand, according to [26], every effect algebra with RDP is an interval effect algebra and its universal group is an interpolation group. The technique used in the proof (so-called word method) goes back to Baer [1]. See [26] or [10] for more details.

The proof of the following theorem can be found in [24]. For the convenience of readers, we repeat it here.

THEOREM 3.5. *Let G be an interpolation group with order unit u. Put $E = G^+[0, u]$. Then (G, u) is the universal group for E.*

Proof: Properties (i) and (ii) of Theorem 3.4 are clear. We have to check (iii). Let K be any abelian group and let $\psi : E \to K$ be a K-valued measure. Take $x \in G^+$, then there are $x_1, x_2, \ldots, x_n \in E$ with $x = x_1 + x_2 + \cdots + x_n$. Define $\psi^*(x) = \sum_{i\leq n} \psi(x_i)$. Then $\psi^* : G^+ \to K$ is a well-defined additive map. Indeed, assume that $x = y_1 + y_2 + \cdots + y_m$ with $y_i \in E$, $i = 1, 2, \ldots, m$, be another representation of x. Owing to the Riesz decomposition property for G, there are $w_{ij} \in G^+$ such that $x_i = \sum_{j\leq m} w_{ij}$ for $i \leq n$, $y_j = \sum i \leq n w_{ij}$ for $j \leq m$. Then we have

$$\psi^*(x) = \sum_{i\leq n} \psi(x_i) = \sum_{i\leq n} \sum_{j\leq m} \psi(w_{ij}) = \sum_{j\leq m} \sum_{i\leq n} \psi(w_{ij}) = \sum_{j\leq m} \psi(y_j).$$

Additivity of ψ^* on G^+ is clear. Since G is directed, any $z \in G$ has a form $z = x - y$ with $x, y \in G^+$. Define $\psi^*(z) = \psi(x) - \psi(y)$. If x_1, y_1 is another pair of elements in G^+ such that $z = x_1 - y_1$, then $x + y_1 = x_1 + y$ are elements of G^+, hence $\psi^*(x) + \psi^*(y_1) = \psi^*(x_1) + \psi^*(y)$, which entails $\psi^*(x) - \psi^*(y) = \psi^*(x_1) - \psi^*(y_1)$. So $\psi^* : G \to K$ is well defined, and additivity of ψ^* is clear. □

The following example shows that not every unital group is a universal group for its unit interval (see [14, Example 11.3, 11.5]).

EXAMPLE 3.6. Let $G = \mathbb{Z}$ with the positive cone $G^+ := \{0, 2, 3, 4, \ldots\}$ and unit $u := 5$. Then G is a (nonarchimedean) unital group, and the unit interval $E = \{0, 2, 3, 5\}$ is a Boolean effect algebra. The only group homomorphisms $G \to \mathbb{R}$ are of the form $n \mapsto cn$ where c is a real constant. But, for any $p \in [0, 1]$, $\pi_p : E \to \mathbb{R}$ defined by $\pi_p(0) = 0, \pi_p(2) = p, \pi_p(3) = 1 - p$ and $\pi_p(5) = 1$ is a state on E. So π_p can be extended to a group homomorphism $G \to \mathbb{R}$ iff $p = 1/5$. Therefore, G is not the universal group for E.

Notice that the universal group for E is $(\mathbb{Z} \times \mathbb{Z}, \gamma)$ with coordinatewise addition and partial order, where $\gamma(2) = (1, 0), \gamma(3) = (0, 1)$, and $\gamma(5) = (1, 1)$. The mapping $\nu : \mathbb{Z} \times \mathbb{Z} \to G$ defined by $\nu(m, n) = 2m + 3n$ is a surjective group homomorphism.

Let E and F be effect algebras. A mapping $\phi : E \to F$ is an *(effect algebra) morphism* if ϕ preserves the unit element and $\oplus$.

LEMMA 3.7. *Let E and F be effect algebras with RDP and let (G_E, u) and (G_F, v) be their respective universal groups with the corresponding units. Every morphism $\phi : E \to F$ uniquely extends to a unital group homomorphism $\phi^* : G_E \to G_F$.*

Proof: Owing to the isomorphism, we may identify F with the unit interval $G_F^+[0, v] \subset G_F$, and consider ϕ as a G_F-valued measure on E. By the properties of universal groups, ϕ uniquely extends to a group homomorphism $\phi^* : G_E \to G_F$. As ϕ maps $G_E^+[0, u]$ to $G_F^+[0, v]$, and u and v are generating, we get $\phi^*(G_E)^+ \subseteq (G_F)^+$, hence ϕ^* is positive, and as ϕ preserves the unit elements, we have $\phi^*(u) = v$. Hence ϕ^* is a unital group homomorphism. □

Let us consider the category **E** of effect algebras with RDP with effect algebra morphisms as morphisms, and the category **IG** of interpolation groups with distinguished order units with unital group homomorphisms as morphisms. Denote by $S : \mathbf{E} \to \mathbf{IG}$ the map that to every effect algebra E assigns its universal group G_E, and let $T : \mathbf{IG} \to \mathbf{E}$ denote the map that to every interpolation group G with unit u assigns the effect algebra $G^+[0, u]$.

THEOREM 3.8. (i) *$S : \mathbf{E} \to \mathbf{IG}$ and $T : \mathbf{IG} \to \mathbf{E}$ are functors;* (ii) *there is a categorical equivalence between the category* **E** *of effect algebras with RDP with effect algebra morphisms, and the category* **IG** *of interpolation groups with order unit.*

Proof: (i) For effect algebras E, F and an effect algebra morphism $\phi : E \to F$, define $S\phi = \phi^* : G_E \to G_F$. For groups (G, u) and (H, v) and a unital group homomorphism $f : G \to H$, define $Tf = f/G^+[0, u] : G^+[0, u] \to H^+[0, v]$. Clearly, the identity mapping on E extends to the identity on G_E, and the identity mapping on G_E restricts to the identity on E. Let E_1, E_2, E_3 be effect algebras in **E**, and let $\phi : E_1 \to E_2$ and $\psi : E_2 \to E_3$ be effect algebra morphisms. Then $(\psi \circ \phi) : E_1 \to E_3$ has a unique extension $(\psi \circ \phi)^* : G_{E_1} \to G_{E_3}$. The mapping $\psi^* \circ \phi^*$ coincides with $\psi \circ \phi$ on E_1 (as the unit interval of G_{E_1}), and by uniqueness of the extension, we obtain $(\psi \circ \phi)^* = \psi^* \circ \phi^*$, which entails $S(\psi \circ \phi) = S\psi \circ S\phi$. Similarly we can show that $T(f \circ g) = Tf \circ Tg$. This proves that both S and T are functors.

(ii) We have $S(E) = G_E$, $T(G_E) = G^+[0, u]$, so that $T \circ S(E) = G^+[0, u] \cong E$ by 3.4 (i). Conversely, $T((G, u)) = G^+[0, u]$, $S(G^+[0, u]) = G_{G^+[0,u]}$, so that $S \circ T((G, u)) = G_{G^+[0,u]} \cong (G, u)$ by Theorem 3.5. □

It was proved in [26] that the universal group for an MV-effect algebra is a lattice ordered group with generating unit. By a well-known result in [22], there is a categorical equivalence between MV-algebras with MV-algebra homomorphisms and lattice ordered groups with generating unit with unital homomorphisms of lattice ordered groups.

4. Simplicial groups and dimension groups

Let $\mathbb{Z}$ denote the ordered abelian group of integers with the standard positive cone $\mathbb{Z}^+ = \mathbb{Z} \cap [0, \infty)$. Here every nonzero element in $\mathbb{Z}^+$ is an order unit. If r is a positive integer, we understand that $\mathbb{Z}^r$ is organized into additive abelian group with coordinatewise addition. The standard partial order is the coordinatewise partial order determined by the standard total order on $\mathbb{Z}$, and the corresponding standard positive cone is $(\mathbb{Z}^+)^r$. An element $\mathbf{v} \in \mathbb{Z}^r$ is an order unit if and only if all of its coordinates are strictly positive. In this ordering $\mathbb{Z}^r$ is a lattice ordered, unperforated archimedean group. If $\mathbf{u} = (k_1, k_2, \dots, k_r)$ is an order unit, then the interval $(\mathbb{Z}^+)^r[\mathbf{0}, \mathbf{u}]$ is an MV-algebra which is the direct product of finite chains $(0, 1, \dots, k_i), i = 1, 2, \dots, r$. It can be shown that every finite effect algebra E with RDP is of this type. Indeed, RDP implies the interpolation property, and since E is finite, it is a lattice satisfying RDP, hence an MV-algebra. Every MV-algebra is a direct product of linearly ordered MV-algebras [7], hence E is a direct product of a finite number of finite chains. As we mentioned above, a partially ordered abelian group G is called *simplicial* if it is isomorphic with $\mathbb{Z}^r$ with standard positive cone $(\mathbb{Z}^+)^r$. Consequently, every finite effect algebra with RDP is a unit interval in a simplicial group.

Unital homomorphisms of simplicial groups can be described as follows. Let $\xi : \mathbb{Z}^m \to \mathbb{Z}^k$ be a positive group homomorphism such that $\xi(\mathbf{u}) = \mathbf{v}$ for some order units $\mathbf{u}$ in $\mathbb{Z}^m$ and $\mathbf{v}$ in $\mathbb{Z}^k$. Let $\mathbf{d}_1 = (1, 0, \dots, 0)$, $\mathbf{d}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{d}_m = (0, 0, \dots, 1)$ be the standard base for $\mathbb{Z}^m$, and similarly, let $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_k)$ be the standard base for $\mathbb{Z}^k$. Let $\xi(\mathbf{d}_i) = \sum_j d_{ij}\mathbf{e}_j$. If $\mathbf{x} = (x_1, x_2, \dots, x_m)$, then $\xi(\mathbf{x}) = \sum_{i \le m} x_i \xi(\mathbf{d}_i) = \sum_{i \le m} x_i \sum_{j \le k} d_{ij}\mathbf{e}_j = \sum_{j \le k}(\sum_{i \le m} x_i d_{ij})\mathbf{e}_j$. Thus ξ is determined by the matrix (d_{ij}) whose elements satisfy the system of linear equations $\sum_{i \le m} u_i d_{ij} = v_j$ obtained from $\xi(\mathbf{u}) = \mathbf{v}$. The restriction $\xi : [\mathbf{0}, \mathbf{u}] \to [\mathbf{0}, \mathbf{v}]$ of ξ is an effect algebra morphism. Conversely, every effect algebra morphism $\phi : [\mathbf{0}, \mathbf{u}] \to [\mathbf{0}, \mathbf{v}]$ of unit intervals extends uniquely to a unital group homomorphism $\phi^* : \mathbb{Z}^m \to \mathbb{Z}^k$.

We recall that dimension groups are defined as direct limits of directed systems of simplicial groups. In analogy, direct limits of directed systems of finite effect algebras with RDP will be called *dimension effect algebras*. From the properties of direct limits it follows that a dimension effect algebra has RDP. Direct limits for effect algebras have been studied and their existence was proved in [25, 19], see also [10, §1.9.7.].

THEOREM 4.1. *Let $(D_i; (f_{ij} : D_i \to D_j), i, j \in I, i \le j)$ be a directed system of effect algebras with RDP and let D be its direct limit. Then the universal group G_D*

of D is the direct limit of the directed system $((G_{D_i}, v_i); f_{ij}^* : G_{D_i} \to G_{D_j}, i \leq j)$ *of the universal groups of* $(G_{D_i}, v_i), i \in I$.

Proof: Every $f_{ij} : D_i \to D_j, i \leq j$ extends to a positive unital group homomorphism $f_{ij}^* : G_{D_i} \to G_{D_j}$, such that $f_{ii}^* = \mathrm{id}_{G_{D_i}}$ for every $i \in I$, and $f_{jm}^* f_{ij}^* = f_{im}^*$ whenever $i \leq j \leq m$. Hence $((G_{D_i}, v_i); f_{ij}^*)$ is a directed system of simplicial groups with order units. For every $i \in I$, $f_i : D_i \to D$ extends uniquely to $f_i^* : G_{D_i} \to G_D$, and if $i \leq j$, then $f_j f_{ij} = f_i$ implies $f_j^* f_{ij}^* = f_i^*$.

Let (H, z) be a unital group and $g_i : G_{D_i} \to H, i \in I$ be any system of unital positive group homomorphisms such that $g_j f_{ij}^* = g_i$ for all $i \leq j$ in I. The restriction $g_i^\flat$ of g_i to D_i (identified with $G_{D_i}[0, v_i]$) is an effect algebra morphism from $D_i \to H[0, z]$ for all $i \in I$. Clearly, $g_j^\flat f_{ij} = g_i^\flat$. Since D is the direct limit of $(D_i; (f_{ij}))$, there exists a unique effect algebra morphism $g^\flat : D \to H[0, z]$ such that $g^\flat f_i = g_i^\flat$. This $g^\flat$ uniquely extends to a positive unital group homomorphism g from G_D to H such that $g f_i^* = g_i$. Thus G_D is the direct limit of $((G_{D_i}, v_i); f_{ij}^*)$.□

It follows that the universal group of a (countable) dimension effect algebra is a (countable) dimension group. On the other hand, it can be seen that the unit interval of a dimension group is a dimension effect algebra.

Dimension groups have an intrinsic characterization as Riesz groups [11]. The next theorem gives an intrinsic characterization of dimension effect algebras. Clearly, an effect algebra with RDP is a dimension effect algebra iff its universal group is unperforated.

THEOREM 4.2. *Let E be an effect algebra with RDP and let G be its universal group. Then G is unperforated if and only if E satisfies the following condition: Let $n \in \mathbb{N}$ and let $c_{i,j}^{k,l} \in E$, $i, j = 1, \ldots, n$, $k = 1, \ldots, m$, $l = 1, \ldots, p$ be such that*

$$\oplus_{j,k} c_{i,j}^{k,l} = a_l^i = a_l, \qquad \text{for all } i, l, \tag{1}$$
$$\oplus_{i,l} c_{i,j}^{k,l} = b_k^j \leq b_k, \qquad \text{for all } j, k,$$

for some elements $a_l, b_k \in E$. Then there is some $q \in \mathbb{N}$ and some elements $d_{k,l} \in E$, $k = 1, \ldots, m$, $l = 1, \ldots, p+q$, such that

$$\oplus_k d_{k,l} = a_l, \qquad l = 1, \ldots, p, \tag{2}$$
$$\oplus_l d_{k,l} = b_k, \qquad k = 1, \ldots, m.$$

Proof: First, note that condition (2) is equivalent with the fact that $\sum_l a_l \leq \sum_k b_k$ in G. Indeed,

$$\sum_{l=1}^{p} a_l = \sum_{l=1}^{p} \sum_k d_{k,l} \leq \sum_{l=1}^{p+q} \sum_k d_{k,l} = \sum_k b_k.$$

Conversely, since $\sum_k b_k - \sum_l a_l \geq 0$ and $E \simeq G[0, u]$ generates G^+, there are some elements $c_1, \ldots, c_q \in E$, such that $\sum_k b_k = \sum_l a_l + \sum_{l'} c_{l'}$. The statement now follows from the RDP.

Suppose now that G is unperforated and let $c_{i,j}^{k,l} \in E$ satisfy condition (1). Let $a = \sum_l a_l$, $b = \sum_k b_k$, then

$$nb = \sum_k nb_k \geq \sum_k \sum_{j=1}^n b_k^j = \sum_{i,j,k,l} c_{i,j}^{k,l} = \sum_{i,l} a_l^i = na.$$

It follows that $n(b-a) \geq 0$, so that $b - a \geq 0$, that is, $\sum_k b_k \geq \sum_l a_l$.

To prove the converse, let $c \in G$ be such that $nc \geq 0$ for some n. Since G is directed, $c = b - a$ for some $a, b \in G^+$ and since E is generating, $a = \sum_l a_l$, $b = \sum_k b_k$ for some $a_l, b_k \in E$. From $na \leq nb$, we have by RDP that there are some elements $e_j \leq b$, $j = 1, \ldots, n$, such that $na = \sum_j e_j$. Moreover, since $e_j \leq \sum_k b_k$, there are some $b_k^j \leq b_k$, such that $e_j = \sum_k b_k^j$. Putting all together, we get

$$n \sum_l a_l = na = \sum_j e_j = \sum_{j,k} b_k^j.$$

Applying the RDP to this, we get that there are elements $c_{i,j}^{k,l} \in G^+$ such that $\sum_{j,k} c_{i,j}^{k,l} = a_l$ and $\sum_{i,l} c_{i,j}^{k,l} = b_k^j \leq b_k$. Then $c_{i,j}^{k,l}$ are elements in E, satisfying (1) and by assumptions, (2) must hold. This implies that $\sum_l a_l \leq \sum_k b_k$, hence $c = b - a \geq 0$. □

5. Effect algebras and AF C*-algebras

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A *-algebra A is called *standard matricial* if $A = M_{n(1)}(\mathbb{C}) \times \cdots \times M_{n(k)}(\mathbb{C})$. It is a well-known fact that every finite dimensional C*-algebra A is isomorphic to a standard matricial C*-algebra (see, e.g. [17]).

It was proved by Bratteli [5] that morphisms of standard matricial C*-algebras can be expressed by means of a so-called *standard maps*. If $A = M_{n(1)}(\mathbb{C}) \times \cdots \times M_{n(k)}(\mathbb{C})$, $B = M_{m(1)}(\mathbb{C}) \times \cdots \times M_{m(\ell)}(\mathbb{C})$, then every standard map $A \to B$ is determined by an $\ell \times k$ matrix (s_{ij}) of nonnegative integers such that

$$s_{i1}n(1) + s_{i2}n(2) + \cdots + s_{ik}n(k) = m(i), \qquad i = 1, 2, \ldots, \ell.$$

The construction of a standard map goes as follows. We first map A to $M_{m(1)}(\mathbb{C})$ by sending any k-tuple $(a_1, a_2, \ldots, a_k)$ from A to the block diagonal matrix with entries a_1 s_{11}-times, a_2 s_{12}- times,…, a_k s_{1k}-times. To obtain *-algebra maps $A \to B$, we paste together the maps from A to each of the matrix algebras $M_{m(i)}$.

According to [17, Lemma 17.1], to every *-algebra homomorphism $\phi : A \to B$ there exists an inner *-algebra automorphism β of B such that $\beta \circ \phi : A \to B$ is a standard map. According to [5], [17, Prop. 17.2], any AF C*-algebra is isomorphic (as a C*-algebra) to a C*-algebra direct limit of standard matricial C*-algebras and standard maps.

Let us consider the algebra $M_n(\mathbb{C})$. Two projections in $M_n(\mathbb{C})$ are equivalent iff they have the same dimension. Hence the range of dimension can be described by a finite chain of integers $(0, 1, \ldots, n)$. This chain can be endowed with a structure of an MV-effect algebra, its universal group is $\mathbb{Z}$, with order unit n, which is the dimension group for $M_n(\mathbb{C})$. The general case of a finite-dimensional C*-algebra is treated in the next lemma.

LEMMA 5.1. *Let A be a finite-dimensional C*-algebra which is isomorphic with $M_{n(1)}(\mathbb{C}) \times M_{n(2)}(\mathbb{C}) \times \cdots \times M_{n(r)}(\mathbb{C})$. Its dimension group is the simplicial group $\mathbb{Z}^r$ with order unit $(n(1), n(2), \ldots, n(r))$. Conversely, to every simplicial group $\mathbb{Z}^r$ with distinguished order unit $u = (u_1, u_2, \ldots, u_n)$ there exists a finite-dimensional C*-algebra A such that $\mathbb{Z}^r$ is its dimension group.*

Proof: The first part is proved in [17, Lemma 20.4]. The range $D(A)$ of the dimension equivalence of A can be described as follows. There are central projections $p_1, \ldots, p_r$, with the dimensions $\dim p_i = n(i), i \leq r$, in A such that $p_i A p_i$ is isomorphic with $M_{n(i)}(\mathbb{C})$, and $\sum_{i \leq r} p_i = 1$. Every projection q can be written as $q = \sum_{i \leq r} q p_i$, and to the equivalence class $[q]$, a sequence $(k_1, k_2, \ldots, k_r)$ can be assigned, where $k_i = \dim q p_i \leq n(i), i \leq r$. So the range of the dimension equivalence corresponds to the direct product of finite chains $(0, 1, \ldots, n(1)) \times \ldots \times (0, 1, \ldots, n(r))$, which is a finite-dimensional MV-algebra corresponding to the unit interval $(\mathbb{Z}^r)^+[0, \mathbf{u}]$, $\mathbf{u} = (n(1), n(2), \ldots, n(r))$.

Conversely, to any simplicial group $(\mathbb{Z}^r)$ with unit $\mathbf{u}$, $\mathbf{u} = (n(1), n(2), \ldots, n(r))$, we may assign the standard matricial algebra $M_{n(1)}(\mathbb{C}) \times M_{n(2)}(\mathbb{C}) \times \cdots \times M_{n(r)}(\mathbb{C})$, which is completely described by the n-tuple $(n(1), n(2), \ldots, n(r))$, and whose dimension group is the given simplicial group. $\square$

Comparing standard maps and positive unital homomorphisms of simplicial groups, we obtain the following (compare with [12, Lemma 5.3, Lemma 5.4]) proposition.

PROPOSITION 1. *Let D, F be finite effect algebras with RDP. Every effect algebra morphism $\phi : D \to F$ is induced by a *-algebra homomorphism $\psi : A \to B$, where A and B are finite-dimensional C*-algebras whose ranges of dimension are isomorphic with D and F, respectively.*

By [12], the situation is the following. To any sequence

$$A_1 \to A_2 \to \cdots$$

of finite-dimensional C*-algebras (unital, with the same unit) there is a canonically induced sequence

$$D(A_1) \to D(A_2) \to \cdots$$

of dimension effect algebras, where $D(A_i)$ denotes the range of dimension of A_i, which induce a sequence

$$G_1 \to G_2 \to \cdots$$

of simplicial groups, where G_i is the universal group for $D(A_i)$. If A is the direct limit of A_i, then $G = K_0(A)$ is the direct limit of $G_i = K_0(A_i)$, and hence the range $D(A)(= G^+[0, u])$ of its dimension is the direct limit of $D(A_i)$.

Conversely, if

$$D_1 \to D_2 \to \ldots$$

is an increasing sequence of finite effect algebras with RDP with direct limit D, then there is an increasing sequence

$$A_1 \to A_2 \to \cdots$$

of finite-dimensional C*-algebras such that

$$D(A_1) \to D(A_2) \to \cdots$$

is isomorphic with

$$D_1 \to D_2 \to \cdots$$

and if A is the direct limit of A_i, then $D(A)$ is isomorphic with D. Moreover, $K_0(A)$ is isomorphic with the universal group G_D of D.

So there is one-to-one correspondence between any two of the following classes: unital AF C*-algebras, countable dimension effect algebras and countable dimension groups. In what follows, we show that there are categorical equivalences between any two of the three corresponding categories. Owing to Theorem 3.8, it suffices to show that there is a categorical equivalence between AF C*-algebras and countable dimension effect algebras.

THEOREM 5.2. *There is a categorical equivalence between the category of unital AF C*-algebras and countable dimension effect algebras.*

Proof: Let $\mathcal{A}$ denote the category of AF C*-algebras with C*-algebra homomorphisms as morphisms, and let $\mathcal{D}$ denote the category of countable dimension effect algebras with effect algebra morphisms as morphisms. Let $S : \mathcal{A} \to \mathcal{D}$ denote the mapping such that $SA = D(A)$. If $\phi : A \to B$ is a C*-algebra homomorphism, its restriction to projections preserves the equivalence of projections, therefore it induces a well-defined mapping $S\phi : D(A) \to D(B)$. Clearly, if $\phi = \mathrm{id}_{\mathcal{A}}$, then $S\phi = \mathrm{id}_{\mathcal{D}}$, and $S(\phi \circ \psi) = S\phi \circ S\psi$, whenever $\phi \circ \psi$ makes sense. It follows that S is a functor.

Conversely, let $T : \mathcal{D} \to \mathcal{A}$ be the mapping which to every countable dimension effect algebra assigns the corresponding AF C*-algebra such that $D = D(A)$. We have to prove that T is a functor. Let $\psi : D \to F$ for $D, F \in \mathcal{D}$ be an effect algebra morphism. Then there are increasing sequences

$$D_1 \to D_2 \to \cdots$$

$$F_1 \to F_2 \to \cdots$$

of finite effect algebras with RDP, such that D is the direct limit of D_i and F is the direct limit of F_i. Let

$$A_1 \to A_2 \to \cdots$$

$$B_1 \to B_2 \to \cdots$$

be sequences of finite-dimensional C*-algebras induced by the sequences D_i and F_i, and let A, B be their corresponding limits.

For every D_i, there is $F_{k(i)}$ such that $\psi_i : D_i \to F_{k(i)}$, where ψ_i is the restriction of ψ to D_i. If $i \leq j$ then since $D_i \subset D_j$, we may assume that $F_{k(i)} \subset F_{k(j)}$. By Proposition 1, $\psi_i : D_i \to F_{k(i)}$ is induced by a C*-algebra morphism $\psi_i^* : A_i \to B_{k(i)}$. Since $\psi_j \circ f_{ij} = \psi_i$, where $f_{ij} : D_i \to D_j$, $i \leq j$, we have $\psi_j^* \circ f_{ij}^* = \psi_i^*$, where $f_{ij}^* : A_i \to A_j$, $i \leq j$. By the properties of direct limits, the morphisms $\psi_i^* : A_i \to B$ then uniquely extend to a morphism $\psi^* : A \to B$. Denoting $\psi^* = T\psi$, it is easily seen that $T : \mathcal{D} \to \mathcal{A}$ is a functor.

Now we have, for $A \in \mathcal{A}$, $TSA = TD(A) \cong A$, and for every dimension effect algebra $STD = D(TD) \cong D$. This proves the statement. $\square$

We note that, taking into account Theorem 3.5 and the fact that $D(A)$ is the unit interval in $K_0(A)$, the K_0 group for an AF C*-algebra A can be constructed from $D(A)$ using the word method, described in [26, 10]. This may replace consideration of all matrix algebras over A in the construction of $K_0(A)$.

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