Rényi relative entropies and sufficiency of quantum channels

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Dedicated to the memory of Dénes Petz
Classical Rényi relative entropies

For $p, q$ probability measures over a finite set $X$, $0 < \alpha \neq 1$:

$$D_\alpha(p\|q) = \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}$$

In the limit $\alpha \to 1$: relative entropy

$$S(p\|q) = \sum_x p(x) \log(p(x)/q(x))$$

- introduced as the unique family of divergences satisfying a set of postulates
- fundamental quantities appearing in many information-theoretic tasks

For density matrices \( \rho, \sigma, 0 < \alpha \neq 1 \),

\[
D_\alpha(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \Tr \rho^\alpha \sigma^{1-\alpha}
\]

In the limit \( \alpha \to 1 \): quantum (Umegaki) relative entropy

\[
S(\rho \| \sigma) = \Tr \rho (\log(\rho) - \log(\sigma))
\]

- obtained from Petz quasi-entropies\(^1,2\)
- defined for normal states of a von Neumann algebra, using the relative modular operator


\(^2\)D. Petz, *Publ. RIMS*, Kyoto Univ., 1985
Standard quantum Rényi relative entropies

It follows from the properties of quasi-entropies that: if \( \alpha \in (0, 2] \)

- **strict positivity:** \( D_\alpha(\rho \| \sigma) \geq 0 \) with equality iff \( \rho = \sigma \);
- **data processing inequality:**

\[
D_\alpha(\rho \| \sigma) \geq D_\alpha(\Phi(\rho) \| \Phi(\sigma))
\]

for any quantum channel \( \Phi \)

- **joint lower semicontinuity**

- **joint (quasi)-convexity:** the map

\[
(\rho, \sigma) \mapsto \exp\{(\alpha - 1)D_\alpha(\rho \| \sigma)\}
\]

is jointly convex.
Equality in DPI: sufficient quantum channels

Let the quantum states $\rho, \sigma$ and a channel $\Phi$ be such that

$$S(\Phi(\rho)||\Phi(\sigma)) = S(\rho||\sigma) < \infty$$

This condition was introduced as a quantum extension of classical sufficient statistics:

A statistic $T$ is sufficient with respect to a pair of probability distributions $\{p, q\}$ if

- the conditional expectation satisfies $E_p[\cdot|T] = E_q[\cdot|T]$
- an equivalent Kullback-Leibler characterization by the classical relative entropy:

$$S(p^T||q^T) = S(p||q) \quad (\text{if} \quad < \infty)$$

D. Petz, CMP, 1986
Sufficient quantum channels

Let $\rho, \sigma$ be quantum states (normal states of a von Neumann algebra), $\sigma$ faithful. Assume that $S(\rho\|\sigma) < \infty$

**Theorem**

The following are equivalent.

1. $S(\rho\|\sigma) = S(\Phi(\rho)\|\Phi(\sigma))$;
2. There is a quantum channel $\Psi$ such that
   
   \[ \Psi \circ \Phi(\rho) = \rho, \quad \Psi \circ \Phi(\sigma) = \sigma \]

We say in this case that $\Phi$ is **sufficient (reversible)** with respect to \{\rho, \sigma\}.
Sufficient quantum channels: divergences

A divergence $D$ characterizes sufficiency if

$$D(\Phi(\rho)\|\Phi(\sigma)) = D(\rho\|\sigma) < \infty$$

implies that $\Phi$ is sufficient with respect to $\{\rho, \sigma\}$.

The following divergences characterize sufficiency:

- relative entropy
- $D_{\alpha}$, with $\alpha \in (0, 2)^{3,4}$
- a class of $f$-divergences (in finite dimension)$^{5,6}$

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$^{4}$ AJ and D. Petz, 2006
$^{5}$ Hiai, Mosonyi, Petz, Bény, 2011
$^{6}$ F. Hiai and M. Mosonyi, 2017
The Petz recovery channel is defined as

\[ \Phi_\sigma(Y) = \sigma^{1/2} \Phi^*(\Phi(\sigma)^{-1/2} Y \Phi(\sigma)^{-1/2}) \sigma^{1/2} \]

Note that we always have \( \Phi_\sigma \circ \Phi(\sigma) = \sigma \).

**Theorem**

\( \Phi \) is sufficient with respect to \( \{\rho, \sigma\} \) if and only if

\[ \Phi_\sigma \circ \Phi(\rho) = \rho. \]
Sufficient quantum channels: a conditional expectation

Note that by the last condition, $\rho$ (and $\sigma$) must be invariant states of the channel $\Phi_\sigma \circ \Phi$.

There is a conditional expectation $E$, $\sigma \circ E = \sigma$, such that $\Phi$ is sufficient with respect to $\{\rho, \sigma\}$ if and only if $\rho \circ E = \rho$.

Structure of the states\textsuperscript{7}: in finite dimensions (or on $B(\mathcal{H})$), there is a decomposition

$$U\sigma U^* = \bigoplus_n \lambda_n \sigma_n^L \otimes \sigma_n^R,$$

$\sigma_n^L, \sigma_n^R$ states, $\lambda_n$ probabilities

such that $\Phi$ is sufficient with respect to $\{\rho, \sigma\}$ iff

$$U\rho U^* = \bigoplus_n \mu_n \rho_n^L \otimes \sigma_n^R,$$

$\rho_n^L$ states, $\mu_n$ probabilities

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\textsuperscript{7}M. Mosonyi and D. Petz, 2004; AJ and D. Petz, 2006
Sufficient quantum channels: applications

Characterization of equality in various entropic inequalities:

- strong subadditivity: characterization of quantum Markov states\(^8\)
- monotonicity of quantum Fisher information, Holevo quantity, etc.

Approximate version - recoverability\(^9\):

\[ S(\rho \parallel \sigma) - S(\Phi(\rho) \parallel \Phi(\sigma)) \geq d(\rho \parallel \tilde{\Phi}_\sigma \circ \Phi(\rho)) \]

\(d\) some divergence measure, \(\tilde{\Phi}_\sigma\) a modification of Petz recovery channel

\(^8\)Hayden, Josza, Petz, Winter, 2004
\(^9\)O. Fawzi and R. Renner, 2014; M. M. Wilde, 2015; Junge et. al, 2015; ...
Sandwiched Rényi relative entropy

Another version:

for density matrices $\rho, \sigma$:

$$
\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \text{Tr} \left[ \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha \right]
$$


- satisfies DPI (+ other properties) if $\alpha \in [1/2, 1) \cup (1, \infty)$
- $\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$
- operational interpretation for $\alpha > 1$: strong converse exponents in quantum hypothesis testing\(^{10}\)

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The purpose of the rest of this talk

Extend the sandwiched Rényi relative entropies to normal states of von Neumann algebras and show some properties

- the standard version $D_\alpha$ (quasi-entropies) is defined in this setting and has an operational interpretation in hypothesis testing as in finite dimensions\footnote{V. Jaksic et al., Rev. Math. Phys., 2012}

In this general setting, prove that $\tilde{D}_\alpha$ characterize sufficiency of channels, for $\alpha \in (1/2, 1) \cup (1, \infty)$. 

\footnote{V. Jaksic et al., Rev. Math. Phys., 2012}
Extensions of $\tilde{D}_\alpha$ to von Neumann algebras

Let $\rho, \sigma$ be normal states on a von Neumann algebra $\mathcal{M}$.

Two constructions, using noncommutative $L_p$-spaces:

- Araki-Masuda divergences, defined for $\alpha \in [1/2, 1) \cup (1, \infty]$, uses Araki-Masuda $L_p$-spaces\(^{12}\)
- sandwiched Rényi relative entropies, defined for $\alpha > 1$, uses Kosaki $L_p$-spaces\(^{13}\)

\(^{12}\)M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018
\(^{13}\)AJ, Ann. H. Poincaré, 2018
Haagerup $L_p$-spaces

For $1 \leq p \leq \infty$, $L_p(M)$ - Haagerup $L_p$-space, with norm $\| \cdot \|_p$

- $M \simeq L_\infty(M)$;
- the predual $M^* \simeq L_1(M)$: $\rho \mapsto h_\rho$, $\text{Tr} h_\rho = \rho(1)$;
- $L_2(M)$ a Hilbert space: $\langle h, k \rangle = \text{Tr} k^* h$

If $M = B(H)$, we can use the Schatten classes:

$$L_p(M) = \{ X \in B(H), \text{Tr} |X|^p < \infty \}, \quad \|X\|_p = (\text{Tr} |X|^p)^{1/p}$$
The standard form

Standard form: $(\lambda(\mathcal{M}), L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$

a representation of $\mathcal{M}$ on $L_2(\mathcal{M})$ by left multiplication:

$$\lambda(x)h = xh, \quad x \in \mathcal{M}, \; h \in L_2(\mathcal{M})$$

Any $\rho \in \mathcal{M}^+_*$ has a unique vector representative $h^1/2_\rho$ in $L_2(\mathcal{M})^+$:

$$\rho(a) = \langle ah^1/2_\rho, h^1/2_\rho \rangle$$
Kosaki $L_p$-spaces with respect to a faithful normal state

Let $\sigma$ be a faithful normal state:

- **Continuous embedding**

  \[ M \to L_1(M), \quad x \mapsto h_{\sigma}^{1/2} x h_{\sigma}^{1/2} \]

- **Put**

  \[ L_{\infty}(M, \sigma) := h_{\sigma}^{1/2} M h_{\sigma}^{1/2}, \quad \| h_{\sigma}^{1/2} x h_{\sigma}^{1/2} \|_{\infty, \sigma} = \| x \|_{\infty} \]

- **Let** $\rho \in M_+^*$, then $h_{\rho} \in L_{\infty}(M, \sigma)$ iff $\rho \leq \lambda \sigma$ and

  \[ \| h_{\rho} \|_{\infty, \sigma} = \inf\{ \lambda > 0, \rho \leq \lambda \sigma \}. \]
Kosaki $L_p$-spaces with respect to a faithful normal state

Let $1 < p < \infty$:

- $(L_\infty(\mathcal{M}, \sigma), L_1(\mathcal{M}))$ compatible pair of Banach spaces
- interpolation space

$$L_p(\mathcal{M}, \sigma) := C_{1/p}(L_\infty(\mathcal{M}, \sigma), L_1(\mathcal{M})), \text{ with norm } \| \cdot \|_{p, \sigma}$$

- Let $1/p + 1/q = 1$, then

$$L_p(\mathcal{M}, \sigma) = \{ h^{1/2q}_\sigma k h^{1/2q}_\sigma, \ k \in L_p(\mathcal{M}) \},$$

$$\| h^{1/2q}_\sigma k h^{1/2q}_\sigma \|_{p, \sigma} = \| k \|_p$$
A definition of $\tilde{D}_\alpha$, $\alpha > 1$

Extension to non-faithful $\sigma$: by restriction to support $s(\sigma) = e$

$$L_p(\mathcal{M}, \sigma) = \{ h \in L_1(\mathcal{M}), \ h = ehe \in L_p(e\mathcal{M}e, \sigma|_{e\mathcal{M}e}) \}.$$ 

For normal states $\rho$, $\sigma$ and $1 < \alpha < \infty$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{\alpha}{\alpha-1} \log(\|h_\rho\|_{\alpha,\sigma}) & \text{if } h_\rho \in L_\alpha(\mathcal{M}, \sigma) \\ \infty & \text{otherwise.} \end{cases}$$
Some properties of $\tilde{D}_\alpha$

Using complex interpolation, we can prove

- **strict positivity**: $\tilde{D}_\alpha(\rho\|\sigma) \geq 0$, with equality iff $\rho = \sigma$.
- **joint lower semicontinuity** (on $L_1(M)^+ \times L_1(M)^+$)
- **if** $\rho \neq \sigma$ and $\tilde{D}_\alpha(\rho\|\sigma) < \infty$, then

$$\alpha' \mapsto \tilde{D}_{\alpha'}(\rho\|\sigma) \text{ is strictly increasing for } \alpha' \in (1, \alpha].$$

- **quasi-convexity**
Relation to the standard version $D_\alpha$, limit values

For normal states $\rho, \sigma$, $\alpha > 1$:

$$D_{2^{-1/\alpha}}(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma)$$

⇓

Limit values:

$$\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$$

$$\lim_{\alpha \to \infty} \tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\infty(\rho\|\sigma) \text{ relative max entropy}$$
Data processing inequality

\( \Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N}) \) positive, trace-preserving. Let \( \sigma_0 = \Phi(\sigma) \).

- \( \Phi \) is a contraction \( L_1(\mathcal{M}) \to L_1(\mathcal{N}) \).
- if \( 0 \leq \rho \leq \lambda \sigma \), then also \( 0 \leq \Phi(\rho) \leq \lambda \sigma_0 \), hence

\[
\Phi(L_\infty(\mathcal{M}, \sigma)^+) \to L_\infty(\mathcal{N}, \sigma_0)^+
\]

- this extends to a map \( x \mapsto y \), \( \Phi(h_\sigma^{1/2} x h_\sigma^{1/2}) = h_\sigma^{1/2} y h_\sigma^{1/2} \) (adjoint of) the Petz recovery channel\(^{14} \) \( \Phi_\sigma \)
- \( \Phi \) defines a contraction \( L_\infty(\mathcal{M}, \sigma) \to L_\infty(\mathcal{N}, \sigma_0) \).

\( \Phi \) defines a contraction \( L_p(\mathcal{M}, \sigma) \to L_p(\mathcal{N}, \sigma_0) \) for all \( 1 \leq p \leq \infty \).

For $\alpha > 1$, normal states $\rho, \sigma$, positive, trace-preserving $\Phi$: 

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

Consequently, by the limit $\alpha \to 1$:

For normal states $\rho, \sigma$,

$$S(\rho\|\sigma) \geq S(\Phi(\rho)\|\Phi(\sigma))$$

holds for any positive trace-preserving map $\Phi$.

A. Müller-Hermes, D. Reeb, Ann. H. Poincaré, 2017 (for $\mathcal{M} = B(\mathcal{H})$)
The Araki-Masuda divergences

The Araki-Masuda $L_p$-norm: defined on $L_2(M)$

- for $2 \leq p \leq \infty$, $\xi \in L_2(M)^+$,
  \[
  \|\xi\|_{p,\sigma}^{AM} = \sup_{\omega \in M^+_*, \omega(1)=1} \|\Delta^{1/2-1/p}_{\omega,\sigma} \xi\|_2
  \]
  if $s(\omega_\xi) \leq s(\sigma)$ and is infinite otherwise

- for $1 \leq p < 2$,
  \[
  \|\xi\|_{p,\sigma}^{AM} = \inf_{\omega \in M^+_*, \omega(1)=1, s(\omega) \geq s(\omega_\xi)} \|\Delta^{1/2-1/p}_{\omega,\sigma} \xi\|_2
  \]

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018
The Araki-Masuda divergences

For normal states $\rho$, $\sigma$ and $\alpha \in [1/2, 1) \cup (1, \infty)$:

$$\tilde{D}_{\alpha}^{AM}(\rho \| \sigma) = \frac{2\alpha}{\alpha - 1} \log(\| h_{\rho}^{1/2} \|_{2\alpha, \sigma}^{AM})$$

- can be defined using any $*$-representation of $\mathcal{M}$ on a Hilbert space $\mathcal{H}$ and any vector $\xi \in \mathcal{H}$ representing $\rho$

- duality relation: for $1/p + 1/q = 1$

$$|\langle \eta, \xi \rangle| \leq \| \eta \|_{p, \sigma}^{AM} \| \xi \|_{q, \sigma}^{AM}, \quad \xi, \eta \in \mathcal{H}$$

- if $1 < p \leq 2$, there is a (unique) unit vector $\eta_0 \in \mathcal{H}$ such that

$$\langle \eta, \eta_0 \rangle = \| \eta \|_{p, \sigma}^{AM} \| \eta_0 \|_{q, \sigma}^{AM}$$

M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018
Araki-Masuda $L_p$-norms can be introduced by interpolation:

- For $2 \leq p \leq \infty$: a continuous embedding $\mathcal{M} \to L_2(\mathcal{M})$
  
  \[ x \mapsto h^{1/2}_\sigma x, \quad x \in \mathcal{M} \]

  the interpolation norm $\| \cdot \|_{p,\sigma}^{\mathcal{AM}}$ in $C_{1/p}(\mathcal{M}, L_2(\mathcal{M}))$.

- For $1 \leq p \leq 2$: a continuous embedding $L_2(\mathcal{M}) \to L_1(\mathcal{M})$
  
  \[ k \mapsto kh^{1/2}_\sigma, \quad k \in L_2(\mathcal{M}) \]

  the interpolation norm $\| \cdot \|_{p,\sigma}^{\mathcal{AM}}$ in $C_{1/p}(L_2(\mathcal{M}), L_1(\mathcal{M}))$. 
\[ \tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha^{AM}(\rho\|\sigma) \]

For \( 1 < \alpha < \infty \),
\[ \tilde{D}_\alpha(\rho\|\sigma) = \tilde{D}_\alpha^{AM}(\rho\|\sigma) \]

For \( 1/2 < \alpha < 1 \):
\[
\frac{1-\alpha}{2\alpha} h^1_\rho h^{1/2}_\sigma \in L_{2\alpha}(\mathcal{M}) \quad \text{and} \quad \\
\tilde{D}_\alpha(\rho\|\sigma) := \tilde{D}_\alpha^{AM}(\rho\|\sigma) = \frac{2\alpha}{\alpha - 1} \log \| h^{1-\alpha}_{\sigma^{2\alpha}} h^{1/2}_\rho \|_{2\alpha} \]

Limit values:

\( \lim_{\alpha \uparrow 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma) \)

\( \lim_{\alpha \rightarrow 1/2} \tilde{D}_\alpha(\rho\|\sigma) = - \log F(\rho, \sigma) \) Uhlmann’s fidelity
Data processing inequality for $\tilde{D}_\alpha$, $\alpha \in (1/2, 1)$

We have to assume that $\Phi : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ is trace preserving and completely positive, with Stinespring representation:

$$\Phi^* = T^* \pi(\cdot) T$$

$\pi$ a $*$-representation, $T$ an isometry

- Let $p = 2\alpha$, $1/p + 1/q = 1$. Let $\rho = \omega_\eta$ and let $\omega := \omega_{\eta_0}$ be such that $\langle \eta, \eta_0 \rangle = \| \eta \|_{p,\sigma} \| \eta_0 \|_{q,\sigma}$. Then

$$\langle \eta, \eta_0 \rangle = \langle T\eta, T\eta_0 \rangle \leq \| T\eta \|_{p,\Phi(\sigma)} \| T\eta_0 \|_{q,\Phi(\sigma)}$$

- we obtain, with $\alpha^* = \alpha/(2\alpha - 1) > 1$:

$$\tilde{D}_\alpha(\rho \| \sigma) \geq \tilde{D}_\alpha(\Phi(\rho) \| \Phi(\sigma)) + \tilde{D}_{\alpha^*}(\omega \| \sigma) - \tilde{D}_{\alpha^*}(\Phi(\omega) \| \Phi(\sigma))$$

$$\geq \tilde{D}_\alpha(\Phi(\rho) \| \Phi(\sigma))$$

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M. Berta, V. B. Scholz, and M. Tomamichel, Ann. H. Poincaré, 2018
Theorem

The sandwiched Rényi relative entropies $\tilde{D}_\alpha$ characterize sufficiency for $\alpha \in (1/2, 1) \cup (1, \infty)$.

That is:

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma) < \infty$$

implies that

$$\Psi \circ \Phi(\rho), \quad \Psi \circ \Phi(\sigma).$$

for some channel $\Psi$. 
The proof in case $\alpha > 1$

Let $\alpha > 1$.

- the assumption

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma) < \infty$$

implies that $h_\rho \in L_\alpha(M, \sigma)$ and $\Phi$ is a contraction preserving its norm.

- An easy proof for $\alpha = 2$:

$L_2(M, \sigma)$ is a Hilbert space, $\Phi_\sigma$ is the adjoint of $\Phi$

By properties of contractions on Hilbert spaces:

$$\Phi_\sigma \circ \Phi(h_\rho) = h_\rho$$

(note that positivity of $\Phi$ is enough for this)
The proof in case $\alpha > 1$

For general $\alpha > 1$, use interpolation:

Let $\tau$ be a normal state, $s(\tau) \leq s(\sigma)$. Put

$$h_\tau(z) = h_\sigma^{(1-z)/2} h_\tau^z h_\sigma^{(1-z)/2} \in L_1(\mathcal{M}), \quad 0 \leq \text{Re}(z) \leq 1,$$

continuous function, analytic in $0 < \text{Re}(z) < 1$.

If the equality

$$\|\Phi(h_\tau(1/\alpha))\|_{\alpha,\Phi(\sigma)} = \|h_\tau(1/\alpha)\|_{\alpha,\sigma}$$

holds for some $\alpha > 1$, then it holds for all $\alpha > 1$. 
The proof in case $\alpha > 1$

By assumptions $h_\rho = th_\tau(1/\alpha)$ for some state $\tau$, $t > 0$ - normalization, and we have

$$\|\Phi(h_\tau(1/\alpha))\|_{\alpha,\Phi(\sigma)} = \|h_\tau(1/\alpha)\|_{\alpha,\sigma}.$$ 

Then the equality holds also for $\alpha = 2$, so that

$$\Phi \text{ is sufficient with respect to } \{\omega, \sigma\}, \text{ where}$$

$$h_\omega = sh_\tau(1/2) = sh_\sigma^{1/4} h_\tau^{1/2} h_\sigma^{1/4}, \quad s > 0.$$ 

Ok, but this is not what we wanted to prove!
The proof in case $\alpha > 1$

Assume $\mathcal{M} = B(\mathcal{H})$, replace $h_\rho$ by the density operator $\rho$.

There is a decomposition $U\sigma U^* = \bigoplus_n \lambda_n \sigma_n^L \otimes \sigma_n^R$ such that $\Phi$ is sufficient with respect to $\{\rho, \sigma\}$ iff

$$U\rho U^* = \bigoplus_n \mu_n \rho_n^L \otimes \sigma_n^R$$

But we have such a decomposition for $\omega = s\sigma^{1/4} \tau^{1/2} \sigma^{1/4}$, hence also for $\rho = t\sigma^{1/2}\beta \tau^{1/\alpha} \sigma^{1/2}\beta$, which implies the result.

(In the general case, we use the characterization by conditional expectations.)
The case $\alpha \in (1/2, 1)$

Let $\alpha \in (1/2, 1)$. Then

$$h^{1-\alpha}_\sigma h^{1/2}_\rho = h^{1/2}_\tau u \in L_{2\alpha}(\mathcal{M})$$

for some $\tau \in \mathcal{M}^+$ and partial isometry $u \in \mathcal{M}$.

- We recall the inequality

$$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) + \tilde{D}_{\alpha^*}(\omega\|\sigma) - \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma))$$

$$\geq \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

for $\alpha^* > 1$ and some state $\omega$.

- Actually, we have $h_\omega = th_\tau(1/\alpha^*)$. 
The case $\alpha \in (1/2, 1)$

- By assumptions, we obtain
  \[ \tilde{D}_{\alpha^*}(\omega\|\sigma) = \tilde{D}_{\alpha^*}(\Phi(\omega)\|\Phi(\sigma)) \]

- Since $\alpha^* > 1$, this implies that $\Phi$ is sufficient with respect to $\{\omega, \sigma\}$.
- Use the decompositions (conditional expectations) as before.