DUALISTIC PROPERTIES OF THE MANIFOLD OF QUANTUM STATES

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Abstract. In the finite dimensional case, we introduce a family of torsion-free affine connections in the manifold of all invertible density matrices. These are the quantum version of Amari's α-connections, based on the α-representation of the tangent space. The dual connections with respect to a monotone metric are, in general, not torsion-free. We compute the torsion and the Riemannian curvature. Finally, geodesics are used to define a divergence.

Monotone metrics

If not stated otherwise, the proofs can be found in [6].

Let $\mathcal{M}$ denote the differentiable manifold of all n-dimensional complex hermitean matrices and let $\mathcal{M}^+ = \{M \in \mathcal{M} \mid M > 0\}$. Let $\tilde{T}_M$ be the tangent space at M. We introduce a Riemannian structure in $\mathcal{M}^+$, defining an inner product in $\tilde{T}_M$ by $\lambda_M(X,Y) = \mathrm{Tr} X J_M(Y)$ where J_M is a suitable superoperator on matrices. The state space of an n-level quantum system can be identified with the submanifold $\mathcal{D} = \{D \in \mathcal{M}^+ \mid \mathrm{Tr} D = 1\}$. The tangent space $T_D \subset \tilde{T}_D$ is the real N-dimensional vector space of all self-adjoint traceless matrices. We require that λ is monotone, in the sense that if T is a stochastic map, then

$$\lambda_{T(M)}(T(X), T(X)) \le \lambda_M(X,X), \ \forall M \in \mathcal{M}^+, X \in \mathcal{M}$$

As it was proved in [8], this is true iff J_M is of the form

$$J_M = (R_M^{\frac{1}{2}} f(L_M R_M^{-1}) R_M^{\frac{1}{2}})^{-1} \tag{1}$$

where $f : R^+ \to R$ is an operator monotone function such that $f(t) = t f(t^{-1})$ for every $t > 0$ and $L_M(A) = MA$ and $R_M(A) = AM$, for each matrix A. We also adopt a normalization condition $f(1) = 1$.

Let T_D^* be the cotangent space of $\mathcal{D}$ at D, then T_D^* is the vector space of all observables A with zero mean at D (i.e. $\mathrm{Tr} DA = 0$). It is easy to see that

$$T_D^* = \{J_D(H) \mid H \in T_D\} \tag{2}$$

Here are some examples of monotone metrics.

1. Let $J_D(H) = G$, where $GD + DG = 2H$, the corresponding metric is called the metric of the symmetric logarithmic derivative, see [5, 9].

2. If $J_D(H) = \frac{d}{dt} \log(D + tH)|_{t=0}$, we obtain the well-known Kubo-Mori metric.

3. Let $J_D(H) = \frac{1}{2}(D^{-1}H + HD^{-1})$. The corresponding metric is the metric of the right logarithmic derivative, see [5, 10]

For more about monotone metrics and their use see [9, 10, 4].

The α-representation

Let $g : R \to R$ be a smooth (strictly) monotone function. We define an operator $L_g[M] : \tilde{T}_M \to \tilde{T}_{g(M)}$ by (see [6])

$$L_g[M](H) = \frac{d}{ds} g(M + sH)|_{s=0}$$

In what follows, we omit the indication of the point in square brackets if no confusion is possible. The vector space

$$T_D^g = \{L_g(H) \mid H \in T_D\}$$

will be called the g-representation of the tangent space T_D. The corresponding inner product in T_D^g is

$$\lambda_D^g(G_1, G_2) = \lambda_D(L_g^{-1}(G_1), L_g^{-1}(G_2)) = \mathrm{Tr} G_1 K_g(G_2)$$

where $K_g = L_g^{-1} J_D L_g^{-1}$. The g-representation of the cotangent space is given by

$$T_D^{g*} = \{K_g(G) \mid G \in T_D^g\} = \{L_{g^{-1}} J_D(H) \mid H \in T_D\}$$

Clearly, if g is the identity function, we obtain the usual tangent and cotangent spaces T_D and T_D^*.

The quantum analogue of Amari's α-representations [1] is obtained if we put

$$g(x) = g_\alpha(x) = \begin{cases} \frac{2}{1-\alpha} x^{\frac{1-\alpha}{2}} & \alpha \neq 1 \\ \log x & \alpha = 1 \end{cases}$$

We have $T_D^{\alpha*} = T_D^{-\alpha}$ for each α so that K_α is an isomorphism $K_\alpha : T_D^\alpha \to T_D^{-\alpha}$. We have

$$K_\alpha^{-1} = K_{-\alpha} \iff J_D = J_\alpha := L_{-\alpha} L_\alpha = J_{-\alpha}.$$

The family of metrics J_α was studied in [4] and it was shown that these are monotone for $\alpha \in [-3, 3]$. For example, the Kubo-Mori metric is obtained if $\alpha = \pm 1$ and the right logarithmic derivative metric if $\alpha = \pm 3$.

The affine connections, torsion and curvature

In this section, we define the α-connections in $\mathcal{M}^+$ and $\mathcal{D}$ and investigate the important notions of duality, stated in [1], namely the dual connections and the existence of dual affine coordinate systems.

Let $g : R \to R$ be a smooth strictly monotone function and let $M, M' \in \mathcal{M}^+$. Then there is an isomorphism $\tilde{T}_M \to \tilde{T}_{M'}$ given by $H \mapsto L_g^{-1}[M'] L_g[M](H)$. This isomorphism induces an affine connection on $\mathcal{M}^+$. Let us denote the corresponding covariant derivative by $\tilde{\nabla}^g$.

Let $x_1, \dots, x_{N+1}$ be a coordinate system in $\mathcal{M}^+$. Let us denote $\partial_i = \frac{\partial}{\partial x_i}$ and let $\tilde{H}_i = \partial_i M(x)$, $\tilde{G}_i = L_g(\tilde{H}_i) = \partial_i g(M(x))$, $i = 1, \dots, N+1$. Then

$$L_g(\tilde{\nabla}^g_{\tilde{H}_j} \tilde{H}_i) = \partial_i \partial_j g(M(x))$$

Hence, the coefficients of the affine connection are

$$\tilde{\Gamma}^g_{ijk}(x) = \lambda_x(\tilde{\nabla}^g_{\tilde{H}_i} \tilde{H}_j, \tilde{H}_k) = \mathrm{Tr} \partial_i \partial_j g(M(x)) K_g(\tilde{G}_k), \; i, j, k = 1, \dots, N+1$$

From this, it follows that this connection is torsion-free.

If we use the functions g_α, we obtain a one-parameter family of torsion-free connections $\tilde{\nabla}^\alpha$ which is the quantum analogue of α-connections defined by Amari

[1]. But, unlike the classical case, the connections $\tilde{\nabla}^{\alpha}$ and $\tilde{\nabla}^{-\alpha}$ are not dual in general.

To obtain the dual connection, consider the isomorphism $\tilde{T}_M \to \tilde{T}_{M'}$, given by $H \mapsto J_{M'}^{-1} L_g[M'] L_g^{-1}[M] J_M(H)$. The induced affine connection on $\mathcal{M}^+$, resp. the covariant derivative will be denoted by $\tilde{\nabla}^{g*}$. We have

$$L_g^{-1}[M] J_M(\tilde{\nabla}^{g*}_{\tilde{H}_i} \tilde{H}_j) = \partial_i K_g(\tilde{G}_j(x))$$

The coefficients of this connection are

$$\tilde{\Gamma}^{g*}_{ijk} = \lambda(\tilde{\nabla}^{g*}_{\tilde{H}_i} \tilde{H}_j, \tilde{H}_k) = \mathrm{Tr} \partial_i K_g(\tilde{G}_j) \tilde{G}_k \tag{3}$$

It is easy to prove that $\tilde{\nabla}^{g}$ and $\tilde{\nabla}^{g*}$ are dual.

The components of the torsion tensor are

$$\tilde{S}^{g*}_{ijk} = \tilde{\Gamma}^{g*}_{ijk} - \tilde{\Gamma}^{g*}_{jik} = \mathrm{Tr}\{\partial_i K_g(\tilde{G}_j(x)) - \partial_j K_g(\tilde{G}_i(x))\} \tilde{G}_k$$

so that this connection is torsion-free iff $\partial_i K_g(\tilde{G}_j) = \partial_j K_g(\tilde{G}_i)$, $\forall i, j = 1, \dots, N+1$. Obviously, this is not always the case.

Consider now $\mathcal{D}$ as an N-dimensional submanifold in $\mathcal{M}$ and let $t_1, \dots, t_N$ be a coordinate system in $\mathcal{D}$. As there is no danger of confusion, we use the symbol ∂_i also for $\frac{\partial}{\partial t_i}$. Let $H_i = \partial_i D(t)$, $G_i = \partial_i g(D(t))$, $i = 1, \dots, N$. The affine structure in $\mathcal{D}$ is obtained by projecting the above affine connections orthogonaly onto $\mathcal{D}$. The covariant derivative is given by

$$\begin{aligned} L_g(\nabla^g_{H_j} H_i) &= \partial_i \partial_j g(D_t) - g'(D_t) D_t \mathrm{Tr}(g'(D_t))^{-1} \partial_i \partial_j g(D_t) \\ L_g^{-1} J_D(\nabla^{g*}_{H_i} H_j) &= \partial_i K_g(G_j(t)) - (g'(D_t))^{-1} \mathrm{Tr} g'(D_t) D_t \partial_i K_g(G_j(t)) \end{aligned}$$

and the coefficients are

$$\Gamma^g_{ijk}(t) = \mathrm{Tr} K_g(G_k) \partial_i \partial_j g(D_t) \quad \Gamma^{g*}_{ijk}(t) = \mathrm{Tr} G_k \partial_i K_g(G_j(t))$$

Proposition 1. *Let $\alpha \in [-3, 3]$. The following statements are equivalent:*

(i) $\tilde{\nabla}^{\alpha*}$ *is torsion-free*
(ii) $\nabla^{\alpha*}$ *is torsion-free*
(iii) $J_D = J_\alpha = L_{-\alpha} L_\alpha$

Proof. The cases (i) $\Rightarrow$ (ii) and (iii)$\Rightarrow$ (i) are rather obvious. We prove (ii)$\Rightarrow$ (iii):

Let us consider the natural affine parametrization in $\mathcal{D}$

$$D_t = D_0 + \sum_i t_i H_i \quad t \in \Theta \subset R^N \quad \mathrm{Tr} H_i = 0, \ i = 1, \dots, N$$

Let $\nabla^{\alpha*}$ be torsion-free, then

$$\mathrm{Tr}\left(\partial_i L_\alpha^{-1} J_D(H_j) - \partial_j L_\alpha^{-1} J_D(H_i)\right) L_\alpha(H_k) = 0 \ i, j, k = 1, \dots, N$$

Clearly, $D \in \tilde{T}_D$, $D \perp T_D$ for each $D \in \mathcal{D}$ and

$$\begin{aligned} \mathrm{Tr} \partial_i \{L_\alpha^{-1} J_D(H_j)\} L_\alpha(D) &= \partial_i \mathrm{Tr} L_\alpha^{-1} J_D(H_j) L_\alpha(D) - \frac{1-\alpha}{2} \mathrm{Tr} L_\alpha J_D(H_j) \partial_i g_\alpha(D) = \\ &= -\frac{1-\alpha}{2} \mathrm{Tr} J_D(H_j) H_i = \mathrm{Tr} \partial_j \{L_\alpha^{-1} J_D(H_i)\} L_\alpha(D) \end{aligned}$$

It follows that $\partial_i L_\alpha^{-1} J_D(H_j) = \partial_j L_\alpha^{-1} J_D(H_i)$, for $i, j = 1, \dots, N$. This implies the existence of a potential function $\Phi : \Theta \to \mathcal{M}$, such that $\frac{d}{dt} \Phi(D + tH) =$

$L_\alpha^{-1}[D+tH]J_{D+tH}(H)$. Let $\alpha\neq -1$. If $DH=HD$, then $L_\alpha^{-1}[D+tH]J_{D+tH}(H)=(D+tH)^{\frac{\alpha-1}{2}}H$. Let $H=D-\frac{1}{n}$. Compute

$$\Phi(D)-\Phi(\frac{1}{n})=\int_0^1\frac{d}{dt}\Phi(\frac{1}{n}+tH)dt=\int_0^1(\frac{1}{n}+tH)^{\frac{\alpha-1}{2}}Hdt=g_{-\alpha}(D)-g_\alpha(\frac{1}{n})$$

It follows that $L_\alpha^{-1}J_D(H)=L_{-\alpha}(H)$. The case $\alpha=-1$ is proved similarly.

Let us now compute the Riemannian curvature. Let $\tilde{R}^g$ ($\tilde{R}^{g*}$) be the Riemannian curvature tensor of $\tilde{\nabla}^g$ ($\tilde{\nabla}^{g*}$) in $\mathcal{M}^+$, then $\tilde{R}^g=\tilde{R}^{g*}=0$. The curvature tensor R^g in $\mathcal{D}$ is equal to

$$R^g_{ijkl}=H^g_{jk1}H^{g*}_{il1}-H^g_{ik1}H^{g*}_{jl1}$$

where H^g_{ij1}, resp. H^{g*}_{ij1}, is the Euler-Shouten imbedding curvature. If $g=g_\alpha$,

$$R^\alpha_{ijkl}(D)=\frac{1-\alpha^2}{4}\{\mathrm{Tr}H_jJ_\alpha(H_k)\mathrm{Tr}H_iJ_D(H_l)-\mathrm{Tr}H_iJ_\alpha(H_k)\mathrm{Tr}H_jJ_D(H_l)\}$$

We see that $R^\alpha=0$ iff $\alpha=\pm1$. The dual curvature $R^{\alpha*}$ is then computed from [2]

$$\tilde{R}(X,Y,Z,W)=-\tilde{R}^*(X,Y,W,Z) \tag{4}$$

We see that, as $\tilde{\nabla}^\alpha$ and $\nabla^{\pm1}$ connections are flat, there exists an affine coordinate system. On the other hand, no dual coordinate system and hence no potential function exists, unless $J_D=J_\alpha$. It means that, in general, we cannot use Amari's theory to define a divergence.

Geodesics and divergence functions

As one dimensional submanifolds are always torsion-free, a divergence exists for each $\nabla^{\pm1*}$ and $\tilde{\nabla}^{\alpha*}$-geodesic. As suggested in [7], we use these to define a divergence in $\mathcal{D}$ ($\mathcal{M}^+$).

For each α, a $\tilde{\nabla}^{\alpha*}$-geodesic is a solution of

$$L_\alpha^{-1}J_{\rho_t}(\dot{\rho}_t)=A \tag{5}$$

where $A\in\mathcal{M}$, it means that each geodesic is determined by the observable A.

Proposition 2. *Let $\tilde{\rho}_t$ be a solution of $L_\alpha^{-1}J_{\rho_t}(\dot{\rho}_t)=A$. Then $\rho_t=\frac{\tilde{\rho}_t}{\mathrm{Tr}\tilde{\rho}_t}$ is a $\nabla^{\alpha*}$ geodesic.*

A divergence measure in $\mathcal{D}$ ($\mathcal{M}^+$) can be defined as follows. Let $\rho_0,\rho_1\in\mathcal{M}^+$. Let $\tilde{\rho}_t$ be the unique $\tilde{\nabla}^{\alpha*}$-geodesic connecting them. The α-divergence is $D_\alpha(\rho_0,\rho_1)=D^{\tilde{\rho}}_\alpha(0,1)$.

Proposition 3. *Let $\alpha\neq1$, $\rho_0,\rho_1\in\mathcal{M}^+$. Let $\tilde{\rho}_t$ be as above and let A be the corresponding observable. Then*

$$D_\alpha(\rho_0,\rho_1)=\frac{2}{1-\alpha}(\mathrm{Tr}\rho_0-\mathrm{Tr}\rho_1)+\mathrm{Tr}g_\alpha(\rho_1)A$$

If $\alpha=-1$ and $\rho_0,\rho_1\in\mathcal{D}$, we may use the $\nabla^{\alpha*}$-geodesic $\rho_t=(\mathrm{Tr}\tilde{\rho}_t)^{-1}\tilde{\rho}_t$ to define -1- divergence. In this way we obtain the restriction of D_α to $\mathcal{D}$.

Examples. 1. Let $J_D=J_\alpha$ for some $\alpha\in[-3,3]$. In this case we have the same situation as in the classical case. Thus for $\alpha=\pm1$, the divergence is given by

$$D(\rho_0,\rho_1)=\mathrm{Tr}\rho_1(\log\rho_1-\log\rho_0)$$

which is the relative entropy.

For $\alpha \neq \pm 1$, $D_\alpha(\rho_0, \rho_1) = \mathrm{Tr} g_\alpha(\rho_1)(g_{-\alpha}(\rho_1) - g_{-\alpha}(\rho_0))$ This α-divergence was defined also in [3]. It is easy to see that the above divergence functions are the same as those from Proposition 3.

2.[7] Let $\alpha = -1$ and let λ be the metric of the symmetric logarithmic derivative. Then

$$\rho_t = \exp\{\frac{1}{2}(tA - \psi(t))\}\rho_0 \exp\{\frac{1}{2}(tA - \psi(t))\}$$

where $\psi(t) = \log \mathrm{Tr}\rho_0 \exp tA$ is a ∇^{-1*}-geodesic. The divergence is

$$D_{-1}(\rho_0, \rho_1) = 2\mathrm{Tr}\rho_1 \log \rho_0^{-\frac{1}{2}}(\rho_0^{\frac{1}{2}}\rho_1\rho_0^{\frac{1}{2}})^{\frac{1}{2}}\rho_0^{-\frac{1}{2}}$$

which coincides with the relative entropy if ρ_0 and ρ_1 commute.

3. Let $\alpha = -1$ and let λ be the metric of the right logarithmic derivative. Let Q_F be the linear operator given by $Q_F^{-1}(A) = \frac{1}{2}(F^{-\frac{1}{2}}AF^{\frac{1}{2}} + F^{\frac{1}{2}}AF^{-\frac{1}{2}})$, then

$$\rho_t = \rho^{\frac{1}{2}} \exp\{(t-1)Q_\rho(A) - \psi(t)\}\rho^{\frac{1}{2}}$$

with $\psi(t) = \log \mathrm{Tr}\rho \exp\{(t-1)Q_\rho(A)\}$ is a ∇^{-1*}-geodesic. The divergence is given by

$$D_{-1}(\rho_0, \rho_1) = \mathrm{Tr}\rho_1 A = \mathrm{Tr}\rho_1 \log \rho_1^{\frac{1}{2}}\rho_0^{-1}\rho_1^{\frac{1}{2}}$$

which is another version of the relative entropy.

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