Comparison of quantum channels and quantum statistical experiments

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Ordering of quantum channels

Let $\Phi$ and $\Psi$ be two channels with the same input space. Ordering by degradability: $\Phi \preceq \Psi$ if

$$\Phi = \alpha \circ \Psi$$

for some channel $\alpha$.

Applications:

- (anti)degradable channels
- for classical-to-quantum channels:

$$\alpha : \{\rho_1, \ldots, \rho_n\} \rightarrow \{\sigma_1, \ldots, \sigma_n\}$$

- Correctable subsystems/subalgebras for a quantum channel
Deficiency of quantum channels

Approximate version: deficiency

\[ \delta(\Phi, \Psi) = \inf_{\alpha} \| \Phi - \alpha \circ \Psi \| \]

- \( \epsilon \)-versions of the previous applications

The aim

The aim is to express \( \delta(\Phi, \Psi) \) in terms of success probabilities for some ensembles.
The classical case: comparison of statistical experiments

**Statistical experiment:** \((X, \Sigma, \{p_\theta, \theta \in \Theta\})\)

- probability distributions over sample space \((X, \Sigma)\)
- in our setting: \(X, \Theta\) finite sets

• Comparison of statistical experiments: (Blackwell, 1951)
  \(S = (X, \{p_\theta, \theta \in \Theta\}), \quad T = (Y, \{q_\theta, \theta \in \Theta\})\)

  \(S \leq T\) if there is a channel \(\alpha : q_\theta \mapsto p_\theta, \quad \forall \theta\)

  \(\equiv S\) is a randomization of \(T\)

• Deficiency of statistical experiments: (Le Cam, 1969)

  \[\delta(S, T) = \inf_{\alpha} \sup_{\theta} \|p_\theta - \alpha(q_\theta)\|_1\]

  - Le Cam distance: \(\Delta(S, T) = \max\{\delta(S, T), \delta(T, S)\}\).
BSS theorem

(Blackwell, Sherman, Stein, 1951)

\[ S \preceq T \iff \]

for any decision space \((D, g)\):

- \(D\) a finite set of decisions
- \(g : \Theta \times D \to \mathbb{R}^+\) a payoff function,

and any decision rule \(\mu\) for \(S\):

- a channel with input \(X\) and output \(D\),

there is a decision rule \(\nu\) for \(T\) such that the average payoffs satisfy:

\[
\sum_{x,d} \mu(x|d)p_\theta(x)g(\theta, d) \leq \sum_{y,d} \nu(y|d)q_\theta(y)g(\theta, d), \quad \forall \theta
\]
Randomization criterion

(Le Cam 1969)

\[ \delta(S, T) \leq \epsilon \iff \]

for any decision space \((D, g)\)

and any decision rule \(\mu\) for \(S\),

there is a decision rule \(\nu\) for \(T\) such that for all \(\theta\):

\[
\sum_{x,d} \mu(x|d)p_{\theta}(x)g(\theta, d) \leq \sum_{y,d} \nu(y|d)q_{\theta}(y)g(\theta, d) + \frac{\epsilon}{2} \sup_d g(\theta, d)
\]
Quantum case: statistical experiments

- **Quantum statistical experiments**: \((\mathcal{H}, \{\rho_\theta, \theta \in \Theta\})\)
  - \(\mathcal{H}\) a Hilbert space, \(\dim(\mathcal{H}) < \infty\)
  - \(\rho_\theta \in \mathcal{S}(\mathcal{H})\) density operators: \(\rho_\theta \geq 0, \text{Tr} \rho_\theta = 1\).

- (classical) decision problems: \((D, g)\)
  - decision rules: POVMs \(\{M_d, d \in D\}, M_d \geq 0, \sum_d M_d = I\).
  - average payoff:
    \[
    \sum_d g(\theta, d) \text{Tr}[M_d \rho_\theta]
    \]

- A straightforward generalization of BSS theorem by comparing the average payoffs - not true (Matsumoto, 2014)
Quantum case

• **Quantum BSS theorem:**
  - requires additional entanglement (Shmaya, 2005)
  - or tensoring with a complete set of states (Buscemi, 2012)

• **Quantum randomization criterion:** (Matsumoto, 2010)
  - uses quantum decision spaces, unclear interpretation

• **Ordering of quantum channels:** (Chefles, 2009)

• **Deficiency of quantum channels:** the main result
  - we also obtain a version of the randomization criterion for quantum experiments
Randomization criterion and MHT

Ensemble: \(E = \{\lambda_i, \pi_i\}_{i=1}^k\),

\(\pi_1, \ldots, \pi_k\) probability distributions, \(\lambda_1, \ldots, \lambda_k\) - prior probabilities

Optimal success probability:

\[
P_{\text{succ}}(E) = \sup \sum_{i} \lambda_i \sum_{x} \mu(x|i)\pi_i(x)
\]

Randomization criterion: \(\delta(S, T) \leq \epsilon \iff \)

for any ensemble \(E = \{\lambda_i, \pi_i\}, \pi_i \in \mathcal{P}(\Theta)\):

\[
P_{\text{succ}}(\{\lambda_i, \sum_{\theta} \pi_i(\theta)p_{\theta}\}) \leq P_{\text{succ}}(\{\lambda_i, \sum_{\theta} \pi_i(\theta)q_{\theta}\}) + \frac{\epsilon}{2}P_{\text{succ}}(E)
\]
Randomization criterion and MHT

For classical channels:

\[ \delta(\Phi, \Psi) \leq \epsilon \iff \]

for any ensemble \( E = \{ \lambda_i, \pi_j \} \) on the input space,

\[ P_{\text{succ}}(\{ \lambda_i, \Phi(\pi_i) \}) \leq P_{\text{succ}}(\{ \lambda_i, \psi(\pi_i) \}) + \frac{\epsilon}{2} P_{\text{succ}}(E) \]
Deficiency of quantum channels

- **Quantum channel**: completely positive trace preserving map
  \[ \Phi : B(\mathcal{H}) \rightarrow B(\mathcal{K}), \]
  \( \mathcal{H}, \mathcal{K} \) finite dimensional Hilbert spaces.
- **\( C(\mathcal{H}, \mathcal{K}) \)** - the set of all channels
- **Deficiency**: \( \Phi \in C(\mathcal{H}, \mathcal{K}), \Psi \in C(\mathcal{H}, \mathcal{K}'), \)
  \[ \delta(\Phi, \Psi) = \min_{\alpha \in C(\mathcal{K}', \mathcal{K})} \| \Phi - \alpha \circ \Psi \|_\diamond \]

- **Quantum ensemble**: \( \mathcal{E} = \{ \lambda_i, \tau_i \} \), \( \tau_i \) are density operators.
  Optimal success probability:
  \[ P_{\text{succ}}(\mathcal{E}) = \max_M \sum_i \lambda_i \text{Tr} [M_i \tau_i] \]
Randomization criterion for quantum channels

Theorem

Let $\Phi \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, $\Psi \in \mathcal{C}(\mathcal{H}, \mathcal{K}')$.

$\delta(\Phi, \Psi) \leq \epsilon \iff$

for any ensemble $\mathcal{E} = \{\lambda_i, \sigma_i\}_{i=1}^{\dim(\mathcal{K})^2}$ on $\mathcal{H} \otimes \mathcal{K}$,

$$P_{\text{succ}}((\Phi \otimes id_\mathcal{K})(\mathcal{E})) \leq P_{\text{succ}}((\Psi \otimes id_\mathcal{K})(\mathcal{E})) + \frac{\epsilon}{2} P_{\text{succ}}(\mathcal{E})$$

The case $\epsilon = 0$: (Chefles, 2009)
The main tools

- properties of the diamond norm $\| \cdot \|^{\diamond}$ and its dual: $\| \cdot \|^{\diamond}$
  - relation of the norms to the order structure of completely positive maps
  - relation of $\| \cdot \|^{\diamond}$ to optimal success probabilities

- the minimax theorem (as in the classical case)
The space of Hermitian maps

\[ \mathcal{L}(\mathcal{H}, \mathcal{K}) \] - space of Hermitian maps \( \phi : B(\mathcal{H}) \to B(\mathcal{K}) , \)

\[ \phi(X^*) = \phi(X)^* \]

- A positive cone \( \mathcal{L}(\mathcal{H}, \mathcal{K})^+ \) - completely positive maps
- A compact convex subset \( \mathcal{C}(\mathcal{H}, \mathcal{K}) \subset \mathcal{L}(\mathcal{H}, \mathcal{K})^+ \) of channels
- The dual space \( \equiv \mathcal{L}(\mathcal{K}, \mathcal{H}) \), with duality

\[ \langle \phi, \psi \rangle = \text{Tr} \ C(\phi) C(\psi^*) \]

\( C(\phi) \) is the Choi matrix of \( \phi \):

\[ C(\phi) = \sum_{i,j} \phi(|i\rangle\langle j|) \otimes |i\rangle\langle j| . \]

- the dual positive cone: \( \mathcal{L}(\mathcal{K}, \mathcal{H})^+ \)
Diamond norm and its dual

- diamond norm in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ (Kitaev, 1997)
  \[ \|\phi\|_\diamond = \sup_{\rho \in \mathcal{S}(\mathcal{H} \otimes \mathcal{H})} \| (\phi \otimes \text{id}_\mathcal{H})(\rho) \|_1 \]

- the dual norm $\| \cdot \|_\diamond$ in $\mathcal{L}(\mathcal{K}, \mathcal{H})$

- alternative form, using the order structure:
  \[ \|\phi\|_\diamond = \inf_{\alpha \in \mathcal{C}(\mathcal{H}, \mathcal{K})} \inf \{ \lambda > 0, -\lambda \alpha \leq \phi \leq \lambda \alpha \} \]
  \[ \|\psi\|_\diamond = \inf_{\sigma \in \mathcal{S}(\mathcal{H})} \inf \{ \lambda > 0, -\lambda \phi_\sigma \leq \psi \leq \lambda \phi_\sigma \} \]

erasure channels $\phi_\sigma : \mathcal{S}(\mathcal{K}) \ni \rho \mapsto \sigma, \sigma \in \mathcal{S}(\mathcal{H})$. 
Properties of the norms

- If $\phi \in \mathcal{L}(\mathcal{H}, \mathcal{K})^+$, then
  \[
  \|\phi\|_\diamond = \sup_{\sigma \in \mathcal{S}(\mathcal{H})} \text{Tr} \left[ \phi(\sigma) \right], \quad \|\phi\|^\diamond = \sup_{\alpha \in \mathcal{C}(\mathcal{K}, \mathcal{H})} \langle \alpha, \phi \rangle.
  \]

- If $\phi, \psi \in \mathcal{C}(\mathcal{H}, \mathcal{K})$, then
  \[
  \|\phi - \psi\|_\diamond = 2 \sup_{\gamma \geq 0, \|\gamma\|^\diamond \leq 1} \langle \gamma, \phi - \psi \rangle
  \]

- If $\chi$ and $\xi$ are channels, then the maps
  \[
  \phi \mapsto \chi \circ \phi, \quad \phi \mapsto \phi \circ \xi
  \]
  are contractions with respect to both norms.
The dual norm and guessing probabilities

Let $\mathcal{E} = \{\lambda_i, \rho_i\}_{i=1}^k$ be an ensemble. It is an easy observation that

$$P_{\text{succ}}(\mathcal{E}) = \|\gamma_\mathcal{E}\|^{\diamond},$$

where $\gamma_\mathcal{E} : X \mapsto \sum_i X_{ii} \lambda_i \rho_i \in \mathcal{L}(\mathbb{C}^k, \mathcal{H})^\perp$.

**Theorem**

Let $\gamma \in \mathcal{L}(\mathcal{K}, \mathcal{H})^\perp$. Then there is an ensemble $\mathcal{E}_\gamma$ on $\mathcal{H} \otimes \mathcal{K}$ such that

$$\|\gamma\|^{\diamond} = \dim(\mathcal{K}) \text{Tr} [\gamma(I)] P_{\text{succ}}(\mathcal{E}_\gamma)$$

Moreover, if $\Phi$ is a channel, we have

$$\mathcal{E}_{\Phi \circ \gamma} = (\Phi \otimes \text{id})(\mathcal{E}_\gamma)$$
Proof of the randomization criterion (sketch)

\[ \implies: \text{Assume } \delta(\Phi, \Psi) \leq \epsilon. \]

- Note that
  \[ \delta(\Phi \otimes \text{id}_{K_0}, \Psi \otimes \text{id}_{K_0}) \leq \delta(\Psi, \Phi), \]
  for any \( K_0 \).

- Use the properties of the norms to show that
  \[ \delta(\Phi \otimes \text{id}_K, \Psi \otimes \text{id}_K) \leq \epsilon \]
  implies
  \[ \| (\Phi \otimes \text{id}_K) \circ \gamma \|^{\circ} \leq \| (\Psi \otimes \text{id}_K) \circ \gamma \|^{\circ} + \frac{\epsilon}{2} \| \gamma \|^{\circ} \]
  for any \( \gamma \in \mathcal{L}(K, H \otimes K)^+ \).

- For any ensemble \( \mathcal{E} \), put \( \gamma = \gamma_\mathcal{E} \) and use \( \| \gamma_\mathcal{E} \|^{\circ} = P_{suc\mathcal{C}}(\mathcal{E}) \).
By definition and properties of $\| \cdot \|\diamond$:

$$\delta(\Phi, \Psi) = 2 \min_{\alpha \in \mathcal{C}(\mathcal{K}', \mathcal{K})} \left\{ \max_{\gamma \in \mathcal{L}(\mathcal{K}, \mathcal{H})^+, \|\gamma\| \leq 1} \langle \gamma, \Phi - \alpha \circ \Psi \rangle \right\}$$

use minimax theorem and get

$$\delta(\Phi, \Psi) = 2 \max \min_{\gamma} \langle \gamma, \Phi - \alpha \circ \Psi \rangle \leq 2 \max_{\gamma} \{ \| \Phi \circ \gamma \| \diamond - \| \Psi \circ \gamma \| \diamond \}.$$ 

use the relation of $\| \cdot \| \diamond$ and $P_{\text{succ}}$ to see that the last expression is less than $\epsilon$. 

Randomization criterion for quantum experiments

By one more use of the minimax theorem, we obtain:

Theorem

Let $S = (\mathcal{H}, \{\rho_\theta, \ \theta \in \Theta\})$ and $T = (\mathcal{K}, \{\sigma_\theta, \ \theta \in \Theta\})$ be quantum experiments, $\epsilon \geq 0$. The following are equivalent:

(i) There is some channel $\alpha$ such that

$$\sup_{\theta \in \Theta} \|\sigma_\theta - \alpha(\rho_\theta)\|_1 \leq \epsilon$$

(ii) For any $\{\theta_1, \ldots, \theta_n\} \subseteq \Theta$ and any ensemble $\mathcal{E} = \{\lambda_i, \tau_i\}_{i=1}^k$ on $\mathbb{C}^n \otimes \mathcal{K}$ with block-diagonal states $\tau_i = \sum |e_j\rangle \langle e_j| \otimes \tau^j_i$,

$$P_{\text{succ}}(\{\lambda_i, \sum_j \sigma_{\theta_j} \otimes \tau^j_i\}) \leq P_{\text{succ}}(\{\lambda_i, \sum_j \rho_{\theta_j} \otimes \tau^j_i\}) + \frac{\epsilon}{2} P_{\text{succ}}(\mathcal{E})$$
Further research

The proof relies on the order structure of completely positive maps and the trace-preserving condition
- this suggests that more general situations can be treated similarly:
  
  - more specific quantum protocols (quantum networks)
  - more general types of deficiency: pre-processings, pre- and post-processings, correlations:
    - In the classical case: (Shannon, 1958)
      \[
      \Phi = \sum_{i} \lambda_i \alpha_i \circ \Psi \circ \beta_i
      \]
    
    Approximate version: (Raginsky, 2011)
  
  - different distance measures