Base norms on subspaces of matrices, with applications in quantum information theory

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Discrimination of quantum states

Let $\mathcal{H}$ be a Hilbert space, $\dim(\mathcal{H}) < \infty$. Let

$$\mathfrak{S}(\mathcal{H}) = \{\rho \in B(\mathcal{H})^+, \operatorname{Tr} \rho = 1\}$$

the set of quantum states (density operators). Let $\rho_0, \rho_1 \in \mathfrak{S}(\mathcal{H})$.

- Assumption: A quantum system represented by $\mathcal{H}$ is in one of the states ρ_0, ρ_1.
- Task: Decide which is the true state.
- Tests: Binary POVMs: $(M, I - M)$, $0 \leq M \leq I$, $\operatorname{Tr} M\rho$ is the probability of choosing ρ_0 if the true state is ρ.
- Alternatively, a test is an affine map $\mathfrak{S}(\mathcal{H}) \to \mathcal{P}(\{0, 1\})$.

Discrimination of quantum states

Optimality: Given a prior $0 \leq \lambda \leq 1$, minimize the average error probability

$$E_\lambda(M) := \lambda \mathrm{Tr}\,(I - M)\rho_0 + (1-\lambda)\mathrm{Tr}\, M\rho_1$$

It is well known that (Helstrom, 1969)

$$\Pi_\lambda(\rho_0, \rho_1) := \min_{0 \leq M \leq I} E_\lambda(M) = \frac{1}{2}(1 - \|\lambda\rho_0 - (1-\lambda)\rho_1\|_1)$$

where

$$\|X\|_1 = \mathrm{Tr}\,|X|, \quad X \in B(\mathcal{H})$$

is the trace norm. Moreover,

- $\mathfrak{S}(\mathcal{H})$ is a base of the positive cone $B(\mathcal{H})^+$
- the corresponding base norm on $B(\mathcal{H})^{sa}$ is the trace norm.

Discrimination of quantum channels

Let $\dim(\mathcal{H}), \dim(\mathcal{K}) < \infty$. A quantum channel is a completely positive trace preserving map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$. Let Φ_0, Φ_1 be two channels.

- Assumption: A channel $B(\mathcal{H}) \to B(\mathcal{K})$ is either Φ_0 or Φ_1.
- Task: Decide which.
- Tests: Binary quantum 1-testers: $T = (\mathcal{H}_A, \rho, M)$, where $\rho \in \mathfrak{S}(\mathcal{H} \otimes \mathcal{H}_A)$, $M \in B(\mathcal{K} \otimes \mathcal{H}_A)$, $0 \leq M \leq I$.
  The probability that Φ_0 is chosen if the true channel is Φ is:

$$p(T, \Phi) := \mathrm{Tr}\, M(\Phi \otimes id_{\mathcal{H}_A})(\rho)$$

- Alternatively, tests are affine maps from all channels to $\mathcal{P}(\{0, 1\})$.

Discrimination of quantum channels

Optimality: Given $0 \leq \lambda \leq 1$, minimize the average error probability

$$E_\lambda^C(T) = \lambda(1 - p(T, \Phi_0)) + (1-\lambda)p(T, \Phi_1)$$

It is known that

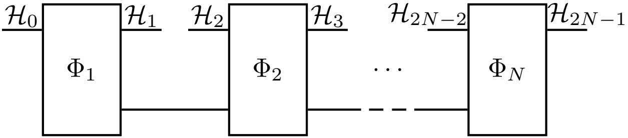
$$\Pi_\lambda^C(\Phi_0, \Phi_1) := \min_T E_\lambda^C(T) = \frac{1}{2}(1 - \|\lambda\Phi_0 - (1-\lambda)\Phi_1\|_\diamond)$$

where for $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$, $\Phi(X^*) = \Phi(X)^*$,

$$\begin{aligned} \|\Phi\|_\diamond &= \sup_{\dim(\mathcal{L}')<\infty} \sup_{\rho\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{L}')} \|\Phi \otimes id_{\mathcal{L}'}(\rho)\|_1 \\ &= \sup_{\rho\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{L})} \|\Phi \otimes id_{\mathcal{L}}(\rho)\|_1, \qquad \dim(\mathcal{L}) = \dim(\mathcal{H}) \end{aligned}$$

Discrimination of quantum networks

Quantum N-combs: (Chiribella,D'Ariano,Perinotti,2008)
Quantum N-round strategies (Gutoski,Watrous,2007,2011)



This defines a channel

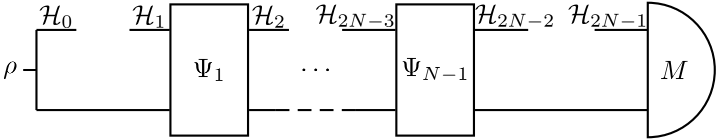
$$\Phi : B(\mathcal{H}_0 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_{2N-2}) \to B(\mathcal{H}_1 \otimes \mathcal{H}_3 \otimes \cdots \otimes \mathcal{H}_{2N-1}).$$

Its Choi matrix X_Φ is a quantum N-comb for $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$.

- Task: Discriminate between two quantum networks Φ^0 and Φ^1

Discrimination of quantum networks

- Tests: Binary quantum N-testers (CDP,2008)
  measuring N-round co-strategies (GW,2007,2011)



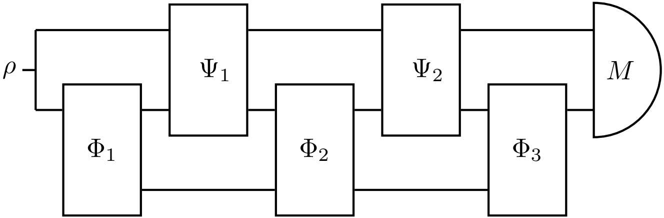
$\rho \in \mathfrak{S}(\mathcal{H}_0 \otimes \mathcal{H}_A)$, $M \in B(\mathcal{H}_{2N-1} \otimes \mathcal{H}_A)$ a binary POVM

- Represented by $T = (T_0, T_1)$, $T_i \in B(\mathcal{H}_{2N-1} \otimes \cdots \otimes \mathcal{H}_0)^+$,

  $T_0 + T_1$ is an $(N+1)$-comb for $(\mathbb{C}, \mathcal{H}_0, \ldots, \mathcal{H}_{2N-1}, \mathbb{C})$.

Discrimination of quantum networks

- The probability of choosing Φ^0 if the true value is Φ is $\mathrm{Tr}\, T_0 X_\Phi$



$N = 3$

- Alternatively, test are affine maps from N-combs to $\mathcal{P}(\{0, 1\})$.

Discrimination of quantum networks

- The minimum average error probability:
  (CDP 2008, Gutoski 2011 - "strategy N-norm")

$$\Pi_\lambda^N(\Phi^0,\Phi^1)=\frac{1}{2}(1-\|\lambda\Phi^0-(1-\lambda)\Phi^1\|_{\diamond N})$$

  where for any Hermitian map Φ,

$$\begin{aligned}\|\Phi\|_{\diamond N} &= \sup_{Y=T_0+T_1}\|Y^{1/2}X_\Phi Y^{1/2}\|_1 \quad \text{(CDP)}\\ &= \sup_T \mathrm{Tr}\, X_\Phi(T_1-T_0), \quad \text{(Gutoski)}\end{aligned}$$

- We will show that all these norms are obtained as certain base norms on subspaces of self-adjoint operators.

Ordered vector spaces, bases and order units

Let $\mathcal{V}$, $\mathcal{V}^*$ be a finite dimensional real vector space and its dual, with duality $\langle\cdot,\cdot\rangle$.

- $Q \subset \mathcal{V}$ a positive cone in $\mathcal{V}$: closed convex cone, $Q \cap -Q = \{0\}$, $\mathcal{V} = Q - Q$
- $(\mathcal{V}, Q)$ ordered vector space: $x \leq_Q y$ if $y - x \in Q$
- $Q^* \subset \mathcal{V}^*$ the dual cone: $\{f \in \mathcal{V}^*, \langle f, q\rangle \geq 0,\ q \in Q\}$, $(\mathcal{V}^*, Q^*)$ is an ordered vector space
- $B \subset Q$ a base: for $q \in Q$, $q \neq 0$, $\exists t > 0, b \in B$: $q = tb$
- $e \in Q$ an order unit: for $x \in \mathcal{V}$, $\exists r > 0$: $re \geq_Q x$, e is an order unit iff $e \in int(Q)$

There is a 1-1 correspondence between bases of Q and order units $e \in Q^*$:

$$B = \{q \in Q,\ \langle e, q\rangle = 1\}$$

Base norms and order unit norms

Let $B \subset Q$ be a base, $e \in Q$ an order unit.

- Base norm in $\mathcal{V}$:
  $\|x\|_B = \inf\{\lambda + \mu : \ x = \lambda b_1 - \mu b_2, \lambda, \mu \geq 0, b_1, b_2 \in B\}$
- Order unit norm in $\mathcal{V}$:
  $\|x\|_e = \inf\{\lambda > 0 : \ -\lambda e \leq_Q x \leq_Q \lambda e\}$

Let $e_B \in Q^*$ be such that $B = \{q \in Q : \ \langle e_B, q\rangle = 1\}$. Then $\| \cdot \|_B^* = \| \cdot \|_{e_B}$ is the dual norm. Hence

$$\|x\|_B = \sup_{-e_B \leq_{Q^*} f \leq_{Q^*} e_B} \langle f, x\rangle = 2 \sup_{0 \leq_{Q^*} f \leq_{Q^*} e_B} \langle f, x\rangle - \langle e_B, x\rangle$$

Discrimination of elements of a base B

Formally, we can consider the discrimination problem in B:

- Given $b_0, b_1 \in B$, distinguish them by tests = affine maps $B \to \mathcal{P}(\{0,1\})$.
- Tests correspond to elements $0 \leq_{Q^*} m \leq_{Q^*} e_B$: the probability of choosing b_0 if the "true value" is b is $\langle m, b \rangle$.
- Given $0 \leq \lambda \leq 1$, the average error probability is

$$\begin{aligned} E_\lambda^B(m) &= \lambda(1 - \langle m, b_0 \rangle) + (1-\lambda)\langle m, b_1 \rangle \\ &= \lambda - \langle m, \lambda b_0 - (1-\lambda) b_1 \rangle \end{aligned}$$

It follows that

$$\Pi_\lambda^B(b_0, b_1) := \min_{0 \leq_{Q^*} m \leq_{Q^*} e_B} E_\lambda^B(m) = \frac{1}{2}(1 - \|\lambda b_0 - (1-\lambda) b_1\|_B)$$

Bases of the cone of positive operators

Let $\mathcal{A}$ be a finite dimensional C^*-algebra, $\mathcal{A}^{sa} = \{a \in \mathcal{A}, a = a^*\}$, $\mathcal{A}^+ = \{a \in \mathcal{A}, a \geq 0\}$. Identify $(\mathcal{A}^{sa})^* = \mathcal{A}^{sa}$, with $\langle a, b \rangle = \mathrm{Tr}\, ab$.

- $(\mathcal{A}^{sa}, \mathcal{A}^+)$ is an ordered vector space.
- $\mathcal{A}^+$ is self-dual: $(\mathcal{A}^+)^* = \mathcal{A}^+$.
- Order units: $a \in int(\mathcal{A}^+) = \{a \geq 0, a \text{ invertible}\}$.
- Bases of $\mathcal{A}^+$: $B \leftrightarrow \tilde{b} \in int(\mathcal{A}^+)$:

$$B = S_{\tilde{b}} = \{a \in \mathcal{A}^+, \mathrm{Tr}\, a\tilde{b} = 1\}$$

- States: $\mathfrak{S}(\mathcal{A}) = S_I$

Base norms and order unit norms in $\mathcal{A}^{sa}$

Let $\tilde{b} \in int(\mathcal{A}^+)$, $B = S_{\tilde{b}}$.

$$
\begin{aligned}
\|x\|_{S_{\tilde{b}}} &= \|\tilde{b}^{1/2} x \tilde{b}^{1/2}\|_1 \\
\|x\|^*_{S_{\tilde{b}}} &= \|x\|_{\tilde{b}} = \inf\{\lambda > 0 : -\lambda\tilde{b} \leq x \leq \lambda\tilde{b}\}
\end{aligned}
$$

For $\tilde{b} = I$,

$$
\|x\|_{\mathfrak{S}(\mathcal{A})} = \|x\|_1 = \mathrm{Tr}\,|x|, \qquad \|x\|_I = \|x\|
$$

If $a \geq 0$, then

$$
\|a\|_{\tilde{b}} = 2^{D_{max}(a\|b)},
$$

$D_{max}(a\|b)$ - relative max-entropy (Datta, 2009)

Bases for subspaces in $\mathcal{A}^{sa}$

Let $J \subseteq \mathcal{A}^{sa}$ be a real linear subspace, $Q = J \cap \mathcal{A}^+$. Suppose $J = Q - Q$.

- (J, Q) is an ordered vector space.
- $J^* \equiv \mathcal{A}^{sa}|_{J^\perp} = \{a + J^\perp, a \in \mathcal{A}^{sa}\}$, $J^\perp = \{a, \mathrm{Tr}\, ax = 0, x \in J\}$
- Duality: $\langle a + J^\perp, x\rangle = \mathrm{Tr}\, ax$, $x \in J$
- $Q^* = \{a + J^\perp,\ a \in \mathcal{A}^+\}$ (AJ, 2012)
- $int(Q^*) = \{a + J^\perp,\ a \in int(\mathcal{A}^+)\}$

Lemma

Bases of Q are precisely sets of the form $B = J \cap S_{\tilde{b}}$, with $\tilde{b} \in int(\mathcal{A}^+)$, $e_B = \tilde{b} + J^\perp$.

Sections of a base of $\mathcal{A}^+$

Let $\tilde{b} \in int(\mathcal{A}^+)$, $L \subseteq \mathcal{A}^{sa}$ be a subspace. Let $B = L \cap S_{\tilde{b}}$.

- B is called a section of a base of $\mathcal{A}^+$.
- Let $J = \mathrm{span}(B)$, $Q = J \cap \mathcal{A}^+$, then B is a base of Q.
- Conversely, if $J \subseteq \mathcal{A}^+$, $Q = J \cap \mathcal{A}^+$, $J = Q - Q$ and B is a base of Q, then B is a section of a base of $\mathcal{A}^+$.
- If B is a section and $B \cap int(\mathcal{A}^+) \neq \emptyset$, then B is called faithful. In this case, $B \cap int(\mathcal{A}^+) = ri(B)$.

The dual section

Let B be a section of a base of $\mathcal{A}^+$, $J = \mathrm{span}(B)$, $Q = J \cap \mathcal{A}^+$.
Let
$$\tilde{B} = \{\tilde{b} \in \mathcal{A}^+ : \ \mathrm{Tr}\, \tilde{b}b = 1, b \in B\}.$$

Then $\tilde{B}$ is convex and closed, and

$$ri(\tilde{B}) = \{\tilde{b} \in int(\mathcal{A}^+) : \ B = J \cap S_{\tilde{b}}\}$$

Lemma

Let B be faithful, $b \in ri(B)$. Let $\tilde{J} = \mathrm{span}(\tilde{B})$.

(i) $\tilde{B} = \tilde{J} \cap S_b$, *so that $\tilde{B}$ is a faithful section of a base of $\mathcal{A}^+$.*

(ii) $\tilde{\tilde{B}} = B$

(iii) $B = \bigcap_{\tilde{b} \in ri(\tilde{B})} S_{\tilde{b}} = \bigcap \{B_0 \text{ is a base of } \mathcal{A}^+, B \subseteq B_0\}$

An extension of the base norm

Let B be a section of a base of $\mathcal{A}^+$, $J = \mathrm{span}(B)$, $Q = J \cap \mathcal{A}^+$. Let

$$\mathcal{O}_B = \{x \in \mathcal{A}^{sa} : \; x = x_1 - x_2, \; x_1, x_2 \geq 0, \; x_1 + x_2 \in B\}$$

$\mathcal{O}_B \cap J$ is the unit ball of the base norm $\| \cdot \|_B$ in (J, Q).

Theorem

Let B be faithful. Then

(i) *$\mathcal{O}_B$ is the unit ball of a norm in $\mathcal{A}^{sa}$. We denote it by $\| \cdot \|_B$.*

(ii) *The dual norm is $\| \cdot \|_{\tilde{B}}$.*

(iii) $\|x\|_B = \sup_{\tilde{b} \in ri(\tilde{B})} \|x\|_{S_{\tilde{b}}} = \max_{\tilde{b} \in \tilde{B}} \|\tilde{b}^{1/2} x \tilde{b}^{1/2}\|_1$

(iv) $\|x\|_B = \inf_{b \in ri(B)} \|x\|_b = \min_{b \in B} \inf\{\lambda > 0 : -\lambda b \leq x \leq \lambda b\}$

Extended base norm in $\mathcal{A}^+$

Let $a \in \mathcal{A}^+$, then

$$\|a\|_B = \max_{\tilde{b} \in \tilde{B}} \mathrm{Tr}\, a\tilde{b} = \min_{b \in B} 2^{D_{max}(a\|b)}$$

Moreover, $\tilde{b}_0 \in \tilde{B}$ is such that $\|a\|_B = \mathrm{Tr}\, a\tilde{b}_0$ if and only if: $\exists q = sb_0 \in Q$, such that

$$a \leq q \quad \text{and} \quad (q - a)\tilde{b}_0 = 0$$

In this case, $s = \|a\|_B = 2^{D_{max}(a\|b_0)}$.

The Choi isomorphism

Let $\dim(\mathcal{H}), \dim(\mathcal{K}) < \infty$.

- Choi isomorphism:
  $X \in B(\mathcal{K} \otimes \mathcal{H}) \leftrightarrow$ linear maps $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$:

  $$X_\Phi = (\Phi \otimes id_{\mathcal{H}})(|\psi\rangle\langle\psi|), \quad \Phi_X(a) = \mathrm{Tr}_{\mathcal{H}}[(I_{\mathcal{K}} \otimes a^{\mathsf{T}})X]$$

  $|\psi\rangle = \sum_i |i\rangle \otimes |i\rangle$, $|i\rangle$ an ONB in $\mathcal{H}$, a^{T} transpose of a
- Φ is completely positive if and only if $X_\Phi \geq 0$.
- Φ is Hermitian $\Phi(x^*) = \Phi(x)^*$ if and only if $X_\Phi = X_\Phi^*$.
- Φ is trace preserving if and only if $\mathrm{Tr}_{\mathcal{K}} X_\Phi = I_{\mathcal{H}}$.

Generalized channels

Let B be a faithful section of a base of $B(\mathcal{H})^+$.

- A generalized channel with respect to B: $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$, completely positive and $\Phi(B) \subseteq \mathfrak{S}(\mathcal{K})$.
- Φ is a generalized channel with respect to B if and only if

$$X_\Phi \in \mathcal{C}_B(\mathcal{H}, \mathcal{K}) := \{X \in B(\mathcal{K} \otimes \mathcal{H})^+,\ (\mathrm{Tr}_{\mathcal{K}} X)^{\mathsf{T}} \in \tilde{B}\}$$

- $\tilde{B} = \mathcal{C}_B(\mathcal{H}, \mathbb{C})$

Theorem

$\mathcal{C}_B(\mathcal{H}, \mathcal{K})$ is a faithful section of a base of $B(\mathcal{K} \otimes \mathcal{H})^+$. The dual section is $\widetilde{\mathcal{C}_B(\mathcal{H}, \mathcal{K})} = \{I_{\mathcal{K}} \otimes b^{\mathsf{T}}, b \in B\} = I_{\mathcal{K}} \otimes B^{\mathsf{T}}$.

Generalized channels: base norm and its dual

Let $X = X^* \in B(\mathcal{K} \otimes \mathcal{H})$, $X = (\Phi \otimes id_{\mathcal{H}})(|\psi\rangle\langle\psi|)$.

Theorem

$$\begin{aligned}
\|X\|_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} &= \sup_{\dim(\mathcal{L}')<\infty} \ \sup_{Y\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L}')} \|(\Phi \otimes id_{\mathcal{L}'})(Y)\|_1 \\
&= \sup_{Y\in\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{L})} \|(\Phi \otimes id_{\mathcal{L}})(Y)\|_1, \quad \dim(\mathcal{L}) = \dim(\mathcal{H}) \\
\|X\|^*_{\mathcal{C}_B(\mathcal{H},\mathcal{K})} &= \|X\|_{I\otimes B^{\mathsf{T}}} \\
&= \inf_{b\in B} \inf\{\lambda > 0, -\lambda(I \otimes b^{\mathsf{T}}) \leq X \leq \lambda(I \otimes b^{\mathsf{T}})\}
\end{aligned}$$

If $X \geq 0$, then

$$\|X\|_{I\otimes B^{\mathsf{T}}} = \inf_{b\in B} 2^{D_{max}(X\|I\otimes b^{\mathsf{T}})}$$

Quantum channels

Let $B = \mathfrak{S}(\mathcal{H})$. Then

- $\mathcal{C}_B(\mathcal{H},\mathcal{K}) =: \mathcal{C}(\mathcal{H},\mathcal{K})$ Choi matrices of channels
- $\mathcal{C}_{\tilde{B}}(\mathcal{H},\mathcal{K}) = \mathfrak{S}(\mathcal{K}\otimes\mathcal{H})$
- $\widetilde{\mathcal{C}(\mathcal{H},\mathcal{K})} = \{I_{\mathcal{K}}\otimes\rho,\ \ \rho\in\mathfrak{S}(\mathcal{H})\}$

Hence if $X = (\Phi\otimes id_{\mathcal{H}})(|\psi\rangle\langle\psi|) \in B(\mathcal{K}\otimes\mathcal{H})^{sa}$,

$$\begin{aligned}
\|X\|_{\mathcal{C}(\mathcal{H},\mathcal{K})} &= \sup_{\dim(\mathcal{L}')<\infty}\ \sup_{\sigma\in\mathfrak{S}(\mathcal{H}\otimes\mathcal{L}')} \|(\Phi\otimes id_{\mathcal{L}'})(\sigma)\|_1 = \|\Phi\|_\diamond \\
\|X\|^*_{\mathcal{C}(\mathcal{H},\mathcal{K})} &= \inf_{\rho\in\mathfrak{S}(\mathcal{H})} \inf\{\lambda>0, -\lambda(I\otimes\rho)\le X\le\lambda(I\otimes\rho)\}
\end{aligned}$$

If $\sigma\in\mathfrak{S}(\mathcal{K}\otimes\mathcal{H})$ is a state, then

$$\|\sigma\|^*_{\mathcal{C}(\mathcal{H},\mathcal{K})} = 2^{-H_{min}(\mathcal{K}|\mathcal{H})_\sigma}$$

$H_{min}(\mathcal{K}|\mathcal{H})_\sigma$ is the conditional min-entropy of σ, (Renner,2008)

Quantum supermaps

Let $\mathcal{H}_0, \mathcal{H}_1, \dots$ be a sequence of finite dimensional Hilbert spaces. Define

1. $\mathcal{C}(\mathcal{H}_0) := \mathfrak{S}(\mathcal{H}_0)$.
2. For $n \geq 1$, $\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n) \subseteq B(\mathcal{H}_n \otimes \cdots \otimes \mathcal{H}_0)^+$,

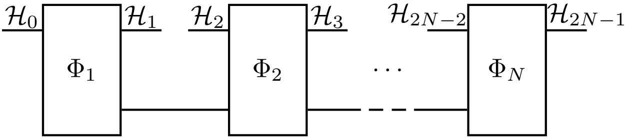
$$\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n) := \{X_\Phi \geq 0,\ \Phi(\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_{n-1})) \subseteq \mathfrak{S}(\mathcal{H}_n)\}$$

Then

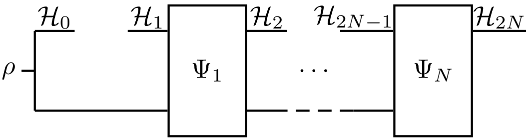
- $\mathcal{C}_n := \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n)$ is a faithful section of a base of $B(\mathcal{H}_n \otimes \cdots \otimes \mathcal{H}_0)^+$.
- $\mathcal{C}_n = \mathcal{C}_{\mathcal{C}_{n-1}}(\mathcal{H}_{n-1} \otimes \cdots \otimes \mathcal{H}_0, \mathcal{H}_n)$
- $\widetilde{\mathcal{C}}_n = I_n \otimes \mathcal{C}_{n-1} = \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n, \mathbb{C})$
- $\mathcal{C}_{\widetilde{\mathcal{C}}_n}(\mathcal{H}_n \otimes \cdots \otimes \mathcal{H}_0, \mathcal{L}') = \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_{n-1}, \mathcal{H}_n \otimes \mathcal{L}')$

Quantum supermaps

- $\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_{2N-1}) \equiv N$-combs for $(\mathcal{H}_0, \dots, \mathcal{H}_{2N-1})$:



- $\mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_{2N}) \equiv (N+1)$-combs for $(\mathbb{C}, \mathcal{H}_0, \dots, \mathcal{H}_{2N})$:



Supermap norms and their duals

Let $n \geq 2$. Let $X \in B(\mathcal{H}_n \otimes \cdots \otimes \mathcal{H}_0)^{sa}$ and let $\Phi : B(\mathcal{H}_{n-1} \otimes \cdots \otimes \mathcal{H}_0) \to B(\mathcal{H}_n)$ be the corresponding Hermitian map. Then

$$\begin{aligned} \|X\|_{\mathcal{C}_n} &= \sup_{\dim(\mathcal{L}')<\infty} \sup_{Y \in \mathcal{C}(\mathcal{H}_0,\ldots,\mathcal{H}_{n-2},\mathcal{H}_{n-1}\otimes\mathcal{L}')} \|(\Phi \otimes id_{\mathcal{L}'})(Y)\|_1 \\ &= \sup_{Y \in \mathcal{C}(\mathcal{H}_0,\ldots,\mathcal{H}_{n-2},\mathcal{H}_{n-1}\otimes\mathcal{L})} \|(\Phi \otimes id_{\mathcal{L}})(Y)\|_1 \end{aligned}$$

where $\dim(\mathcal{L}) = \Pi_{k=0}^{n-1} \dim(\mathcal{H}_k)$. The dual norm is

$$\|X\|^*_{\mathcal{C}_n} = \|X\|_{\mathcal{C}(\mathcal{H}_0,\ldots,\mathcal{H}_n,\mathbb{C})}$$

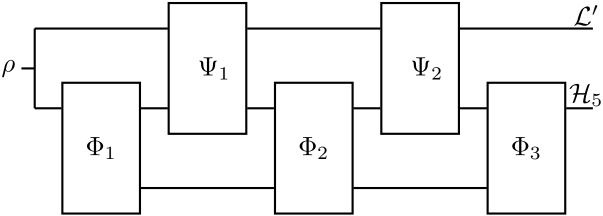
We have

$$\|X\|_{\mathcal{C}_{2N-1}} = \|\Phi\|_{N\diamond}$$

Supermap norms and their duals

Let $X = X_\Phi \in \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_n)$, $Y \in \mathcal{C}(\mathcal{H}_0, \dots, \mathcal{H}_{n-1} \otimes \mathcal{L}')$.

Then $(\Phi \otimes id_{\mathcal{L}'})(Y)$ has the form



$n = 5$

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Measurements on B and generalized POVMs

Let B be a section of a base of $B(\mathcal{H})^+$, U a finite set. Let $J = \mathrm{span}(B)$, $Q = J \cap B(\mathcal{H})^+$.

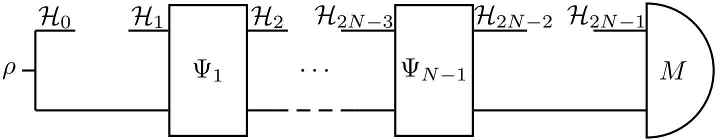
- A measurement on B is an affine map $\mathbf{m} : B \to \mathcal{P}(U)$.
- $\mathbf{m} \equiv \{f_u \in Q^*,\ u \in U\}$, $\sum_u f_u = e_B$: $\mathbf{m}(b)_u = \langle f_u, b\rangle$.
- $f_u = M_u + J^\perp$, for some $M_u \in B(\mathcal{H})^+$, $\sum_u M_u \in \tilde{B}$.
- $\mathcal{M}_B(\mathcal{H}, U) = \{\{M_u, u \in U\}, M_u \in B(\mathcal{H})^+, \sum_u M_u \in \tilde{B}\}$ - generalized POVMs with respect to B
- $\mathcal{M}_B(\mathcal{A}, U) \subset \mathcal{C}_B(\mathcal{H}, \mathcal{H}_U)$, $\dim(\mathcal{H}_U) = |U|$:

$$\{M_u, u \in U\} \mapsto M = \sum_{u \in U} |u\rangle\langle u| \otimes M_u^{\mathsf{T}}$$

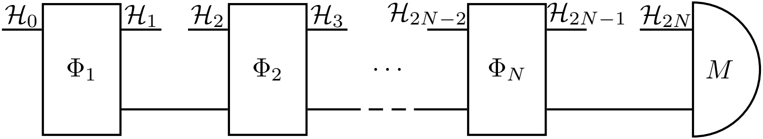
$\{|u\rangle, u \in U\}$ an ONB in $\mathcal{H}_U$.

Quantum testers

If $B = \mathcal{C}(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$, then elements in $\mathcal{M}_B(\mathcal{A}, U)$ are quantum N-testers for $(\mathcal{H}_0, \ldots, \mathcal{H}_{2N-1})$:



If $B = \mathcal{C}(\mathcal{H}_0, \ldots, \mathcal{H}_{2N})$:



$M = \{M_u, u \in U\}$, $M_u \in B(\mathcal{H}_n \otimes \mathcal{H}_A)^+$, $\sum_u M_u = I$ a POVM

Decision theory

Let B be a section of a base of $B(\mathcal{H})^+$.
Let $\{b_\theta, \theta \in \Theta\} \subset B$, $|\Theta| = m$, be the set of possible values of $b \in B$. Let

- D, $|D| = k$ be a set of decisions,
- $w : \Theta \times D \to [0,1]$ a loss function,
- (D, w) a (classical) decision problem,
- $M = \{M_d, d \in D\} \in \mathcal{M}_B(\mathcal{H}, D)$ a decision function.
- Given a prior $\lambda \in \mathcal{P}(\Theta)$, the average loss for M is

$$\mathcal{W}^B_{w,\lambda}(M) = \sum_\theta \sum_d \lambda_\theta w(\theta, d) \mathrm{Tr}\, b_\theta M_d$$

- Minimize $\mathcal{W}^B_{w,\lambda}(M)$ over all decision functions.

Quantum decision problems

Let $\{b_\theta, \theta \in \Theta\} \subset B$, $|\Theta| = m$. Let

- $\mathcal{D}$, $\dim(\mathcal{D}) = k$ be a Hilbert space,
- $W : \Theta \ni \theta \mapsto W_\theta \in B(\mathcal{D})$, $0 \leq W_\theta \leq I_{\mathcal{D}}$ loss operators,
- $(\mathcal{D}, W)$ a quantum decision problem,
- $X \in \mathcal{C}_B(\mathcal{H}, \mathcal{D})$ a quantum decision function.
- Given a prior $\lambda \in \mathcal{P}(\Theta)$, the average loss for $X = X_\Phi$ is

$$\mathcal{W}^B_{W,\lambda}(X) = \sum_\theta \lambda_\theta \mathrm{Tr}\, \Phi(b_\theta) W_\theta = \sum_\theta \lambda_\theta \mathrm{Tr}\, (W_\theta \otimes b_\theta^{\mathsf{T}}) X$$

- Minimize this over all X.

Classical and quantum decision problems

Let (D, w) be a classical decision problem. Let

- $\mathcal{H}_D$ a Hilbert space, $\dim(\mathcal{H}_D) = |D|$, $\{|d\rangle, d \in D\}$ ONB,
- $W_\theta = \sum_{d \in D} w(\theta, d)|d\rangle\langle d| \in B(\mathcal{H}_D)$, $0 \leq W_\theta \leq I$.

Then $(\mathcal{H}_D, W)$ is a quantum decision problem, with

$$W_\theta W_{\theta'} = W_{\theta'} W_\theta, \qquad \theta, \theta' \in \Theta \tag{$*$}$$

Conversely, let $(\mathcal{D}, W)$ be a quantum decision problem with $(*)$, then

$$W_\theta = \sum_d w(\theta, d)|e_d\rangle\langle e_d|, \qquad \theta \in \Theta$$

for some ONB, and (D, w) is a classical decision problem. If $X \in \mathcal{C}_B(\mathcal{H}, \mathcal{D})$, then $\mathcal{W}^B_{W,\lambda}(X) = \mathcal{W}^B_{w,\lambda}(M)$, with

$$M \in \mathcal{M}_B(\mathcal{H}, D), \quad M_d = (\langle e_d| \otimes I_{\mathcal{H}})X^{\mathsf{T}}(|e_d\rangle \otimes I_{\mathcal{H}})$$

Minimal average loss

Theorem

Minimal average loss is

$$\begin{aligned}\Pi^B_{W,\lambda} &:= \min_{X\in\mathcal{C}_B(\mathcal{H},\mathcal{D})} \mathcal{W}^B_{W,\lambda}(X) = 1 - \|\xi_{W,\lambda}\|_{I_\mathcal{D}\otimes B} \\ &= 1 - \inf_{b\in B} 2^{D_{max}(\xi_{W,\lambda}\|I_\mathcal{D}\otimes b)}\end{aligned}$$

where $\xi_{W,\lambda} = \sum_\theta \lambda_\theta (I - W_\theta^\mathsf{T}) \otimes b_\theta$. *If* $W_\theta = \sum_d w(\theta, d)|d\rangle\langle d|$,

$$\xi_{W,\lambda} = \sum_d |d\rangle\langle d| \otimes \sum_\theta \lambda_\theta w(\theta, d) b_\theta = \sum_d |d\rangle\langle d| \otimes \xi_{w,\lambda,d}$$

and

$$\|\xi_{W,\lambda}\|_{I_\mathcal{D}\otimes B} = \inf_{b\in B} \sup_{d\in D} 2^{D_{max}(\xi_{w,\lambda,d}\|b)}$$

Proof

We have

$$\mathcal{W}^{B}_{W,\lambda}(X) = \sum_{\theta} \lambda_\theta \mathrm{Tr}\,(W_\theta \otimes b_\theta^{\mathsf{T}})X = 1 - \mathrm{Tr}\,(X^{\mathsf{T}} \xi_{W,\lambda})$$

Hence

$$\begin{aligned} \min_{X \in \mathcal{C}_B(\mathcal{H},\mathcal{D})} \mathcal{W}^{B}_{W,\lambda}(X) &= 1 - \max_{X \in \mathcal{C}_{B^{\mathsf{T}}}(\mathcal{H},\mathcal{D})} \mathrm{Tr}\,(X \xi_{W,\lambda}) \\ &= 1 - \|\xi_{W,\lambda}\|^{*}_{\mathcal{C}_{B^{\mathsf{T}}}(\mathcal{H},\mathcal{D})} = 1 - \|\xi_{W,\lambda}\|_{I \otimes B} \end{aligned}$$

$\square$

Optimal Bayes decision function

Corollary

- $X_0 \in \mathcal{C}_B(\mathcal{H}, \mathcal{D})$ *satisfies* $\mathcal{W}^B_{W,\lambda}(X_0) = \Pi^B_{W,\lambda}$ *if and only if there* $\exists$ *some* $b_0 \in B$, $s \geq 0$ *such that*

$$s(I_{\mathcal{D}} \otimes b_0) \geq \xi_{W,\lambda} \quad \textit{and} \quad (s(I \otimes b_0) - \xi_{W,\lambda})X_0^{\mathsf{T}} = 0$$

In this case, $s = 2^{D_{max}(\xi_{W,\lambda} \| I \otimes b_0)} = 1 - \Pi^B_{W,\lambda}$.

- *If* W *is classical,* $M^0 \in \mathcal{M}_B(\mathcal{H}, U)$ *is optimal if and only if there* $\exists$ $b_0 \in B$, $s \geq 0$ *such that for all* $d \in D$,

$$sb_0 \geq \xi_{w,\lambda,d} \quad \textit{and} \quad (sb_0 - \xi_{w,\lambda,d})M_d = 0$$

Hypothesis testing

Let $\Theta = D = \{0,1\}$ and $w(\theta, d) = 1 - \delta_{\theta d}$, the average loss is the average error probability

$$\mathcal{W}^B_{w,\lambda}(M) = E^B_\lambda(M) = \lambda \mathrm{Tr}\, b_0 M_1 + (1-\lambda)\mathrm{Tr}\, b_1 M_0$$

and

$$\Pi^B_{w,\lambda} = 1 - \|\xi_{w,\lambda}\|_{l_2 \otimes B}, \quad \xi_{w,\lambda} = |0\rangle\langle 0| \otimes \lambda b_0 + |1\rangle\langle 1| \otimes (1-\lambda) b_1$$

Lemma

Let $s, t > 0$, $b_0, b_1 \in B$. Then

$$\||0\rangle\langle 0| \otimes s b_0 + |1\rangle\langle 1| \otimes t b_1\|_{l_2 \otimes B} = \frac{1}{2}(\|s b_0 - t b_1\|_B + s + t)$$

Hence $\Pi^B_{w,\lambda} = \frac{1}{2}(1 - \|\lambda b_0 - (1-\lambda b_1)\|_B)$.

Multiple hypothesis testing

Task: given $b_1, \ldots, b_m \in B$, with a prior λ, decide which is the true value, with minimal probablity of error

- $\Theta = D = \{1, \ldots, m\}$, $w(\theta, d) = 1 - \delta_{\theta d}$
- $\xi_{w,\lambda} = \sum_{d \in D} |d\rangle\langle d| \otimes \lambda_d b_d$
- $\Pi^B_{w,\lambda} = 1 - \|\xi_{w,\lambda}\|_{I_m \otimes B} = 1 - \min_{b \in B} \max_{\theta \in \Theta} \lambda_\theta 2^{D_{max}(b_\theta \| b)}$

If $B = \mathfrak{S}(\mathcal{H})$, $\{\rho_1, \ldots, \rho_m\} \subset \mathfrak{S}(\mathcal{H})$:

$$\begin{aligned}\Pi_\lambda &= 1 - 2^{-H_{min}(\mathcal{H}|\mathcal{H}_D)_\xi} && \text{(Renner, 2008)}\\ &= 1 - \min_{\sigma \in \mathfrak{S}(\mathcal{H})} \max_{1 \le i \le m} \lambda_i 2^{D_{max}(\rho_i \| \sigma)} && \text{(Datta,2009)}\end{aligned}$$