Higher order operations in Boxworld?

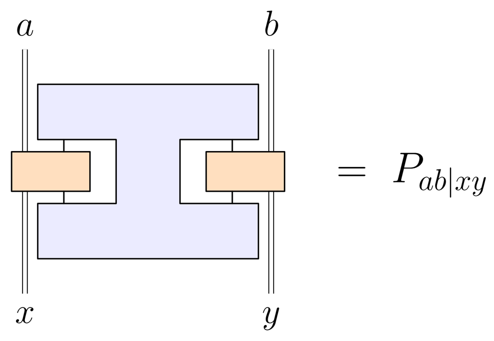
Anna Jenčová

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia
jencaster@gmail.com

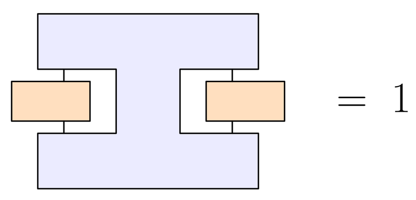
Higher order quantum operations

Higher order quantum operations form a hierarchy constructed starting from the set of quantum states, including at each step all transformations between objects built at the previous steps. The hierarchy contains states, channels, combs, process matrices, etc.

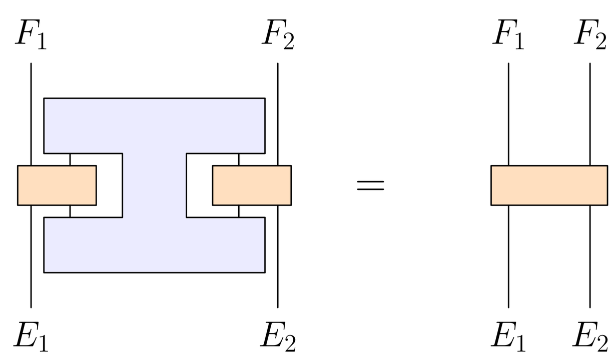
Process matrices transform pairs of quantum instruments to conditional probabilities (classical channels). Admissibility conditions - must return valid channels if applied only partially:



A: Process matrix definition



B: An affine condition



C: Admissibility

Choi representation: $W \in B(\mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2})$, such that

$$W * C_1 * C_2 \in \mathcal{C}(E_1E_2 \to F_1F_n)$$

for all Choi matrices of channels $C_i \in \mathcal{C}(A_iE_i \to B_iF_i)$ and the **link product** $*$. Equivalently,

$$W \geq 0, \qquad \mathrm{Tr}(W) = d_{B_1}d_{B_2}$$
$$P_{A_1B_1}(W) = P_{B_2A_1B_1}(W), \qquad P_{A_2B_2}(W) = P_{B_1A_2B_2}(W)$$

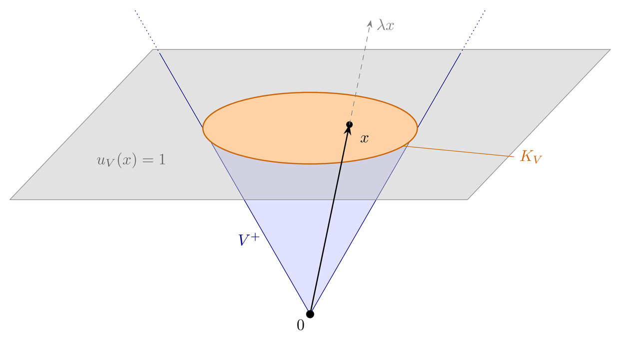
here $P_A(W) = \mathrm{Tr}_A(W) \otimes d_A^{-1} I_A$.

Higher order operations in other theories?

General probabilistic theories (GPT):

A **system** is a triple (V, V^+, u_V), V a real vector space, $\dim(V) < \infty$, V^+ a positive cone, u_V the unit functional. The **state space** is determined as

$$K_V := \{x \in V^+ : u_V(x) = 1\}.$$



Channels are defined as (some) affine maps $K_V \to K_W$. **Joint systems** are represented by a tensor product

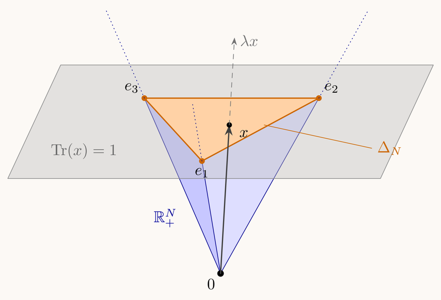
$$(V \otimes W, V^+ \otimes W^+, u_V \otimes u_W), \text{ with } V^+ \otimes_{min} W^+ \subseteq V^+ \otimes W^+ \subseteq V^+ \otimes_{max} W^+.$$

In principle, higher order operations could be represented using the analog of Choi matrices and link product present in the category of f. d. vector spaces.

Classical theory:

Classical systems $(\mathbb{R}^N, \mathbb{R}^N_+, \mathrm{Tr})$, state spaces are simplexes

$$K_V = \Delta_N = \{p \in \mathbb{R}^N_+ : \sum_i p_i = 1\}$$



Joint systems: $\mathbb{R}^N_+ \otimes_{min} \mathbb{R}^M_+ = \mathbb{R}^{MN}_+ = \mathbb{R}^N_+ \otimes_{max} \mathbb{R}^M_+$

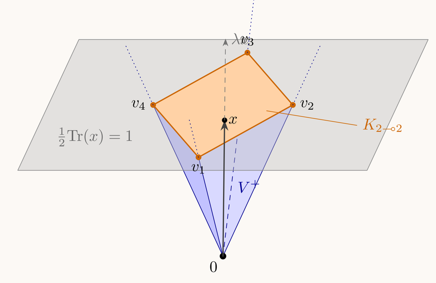
Classical channels = positive TP maps = conditional probabilities

Higher order classical theory (HOCT) similar to the quantum case

Boxworld:

Basic state spaces ≡ sets of classical channels

$$K_{N_I \multimap N_O} := \{p_{i|j} \geq 0 : \sum_i = 1, \forall j\} = \Delta_{N_O}^{N_I}$$



Joint systems:

$K_{M_I \multimap M_O} \otimes_{max} K_{N_I \multimap N_O} \equiv$ no-signalling classical channels

Boxworld channels = all affine maps between state spaces

Process matrices in Boxworld

There exist different notions of Boxworld process matrices considered previously [1, 3, 4], with different definitions of instruments and different admissibility conditions.

Minimal admissibility

1-preserving Boxworld process matrices [3]

- Uses only the weakest admissibility condition (Fig. A)
- Instruments:

  Boxworld channels $K_1 \otimes \Delta_M \to K_2 \otimes \Delta_N$ (with a Boxworld and classical input/output)
- Condition in Fig. A $\iff$ condition in Fig. B + positivity-preserving
- Process matrices can be obtained as affine mappings between polytopes

Maximal admissibility

NSWSE process tensors [1]

- Uses the relation Boxworld $\to$ Higher order classical theory
- Instruments: $\{T^x_a\}$, $T^x_a \in \mathbb{R}^M_+$, $\sum_a T^x_a = T^x$ restricts to a Boxworld channel
- No-signalling instruments: $\{T^x_a\}$, $T^x = T^{x'}$ as Boxworld channels
- Process matrices: classical tensors $W \in \mathbb{R}^{N^A N^B}_+$ satisfying NSWSE:

  $\{T^x_a\}$, $\{T^y_b\}$ no-signalling $\implies$ $\{W * T^x_a * T^y_b\} = P_{ab|xy}$ no-signalling classical channel

Process matrices in a Polytope GPT

Polytopes:

Polytope systems: a subspace $S \subseteq \mathbb{R}^N$, generating $\equiv S \cap \mathrm{int}(\mathbb{R}^N_+)) \neq \emptyset$

$$(S, S^+ := S \cap \mathbb{R}^N_+, e_S \in \mathrm{int}(\mathbb{R}^N_+)),$$

Polytope state spaces: $K_S = \{x \in S^+ : \langle e_S, x\rangle = 1\}$ a polytope

Joint systems: $S^+ \otimes T^+ = (S \otimes T)^+ = S^+ \otimes_{max} T^+$, $e_{S\otimes T} = e_S \otimes e_T$

Polytope channels = all affine maps $K_S \to K_T$

Instruments: channels $K_S \otimes \Delta_N \to K_T \otimes \Delta_M$ (as in [3] for Boxworld)

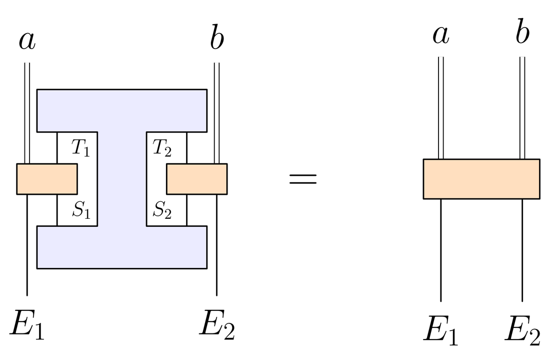
Useful inclusions and relations:

Classical theory $\subseteq$ Boxworld $\to$ HOCT $\subsetneq$ $\mathrm{Caus}[\mathrm{Mat}(\mathbb{R}_+)]$ [2]

Boxworld $\subsetneq$ Polytopes $\to$ $\mathrm{Caus}[\mathrm{Mat}(\mathbb{R}_+)]$ [2]

Admissibility conditions

For the admissibility conditions (Fig. C) in Polytopes, we may always assume F_i to be classical:



D: Admissibility for Polytopes

With a fixed subset $\mathcal{E} \subseteq$ Polytopes:

- $\mathcal{E}$-admissibility: the condition holds for all $E_1, E_2 \in \mathcal{E}$
- minimal admissibility: $\mathcal{E}$ = Classical theory
- maximal admissibility: $\mathcal{E}$ = Polytopes
- for Boxworld process matrices: $\mathcal{E}$ = Boxworld

Notation:

- for $S \subseteq \mathbb{R}^{N_S}$, $T \subseteq \mathbb{R}^{N_T}$: $[S, T] = (S \otimes T^\perp)^\perp$,
- $L_i := [S_i, T_i]$, $L_i^+ := L_i \cap \mathbb{R}^{N_{S_i}N_{T_i}}_+$, $i = 1, 2$,
- Π_T - orthogonal projection in $\mathbb{R}^{N_T}$ onto T, $\Pi_i = \Pi_{S_i \otimes T_i} = \Pi_{S_i} \otimes \Pi_{T_i}$, $i = 1, 2$.

Some results on $\mathcal{E}$-admissibility

Consider Polytope systems given by generating subspaces $S_i \subseteq \mathbb{R}^{N_{S_i}}$, $T_i \subseteq \mathbb{R}^{N_{T_i}}$, $i = 1, 2$. Define

$$\widetilde{\mathcal{R}}_{\mathcal{E}} := \{(P_1 \otimes P_2) * \sigma : P_i \in [E_i, L_i]^+,\ \sigma \in (E_1 \otimes E_2)^+,\ E_1, E_2 \in \mathcal{E}\}, \quad \mathcal{R}_{\mathcal{E}} := (\Pi_1 \otimes \Pi_2)(\widetilde{\mathcal{R}}_{\mathcal{E}})$$

$\mathcal{E}$-admissibility

Let $W \in S_1 \otimes T_1 \otimes S_2 \otimes T_2$. Then W defines an $\mathcal{E}$-admissible process matrix if and only if

- W satisfies Fig. B (affine condition),
- $\langle W, \xi\rangle \geq 0$ for all $\xi \in \mathcal{R}_{\mathcal{E}}$ (positivity).

The cone $\mathcal{R}_{\mathcal{E}}$

Assume that $\mathcal{E}$ contains all classical systems and is closed under $\otimes_{max}$. Then $\mathcal{R}_{\mathcal{E}}$ is a convex cone in $S_1 \otimes T_1 \otimes S_2 \otimes T_2$, such that

$$\mathcal{R}_{\min} = \Pi_1(L_1^+) \otimes_{min} \Pi_2(L_2^+) \subseteq \mathcal{R}_{\mathcal{E}} \subseteq (\Pi_1 \otimes \Pi_2)(L_1^+ \otimes_{max} L_2^+) = \mathcal{R}_{\max}. \qquad (*)$$

In particular, this is true for $\mathcal{E}$ = Boxworld. We further have:

- If one of T_1 or T_2 is classical, then $\mathcal{R}_{\min} = \mathcal{R}_{\max}$.
- If none of T_1, T_2 is classical and $T_1, T_2 \in \mathcal{E}$, then the first inclusion in (*) is strict.
- W is a maximally admissible process matrix if and only if

  $$W = (\Pi_{T_1} \otimes \Pi_{T_2})(Q), \quad Q \in \mathbb{R}^{N_1N_2}_+,\ \Pi_{S_i^\perp}(Q) = 0,\ \Pi_{T_i}(Q) = \Pi_{\mathbb{R}\Pi_{T_i}(e_{T_i})}(Q),\ \langle Q, e\rangle = c$$

  In particular, if S_i, T_i are Boxworld systems, Q is a NSWSE process tensor.

An open question

It is not clear whether the second inclusion in (*) is strict for Boxworld.

References

[1] J. Bavaresco et al. *Indefinite causal order in boxworld theories.* arXiv:2411.00951.

[2] A. Kissinger and S. Uijlen. "A categorical semantics for causal structure". In: *Logical Methods in Computer Science* 15 (2019).

[3] V. Kunte. *Higher-Order Operations Beyond Quantum Theory With Applications to Indefinite Causality.* Phd thesis. 2026.

[4] M. Wilson, J. Hefford, and T. Hoffreumon. *Supermaps on generalised theories.* arXiv:2602.23865.