Generalized channels and quantum networks

Anna Jenčová
Mathematical Institute, Slovak Academy of Sciences

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Measurement of quantum channels

Let $\mathcal{H}_0$, $\mathcal{H}_1$ be finite dimensional Hilbert spaces,
$\mathcal{E} : B(\mathcal{H}_0) \to B(\mathcal{H}_1)$ a channel: (cp trace preserving map).
Suppose $\mathcal{E}$ is unknown, we want to identify it by a measurement:

1. prepare of a state $\rho \in B(\mathcal{H}_0 \otimes \mathcal{H}_A)$, $\mathcal{H}_A$ an ancillary Hilbert space,
2. apply $\mathcal{E} \otimes id_{\mathcal{H}_A}$ to ρ
3. measure the output by a POVM $\Lambda : \Omega \to B(\mathcal{H}_1 \otimes \mathcal{H}_A)$i

This produces an outcome $i \in \Omega$ with probability

$$p_i(\mathcal{E}) = \mathrm{Tr} \Lambda_i (\mathcal{E} \otimes id_{\mathcal{H}_A})(\rho)$$

A general channel mesurement

A general definition of channel measurement: an affine map

$$\mathcal{E} \mapsto p(\mathcal{E}) \in \mathcal{P}(\Omega) = \text{set of probability measures on } \Omega$$

Question

Are all channel measurements given by some triple $(\mathcal{H}_A, \rho, \Lambda)$?

Quantum 1- testers (PPOVMs)

Let $X_{\mathcal{E}} \in B(\mathcal{H}_1 \otimes \mathcal{H}_0)$ denote the Choi represenation of $\mathcal{E}$:

$$X_{\mathcal{E}} = (\mathcal{E} \otimes id_{\mathcal{H}_0})(\Psi), \quad \Psi = \sum_{i,j} |i\rangle\langle j| \otimes |i\rangle\langle j|$$

A quantum 1-tester: $\{M_i, i \in \Omega\}$, $M_i \in B(\mathcal{H}_1 \otimes \mathcal{H}_0)^+$,

$$\sum_i M_i = I_{\mathcal{H}_1} \otimes \omega, \qquad \omega \in \mathfrak{S}(\mathcal{H}_0)$$

This defines a channel measurement $\mathcal{E} \mapsto p_i(\mathcal{E}) = \mathrm{Tr} M_i X_{\mathcal{E}}$, $i \in \Omega$.

Theorem

Each triple $(\mathcal{H}_A, \rho, \Lambda)$ is associated with a 1-tester. Each 1-tester is implemented by a triple $(\mathcal{H}_A, \rho, \Lambda)$.

(Chiribella, D'Ariano, Perinotti, Phys. Rev. Lett.(2008)
Ziman, Phys. Rev. A(2008) (1-testers are called proces POVMs))

The set $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$

Let $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) \subset B(\mathcal{H}_1 \otimes \mathcal{H}_0)$ denote the set of Choi representations of channels $B(\mathcal{H}_0) \to B(\mathcal{H}_1)$:

$$\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) = \{X \in B(\mathcal{H}_1 \otimes \mathcal{H}_0)^+, \mathrm{Tr}_{\mathcal{H}_1} X = I_{\mathcal{H}_0}\}$$

Question

Are all affine maps $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_0) \to \mathcal{P}(\Omega)$ given by 1-testers?

Note that

$$\mathcal{C}(\mathcal{H}_0, \mathcal{H}_1) = J \cap d_0 \mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_0), \qquad d_0 = \dim(\mathcal{H}_0)$$

where $J \subset B(\mathcal{H}_1 \otimes \mathcal{H}_0)$ is the subspace

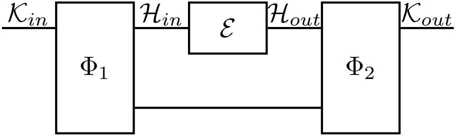
$$J = \{X \in B(\mathcal{H}_1 \otimes \mathcal{H}_0), \mathrm{Tr}_{\mathcal{H}_1} X = z I_{\mathcal{H}_0},\ z \in \mathbb{C}\}$$

Quantum supermaps

- A quantum supermap is a map that maps channels $B(\mathcal{H}_{in}) \to B(\mathcal{H}_{out})$ to channels $B(\mathcal{K}_{in}) \to B(\mathcal{K}_{out})$.
- Such maps are given by cp maps $\mathcal{S} : B(\mathcal{H}_{out} \otimes \mathcal{H}_{in}) \to B(\mathcal{K}_{out} \otimes \mathcal{K}_{in})$, such that

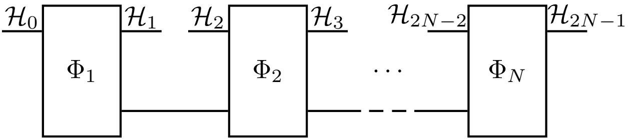
$$\mathcal{S}(\mathcal{C}(\mathcal{H}_{in}, \mathcal{H}_{out})) \subset \mathcal{C}(\mathcal{K}_{in}, \mathcal{K}_{out})$$

- Any such map has the form:



(Chiribella, D'Ariano, Perinotti, Europhysics Letters 2008)

Quantum networks and combs



Quantum network

This defines a channel

$$\Phi : B(\mathcal{H}_0 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_{2N-2}) \to B(\mathcal{H}_1 \otimes \mathcal{H}_3 \otimes \cdots \otimes \mathcal{H}_{2N-1}).$$

Its Choi representation X_Φ is a (deterministic) quantum N-comb.
Let

$$\mathrm{Comb}(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$$

denote the set of (deterministic) N combs.

Characterization of quantum combs

(Gutoski& Watrous, STOC 2007 (quantum strategies), Chiribella, D'Ariano, Perinotti, Phys. Rev. A 2009)

Theorem

An element $X \in B(\otimes_{j=0}^{2N-1}\mathcal{H}_j)$ is a quantum N-comb if and only if there are $X^{(k)} \in B(\otimes_{j=0}^{2k-1}\mathcal{H}_j)^+$, $X^{(N)} = X$, such that

$$\begin{aligned} \mathrm{Tr}_{\mathcal{H}_{2k-1}} X^{(k)} &= I_{\mathcal{H}_{2k-2}} \otimes X^{(k-1)}, \quad k = 2, \ldots, N \\ \mathrm{Tr}_{\mathcal{H}_1} X^{(1)} &= I_{\mathcal{H}_0} \end{aligned}$$

It follows that

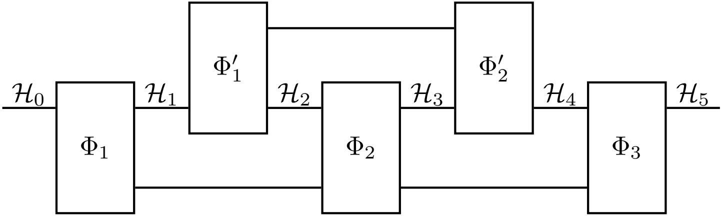
$$\mathrm{Comb}(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1}) = J_N \cap d_N \mathfrak{S}(\otimes_{j=0}^{2N-1}\mathcal{H}_j)$$

for some subspace J_N and $d_N > 0$.

Quantum N combs as supermaps

Quantum N-combs can be defined as follows:

1. $\mathrm{Comb}(\mathcal{H}_0, \mathcal{H}_1) = \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$.
2. $\mathrm{Comb}(\mathcal{H}_0, \dots, \mathcal{H}_{2N-1})$ is the set of Choi operators of cp maps that transform $\mathrm{Comb}(\mathcal{H}_1, \dots, \mathcal{H}_{2N-2})$ to $\mathcal{C}(\mathcal{H}_0, \mathcal{H}_{2N-1})$.



Conclusion

Quantum networks (supermaps, testers,...) are given by (constant multiples of) cp maps that map $J \cap \mathfrak{S}(\mathcal{H})$ into $\mathfrak{S}(\mathcal{K})$.

Algebras and subspaces

Let $\mathcal{A}$ be a finite dim C^*-algebra. We suppose

$$\mathcal{A} = \oplus_i B(\mathcal{H}_i) \subset B(\mathcal{H}), \qquad \mathcal{H} = \oplus_i \mathcal{H}_i.$$

We denote

- $\mathrm{Tr}_{\mathcal{A}}$ the trace inherited from $B(\mathcal{H})$
- a^T the transpose of a
- $\mathcal{A}^+$ the positive cone in $\mathcal{A}$
- $\mathfrak{S}(\mathcal{A}) = \{\rho \in \mathcal{A}^+, \mathrm{Tr}_{\mathcal{A}}\rho = 1\}$ the set of density operators

Let $J \subset \mathcal{A}$ be a positively generated subspace. We denote

- $J^T = \{a^T, a \in J\}$ is positively generated.
- $J^\perp = \{x \in \mathcal{A}, \mathrm{Tr}_{\mathcal{A}}(x^* a) = 0,\ a \in J\}$ self-adjoint subspace

Channels

Let $K \subset \mathfrak{S}(\mathcal{A})$ be a convex subset, $J = \mathrm{span}(K)$.

Definition

A channel $\Xi : K \to \mathfrak{S}(\mathcal{B})$ is an affine map, extending to a cp map $\Xi : J \to \mathcal{B}$.

- $\Xi : J \to \mathcal{B}$ is trace preserving
- we may suppose that K is a section of $\mathfrak{S}(\mathcal{A})$:

$$K = J \cap \mathfrak{S}(\mathcal{A})$$

- If $K = \mathfrak{S}(\mathcal{A})$, then Ξ is a channel $\mathcal{A} \to \mathcal{B}$.

An extension theorem

The following is an easy consequence of Arveson's extension theorem:

Theorem

Let $J \subset \mathcal{A}$ be a positively generated subspace and let $\Xi : J \to \mathcal{B}$ be a cp map. There is a cp map $\Phi : \mathcal{A} \to \mathcal{B}$ such that $\Phi(a) = \Xi(a)$, $a \in J$.

Any channel $\Xi : K \to \mathfrak{S}(\mathcal{B})$ is the restriction of a generalized channel with respect to J:
a cp map $\Phi : \mathcal{A} \to \mathcal{B}$ that preserves trace on J

The Choi representation

Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear map. The Choi representation of Φ: $X_\Phi \in \mathcal{B} \otimes \mathcal{A}$, such that

$$\Phi(a) = \mathrm{Tr}_{\mathcal{A}}(I_{\mathcal{B}} \otimes a^T)X_\Phi$$

Φ is cp if and only if $X_\Phi \geq 0$

Theorem

$\Phi : \mathcal{A} \to \mathcal{B}$ is a generalized channel with respect to J if and only if

$$X_\Phi \geq 0, \qquad \mathrm{Tr}_{\mathcal{B}} X_\Phi \in I_{\mathcal{A}} + (J^T)^\perp$$

We denote by $\mathcal{C}_J(\mathcal{A}, \mathcal{B})$ the set of all (Choi representations of) generalized channels, $\mathcal{C}_{\mathcal{A}}(\mathcal{A}, \mathcal{B}) = \mathcal{C}(\mathcal{A}, \mathcal{B})$.

Decomposition of generalized channels

Let $c \in \mathcal{A}$, define $\chi_c : \mathcal{A} \to \mathcal{A}$, $\chi_c(a) = cac^*$. Then $\chi_c \in \mathcal{C}_J(\mathcal{A}, \mathcal{A})$ if and only if

$$c^* c \in \mathcal{A}^+ \cap (I_\mathcal{A} + J^\perp)$$

Such generalized channels are called simple.

Theorem

$\Phi \in \mathcal{C}_J(\mathcal{A}, \mathcal{B})$ if and only if there is a pair (χ_c, Λ), $\chi_c \in \mathcal{C}_J(\mathcal{A}, \mathcal{A})$ is simple and $\Lambda \in \mathcal{C}(\mathcal{A}, \mathcal{B})$, such that

$$\Phi = \Lambda \circ \chi_c$$

Moreover, there is a minimal decomposition with $p = \mathrm{supp}(cc^*)$ and $\Lambda \in \mathcal{C}(p\mathcal{A}p, \mathcal{B})$, which is unique (up to a unitary transformation).

Note that we have $c^* c = \Phi^*(I_\mathcal{B}) = (\mathrm{Tr}_\mathcal{B} X_\Phi)^T$.

A general example

Suppose that $J \subset \mathcal{A}$ is a positively generated subspace and $J = S^{-1}(J_0)$, where $S : \mathcal{A} \to \mathcal{A}_0$ is a channel and $J_0 \subset \mathcal{A}_0$ is a subspace. Then $\Phi : \mathcal{A} \to \mathcal{B}$ is a generalized channel with respect to J if and only if

$$\mathrm{Tr}_{\mathcal{B}} X_\Phi \in (S^T)^*(I_{\mathcal{A}_0} + (J_0^T)^\perp)$$

where $S^T : \mathcal{A} \to \mathcal{A}_0$ is defined as $S^T(a) = [S(a^T)]^T$. In particular, if $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_0$ and S is a partial trace, then

$$\mathrm{Tr}_{\mathcal{B}} X_\Phi = I_{\mathcal{A}_1} \otimes a, \qquad a \in \mathcal{A}_0^+ \cap (I_{\mathcal{A}_0} + (J_0^T)^\perp)$$

Example: "Channels on channels"

Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_0$, $S = \mathrm{Tr}_{\mathcal{A}_1}$, $J_0 = \mathbb{C} I_{\mathcal{A}_0}$. Then $J = S^{-1}(J_0)$ is generated by $\mathcal{C}(\mathcal{A}_0, \mathcal{A}_1)$ and

$$\mathcal{C}(\mathcal{A}_0, \mathcal{A}_1) = J \cap t_0 \mathfrak{S}(\mathcal{A}_1 \otimes \mathcal{A}_0), \qquad t_0 = \mathrm{Tr}_{\mathcal{A}_0}(I_{\mathcal{A}_0})$$

Channels $\Xi : \mathcal{C}(\mathcal{A}_0, \mathcal{A}_1) \to \mathfrak{S}(\mathcal{B})$ are given by cp maps $\Phi : \mathcal{A} \to \mathcal{B}$, such that $t_0 \Phi \in \mathcal{C}_J(\mathcal{A}, \mathcal{B})$, that is, such that

$$\mathrm{Tr}_{\mathcal{B}} X_\Phi \in I_{\mathcal{A}_1} \otimes (\mathcal{A}_0^+ \cap (t_0^{-1} I_{\mathcal{A}_0} + \{I_{\mathcal{A}_0}\}^\perp)) = I_{\mathcal{A}_1} \otimes \mathfrak{S}(\mathcal{A}_0)$$

"Channels on channels": decomposition

Suppose $\mathcal{A}_0 = B(\mathcal{H}_0)$. Let

$$\mathrm{Tr}_{\mathcal{B}} X_\Phi = I_{\mathcal{A}_1} \otimes \omega, \qquad \omega \in \mathfrak{S}(\mathcal{H}_0).$$

Then there is a a minimal decomposition of Φ:

$$\Phi = \Lambda \circ \chi_{I \otimes c},$$

where $c^* c = \omega^T$, $\Lambda \in \mathcal{C}(\mathcal{A}_1 \otimes p\mathcal{A}_0 p, \mathcal{B})$, $p = \mathrm{supp}(cc^*)$. Let $X_{\mathcal{E}} \in \mathcal{C}(\mathcal{A}_0, \mathcal{A}_1)$, then

$$\chi_{I \otimes c}(X_{\mathcal{E}}) = \chi_{I \otimes c}(\mathcal{E} \otimes id_{\mathcal{H}_0})(\Psi) = (\mathcal{E} \otimes id_{p\mathcal{H}_0})(\rho)$$

where $\rho = \chi_{I \otimes c}(\Psi) \in \mathfrak{S}(\mathcal{H}_0 \otimes p\mathcal{H}_0)$ is a pure state such that $\mathrm{Tr}_2 \rho = \omega^T$.

"Chanels on channels": implementation

Let $\mathcal{A}_0 = B(\mathcal{H}_0)$.

Theorem

$\Phi : \mathcal{A}_1 \otimes \mathcal{A}_0 \to \mathcal{B}$ defines a channel $\mathcal{C}(\mathcal{A}_0, \mathcal{A}_1) \to \mathfrak{S}(\mathcal{B})$ if and only if there is an ancilary Hilbert space $\mathcal{H}_A$, a pure state $\rho \in \mathfrak{S}(\mathcal{H}_0 \otimes \mathcal{H}_A)$ and a channel $\Lambda : \mathcal{A}_1 \otimes B(\mathcal{H}_A) \to \mathcal{B}$ such that

$$\Phi(X_{\mathcal{E}}) = \Lambda \circ (\mathcal{E} \otimes id_{\mathcal{H}_A})(\rho)$$

Moreover, the triple $(\mathcal{H}_A, \rho, \Lambda)$ is unique (up to a unitary transformation).

Measurements on K and generalized POVMs

Let $K \subset \mathfrak{S}(\mathcal{A})$ be any convex subset. A measurement on K with values in Ω, $|\Omega| = m$ is an affine map

$$K \to \mathcal{P}(\Omega)$$

Let $J = \mathrm{span}(K)$, then any $\Phi \in \mathcal{C}_J(\mathcal{A}, \mathbb{C}^m)$ defines a measurement on K. We have $X_\Phi = \sum_j |j\rangle\langle j| \otimes M_j^T$ and

$$\Phi(a) = (\mathrm{Tr}_{\mathcal{A}}(M_1 a), \dots, \mathrm{Tr}_{\mathcal{A}}(M_m a)), \qquad a \in K$$

where

$$M_i \in \mathcal{A}^+, \qquad \sum_i M_i \in I_{\mathcal{A}} + J^{\perp}$$

$M = (M_1, \dots, M_m)$ is called a generalized POVM with respect to J.

Measurements on sections

Let $K \subseteq \mathfrak{S}(\mathcal{A})$ be a section of $\mathfrak{S}(\mathcal{A})$:

$$K = J \cap \mathfrak{S}(\mathcal{A}), \qquad J = \mathrm{span}(K)$$

Equivalently

(i) $K = L \cap \mathfrak{S}(\mathcal{A})$, $L \subset \mathcal{A}$ is any subspace

(ii) Let $V = \cup_{\lambda \geq 0} \lambda K$, then $a, b \in V$, $a - b \geq 0$ implies $a - b \in V$.

Theorem

All measurements on K are given by generalized POVMs if and only if K is a section of $\mathfrak{S}(\mathcal{A})$, that is, $K = J \cap \mathfrak{S}(\mathcal{A})$.

Example: quantum 1-testers (PPOVMs)

Let $K = \mathcal{C}(\mathcal{H}_0, \mathcal{H}_1)$, then K is a (multiple of) a section of $\mathfrak{S}(\mathcal{H}_1 \otimes \mathcal{H}_0)$. Hence any channel measurement is given by a generalized POVM with respect to $J = \mathrm{span}(K)$:

$$(M_1, \dots, M_m), \quad M_i \in B(\mathcal{H}_1 \otimes \mathcal{H}_0)^+, \ \sum_i M_i = I_{\mathcal{H}_1} \otimes \omega$$

where $\omega \in \mathfrak{S}(\mathcal{H}_0)$, this is a quantum 1-tester (PPOVM).

Theorem

There exists an ancilla $\mathcal{H}_A$, a pure state $\rho \in \mathfrak{S}(\mathcal{H}_0 \otimes \mathcal{H}_A)$ and a POVM $\Lambda : \{1, \dots, m\} \to B(\mathcal{H}_1 \otimes \mathcal{H}_A)$ (unique up to a unitary) such that

$$\mathrm{Tr}(X_{\mathcal{E}} M_i) = \mathrm{Tr}(\Lambda_i(\mathcal{E} \otimes id_{\mathcal{H}_A})(\rho)), \quad i = 1, \dots, m$$

Example: unital channels

Let $\mathcal{A}_0 = \mathcal{A}_1 = B(\mathcal{H})$, $\dim(\mathcal{H}) = d$.
Let $K = \mathcal{C}^U(\mathcal{H})$ = unital channels $B(\mathcal{H}) \to B(\mathcal{H})$. Then

$$K = \mathrm{Tr}_1^{-1}(\mathbb{C}I_{\mathcal{H}}) \cap \mathrm{Tr}_2^{-1}(\mathbb{C}I_{\mathcal{H}}) \cap d\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$$

is (d-times) a section of $\mathfrak{S}(\mathcal{H} \otimes \mathcal{H})$. Hence any measurement on unital channels is given by a generalized POVM with respect to $J = \mathrm{Tr}_1^{-1}(\mathbb{C}I_{\mathcal{H}}) \cap \mathrm{Tr}_2^{-1}(\mathbb{C}I_{\mathcal{H}})$.
These are collections $(M_1, \ldots, M_m)$, $M_i \in B(\mathcal{H} \otimes \mathcal{H})^+$, such that

$$\sum_i M_i = \lambda \omega_1 \otimes I_{\mathcal{H}} + (1 - \lambda) I_{\mathcal{H}} \otimes \omega_2$$

where $\lambda \in [0, 1]$, $\omega_1, \omega_2 \in \mathfrak{S}(\mathcal{H})$.

Implementation ?

The set $\mathcal{C}_J(\mathcal{A},\mathcal{B})$

Let $J \subset \mathcal{A}$ be a positively generated subspace. Let

$$\tilde{J} := \mathrm{span}((J^T)^\perp \cup \{I_\mathcal{A}\})$$

Then we have

$$\mathcal{C}_J(\mathcal{A},\mathcal{B}) = \mathrm{Tr}_\mathcal{B}^{-1}(\tilde{J}) \cap \mathfrak{S}_{I_\mathcal{B}\otimes\rho^T}(\mathcal{B}\otimes\mathcal{A})$$

for any $\rho \in J \cap \mathfrak{S}(\mathcal{A})$, where

$$\mathfrak{S}_{I_\mathcal{B}\otimes\rho^T}(\mathcal{B}\otimes\mathcal{A}) = \{a \in (\mathcal{B}\otimes\mathcal{A})^+,\ \mathrm{Tr}((I_\mathcal{B}\otimes\rho^T)a) = 1\}$$

In particular, if $I_\mathcal{A} \in J$ then $\mathcal{C}_J(\mathcal{A},\mathcal{B})$ is a (multiple of) a section of $\mathfrak{S}(\mathcal{B}\otimes\mathcal{A})$:

$$\mathcal{C}_J(\mathcal{A},\mathcal{B}) = \mathrm{Tr}_\mathcal{B}^{-1}(\tilde{J}) \cap t_\mathcal{A}\mathfrak{S}(\mathcal{B}\otimes\mathcal{A}), \quad t_\mathcal{A} := \mathrm{Tr}_\mathcal{A}(I_\mathcal{A})$$

Generalized supermaps

Let $\mathcal{B}_0, \mathcal{B}_1, \ldots$ be finite dimensional C^*-algebras.

- Suppose $J \subset \mathcal{B}_0$ is positively generated, $I_{\mathcal{B}_0} \in J$.
- Then $\mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_1)$ is (multiple of) a section of $\mathfrak{S}(\mathcal{B}_1 \otimes \mathcal{B}_0)$.
- Channels $\mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_1) \to \mathfrak{S}(\mathcal{B}_2)$ are given (multiples of) generalized channels with respect to $\mathrm{Tr}_{\mathcal{B}_1}^{-1}(\tilde{J})$.
- $I_{\mathcal{B}_1 \otimes \mathcal{B}_0} \in \mathrm{Tr}_{\mathcal{B}_1}^{-1}(\tilde{J})$
- ...

Generalized supermaps

We denote $\mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_n)$ the set of (Choi representations of) cp maps $\mathcal{B}_{n-1} \otimes \cdots \otimes \mathcal{B}_0 \to \mathcal{B}_n$ that map $\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{n-1})$ to $\mathfrak{S}(\mathcal{B}_n)$.

Characterization of generalized supermaps I

Let us denote $\mathcal{A}_n = \mathcal{B}_n \otimes \mathcal{B}_{n-1} \otimes \cdots \otimes \mathcal{B}_0,\ n = 0, 1, \ldots,$
$S_n : \mathcal{A}_n \to \mathcal{A}_{n-1}$ the partial trace $\mathrm{Tr}_{\mathcal{B}_n},\ n = 1, 2, \ldots.$

$$\mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_n) = J_n \cap c_n \mathfrak{S}(\mathcal{A}_n)$$

where $c_n = \Pi_{l=0}^{\lfloor \frac{n-1}{2} \rfloor} t_{\mathcal{B}_{n-1-2l}}$ and

$$\begin{aligned} J_{2k-1} &= S_{2k-1}^{-1}(S_{2k-2}^{*}(S_{2k-3}^{-1}(\ldots S_1^{-1}(\tilde{J}) \ldots))) \\ J_{2k} &= S_{2k}^{-1}(S_{2k-1}^{*}(S_{2k-2}^{-1}(\ldots S_1^{*}(J) \ldots))) \end{aligned}$$

$k = 1, 2, \ldots.$

Characterization of generalized supermaps II

Let $k := \lfloor \frac{n}{2} \rfloor$. Then $X \in \mathcal{C}_J(\mathcal{B}_0, \dots, \mathcal{B}_n)$ if and only if there are $Y^{(m)} \in \mathcal{A}^+_{n-2m}$ for $m = 0, \dots, k$, such that

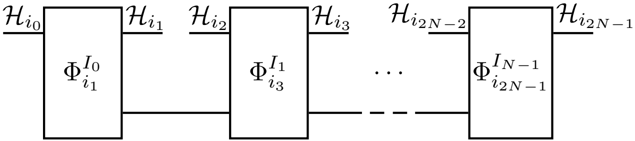
$$\mathrm{Tr}_{\mathcal{B}_{n-2m}} Y^{(m)} = I_{\mathcal{B}_{n-2m-1}} \otimes Y^{(m+1)}, m = 0, \dots, k-1$$

$Y^{(0)} := X$ and

$$\begin{aligned} Y^{(k)} &\in \mathcal{C}_J(\mathcal{B}_0, \mathcal{B}_1) \quad \text{if } n = 2k+1 \\ Y^{(k)} &\in J \cap \mathfrak{S}(\mathcal{B}_0) \quad \text{if } n = 2k \end{aligned}$$

Decomposition of generalized supermaps

Let $n = 2N - 1$.
Then any $\Phi \in \mathcal{C}_J(\mathcal{B}_0, \ldots, \mathcal{B}_{2N-1})$ is implemented as a sequence of networks with classical inputs and outputs:



$i_0, i_1, \ldots$ classical inputs/outputs, $I_j = i_0 i_1 \ldots i_{2j}$

where Φ^{I_0} is a generalized channel, Φ^{I_j}, $j > 0$ are channels, with classical inputs and outputs, labelled by all classical outcomes before step j.

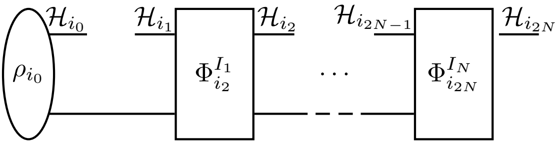
Decomposition of generalized supermaps

Let $\mathcal{B}_j = B(\mathcal{H}_j)$, $j = 0, 1, \dots$ and let $J = \mathcal{B}_0$. Then

$$\mathcal{C}(\mathcal{B}_0, \dots, \mathcal{B}_{2N-1}) = \mathrm{Comb}(\mathcal{H}_0, \dots, \mathcal{H}_{2N-1}), \quad N = 1, 2, \dots$$

If $\mathcal{B}_0, \mathcal{B}_1, \dots$ (finite dim) C^*-algebras, then $\mathcal{C}(\mathcal{B}_0, \dots, \mathcal{B}_{2N-1})$ is the set of conditional combs,
(Chiribella et al, arxiv:0905.3801)

Let $n = 2N$, $\mathcal{B}_0, \mathcal{B}_1, \dots,$ $J \subset \mathcal{B}_0$ arbitrary. Then



$i_0, i_1, \dots$ classical inputs/outputs, $I_j = i_0 i_1 \dots i_{2j-1}$.

$\rho \in \mathrm{Tr}^{-1}_{\mathcal{H}_A}(J) \cap \mathfrak{S}(\mathcal{B}_0 \otimes B(\mathcal{H}_A))$.

Extremal generalized channels

We find extreme points of the set $\mathcal{C}_J(\mathcal{A},\mathcal{B})$:

$$\mathcal{C}_J(\mathcal{A},\mathcal{B}) = (\mathcal{B}\otimes\mathcal{A})^+ \cap L$$

where $L = \mathrm{Tr}_{\mathcal{B}}^{-1}(I_{\mathcal{A}} + (J^T)^\perp)$ is an affine subspace. Let

$$L_0 := \mathrm{Lin}(L) = \mathrm{Tr}_{\mathcal{B}}^{-1}((J^T)^\perp)$$

Theorem

Let $X \in \mathcal{C}_J(\mathcal{A},\mathcal{B})$, $P = \mathrm{supp}(X)$. Then X is an extreme point if and only if

$$P(\mathcal{B}\otimes\mathcal{A})P \cap L_0 = \{0\}$$

Extremal generalized channels: decomposition

Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a generalized channel with respect to J and

$$\Phi = \Lambda \circ \chi_c$$

be a minimal decomposition. Let X_Λ be the Choi representation of Λ, $p = \mathrm{supp}(cc^*)$.

Theorem

X_Φ is extremal in $\mathcal{C}_J(\mathcal{A}, \mathcal{B})$ if and only if X_Λ is extremal in $\mathcal{C}_{J_c}(p\mathcal{A}p, \mathcal{B})$ where $J_c = \chi_c(J) = cJc^*$.

In particular, Λ must be an extremal channel. (Note that Λ is a generalized channel with respect to any subspace.)

Extremal generalized channels: Kraus operators

Suppose $\mathcal{A} = B(\mathcal{H})$, $\mathcal{B} = B(\mathcal{K})$.
Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a cp map and let

$$\Phi(a) = \sum_k V_k a V_k$$

be a Kraus representation. Then Φ is a generalized channel with respect to J if and only if $\sum_k V_k^* V_k \in I_{\mathcal{A}} + J^{\perp}$.

Theorem

Φ is extremal if and only if there is a Kraus representation such that the set

$$\{V_k^* V_l, k, l\} \cup \{x_1, \dots x_m\}$$

is linearly independent, where $\{x_1, \dots, x_m\}$ is a basis of $J^{\perp}$.

Extremal generalized channels: Kraus operators

Let

$$\Phi = \Lambda \circ \chi_c$$

be a minimal decomposition and let $\Lambda(a) = \sum_k W_k a W_k^*$ be a Kraus representation.

Theorem

Φ is extremal if and only if Λ is an extremal channel and

$$\mathrm{span}\{W_k^* W_l, k, l\} \cap J_c^\perp = \{0\}$$

($J_c = cJc^*$)

Extremal generalized channels: conjugate maps

Let $\Phi(a) = \sum_{k=1}^{n} V_k a V_k^*$ be a minimal Kraus representation. The cojugate map $\Phi^C : B(\mathcal{H}) \to B(\mathbb{C}^n)$ is defined by

$$\Phi^C(a) = \sum_{k,l=}^{n} \mathrm{Tr}(V_k a V_l^*)|k\rangle\langle l|$$

Since $\mathrm{Tr}(\Phi(a)) = \mathrm{Tr}(\Phi^C(a))$, Φ^C is a generalized channel with respect to J if and only if Φ is.

Theorem

Φ is extremal if and only if $\Phi^C(J) = B(\mathbb{C}^n)$.

Extremal quantum networks

Since

$$\mathcal{C}(\mathcal{B}_0, \ldots, \mathcal{B}_n) = \frac{1}{c_{n-1}} \mathcal{C}_{J_{n-1}}(\mathcal{A}_{n-1}, \mathcal{B}_n)$$

extremal quantum networks are obtained from extreme points in $\mathcal{C}_{J_{n-1}}(\mathcal{A}_{n-1}, \mathcal{B}_n)$. Let $\mathcal{T}_n = \{a \in \mathcal{B}_n, \mathrm{Tr}_{\mathcal{B}_n} a = 0\}$, $I_n = I_{\mathcal{B}_n}$. Then we have

$$L_0 = (I_0 \otimes J_{n-1}^{\perp}) \vee (\mathcal{T}_n \otimes \mathcal{A}_{n-1})$$

and

$$\begin{aligned}
J_1^{\perp} &= I_1 \otimes \mathcal{T}_0 \\
J_2^{\perp} &= I_2 \otimes \mathcal{T}_1 \otimes \mathcal{B}_0 \\
J_n^{\perp} &= I_n \otimes \left[(I_{n-1} \otimes J_{n-2}^{\perp}) \vee (\mathcal{T}_{n-1} \otimes \mathcal{A}_{n-2}) \right], \quad n \geq 3
\end{aligned}$$

Extremal "channels on channels"

Let $X \in \mathcal{C}(\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2)$ and let

$$\Phi_X = \Lambda \circ \chi_{I \otimes c}$$

be a minimal decomposition.

Theorem

Let $\Lambda(a) = \sum_k W_k a W_k^*$ be a Kraus representation of Λ. Then X is extremal if and only if Λ is an extremal channel and

$$\mathrm{span}(W_k^* W_l) \cap (I_1 \otimes \mathcal{T}_0) = \{0\}$$

Extremal generalized POVMs

Let

$$\mathcal{M}_J(\mathcal{A},\Omega) = \{(M_i,\ i \in \Omega), M_i \in \mathcal{A}^+, \sum_i M_i \in I_{\mathcal{A}} + J^\perp\},$$

then $\mathcal{M}_J(\mathcal{A},\Omega) \equiv \mathcal{C}_J(\mathcal{A},\mathbb{C}^{|\Omega|})$.

Theorem

Let $M \in \mathcal{M}_J(\mathcal{A},\Omega)$, $p_i := \mathrm{supp}(M_i)$, $i \in \Omega$. Then M is extremal if and only if for any collection $(D_i, i \in \Omega), D_i \in p_i\mathcal{A}p_i$, $\sum_i D_i \in J^\perp$ implies that $D_i = 0$ for all $i \in \Omega$.

Extremal generalized POVMs: decomposition

Let $M \in \mathcal{M}_J(\mathcal{A}, \Omega)$, then there is a minimal decomposition

$$M = \Lambda \circ \chi_c$$

with Λ a POVM in $\mathcal{M}(p\mathcal{A}p, \Omega)$, $p = \mathrm{supp}(cc^*)$, $\sum_i M_i = c^*c$.

Theorem

M is extremal if and only if Λ is extremal in $\mathcal{M}_{cJc^*}(p\mathcal{A}p, \Omega)$.

Theorem

Suppose Λ is a PVM. Then M is extremal if and only if

$$\{\Lambda_i, i \in \Omega\}' \cap (cJc^*)^{\perp} \cap p\mathcal{A}p = \{0\}$$

where A' denotes the commutant of A for $A \subset \mathcal{A}$.

Extremal 1-testers

A quantum 1-tester defines a measurement on the set of channels $B(\mathcal{H}_0) \to B(\mathcal{H}_1)$:

$$M = (M_i,\ i \in \Omega),\ M_i \in B(\mathcal{H}_1 \otimes \mathcal{H}_0),\ \sum_i M_i = I_{\mathcal{H}_1} \otimes \omega, \quad \omega \in \mathfrak{S}(\mathcal{H}_0)$$

Suppose $M = \Lambda \circ \chi_{I \otimes c}$ is a minimal decomposition, $p = \mathrm{supp}(cc^*)$ and Λ a POVM on $B(\mathcal{H}_1 \otimes p\mathcal{H}_0)$.

Theorem

Suppose Λ is a PVM. Then M is not extremal if and only if there is a nontrivial subspace $\mathcal{L} \subset p\mathcal{H}_0$, a PVM P on $B(\mathcal{H}_1 \otimes \mathcal{L})$ and a PVM $P_\perp$ on $B(\mathcal{H}_1 \otimes \mathcal{L}^\perp)$ such that $\Lambda = P + P_\perp$.

Note that this characterizes all 2-outcome 1-testers.

Extremal qubit 1-testers with 2 outcomes

Let $\dim(\mathcal{H}_0) = 2$.

- Let $M = (M_1, M_2)$, $M_i \in B(\mathcal{H}_1 \otimes \mathcal{H}_0)^+$, $M_1 + M_2 = I_{\mathcal{H}_1} \otimes \omega$ for $\omega \in \mathfrak{S}(\mathcal{H}_0)$, ω is invertible.
- Suppose $M = \Lambda \circ \chi_c$ is a minimal decomposition, for some $\Lambda = (\Lambda_1, \Lambda_2)$ a POVM.
- If M is extremal, then Λ must be a PVM.
- Any nontrivial subspace $\mathcal{L} \subset \mathcal{H}_0$ is 1 dimensional.
- M is extremal if and only if Λ is a PVM and Λ_1 is NOT of the form
$$\Lambda_1 = e \otimes |\psi\rangle\langle\psi| + f \otimes |\psi^\perp\rangle\langle\psi^\perp|$$
where $\psi, \psi^\perp \in \mathcal{H}_0$, $\langle\psi, \psi^\perp\rangle = 0$ and $e, f \in B(\mathcal{H}_1)$ are any projections.