On characterizations of quantum incompatibility and steering

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POVMs and compatibility

We work with operators on a Hilbert space with $\dim(\mathcal{H}) = d$.

- A POVM with k-outcomes:

$$M_1, \ldots, M_k \geq 0, \quad \sum_i M_i = I$$

- A collection of POVMs:

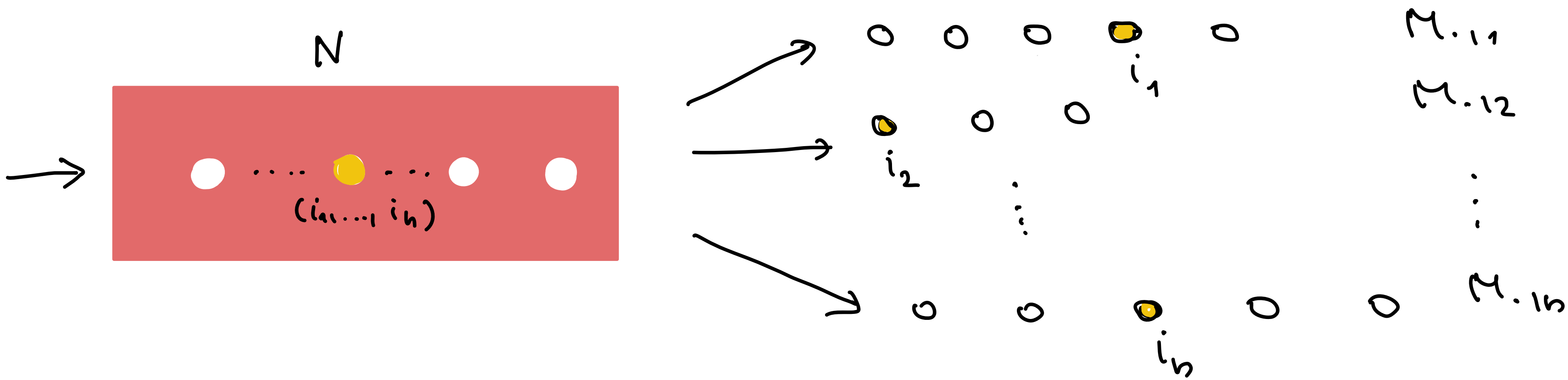
$$M_{\cdot|x} = \{M_{1|x}, \ldots, M_{k_x|x}\}, \quad x \in [n] = \{1, \ldots, n\}.$$

- The POVMs are compatible if all can be simulated by post-processing of a single joint POVM

$$N_1, \ldots, N_m.$$

POVMs and compatibility

- marginals: outcomes of N in $[k_1] \times \cdots \times [k_n]$



- more general post-processings: $\{p(i|j,x)\}$, $p(i|j,x) \geq 0$, $\sum_i p(i|j,x) = 1$:

$$M_{i|x} = \sum_{j=1}^{m} p(i|j,x) N_j, \qquad i = 1, \ldots, k_x, \; x = 1, \ldots, n.$$

Characterizations of compatibility

- SDP

  D. Cavalcanti, P. Skrypzcyk, PRA, 2016

- success probabilities in guessing games

  C. Carmeli, T. Heinosaari, A. Toigo, PRL, 2019

- tensor product of convex cones

  AJ, PRA, 2018

- free spectrahedra, matrix convex sets

  A. Bluhm, I. Nechita, JMP, 2018; Quantum, 2022

- tensor crossnorms

  A. Bluhm, AJ, I. Nechita, CMP, 2022; A. Bluhm, I. Nechita, JMP, 2022

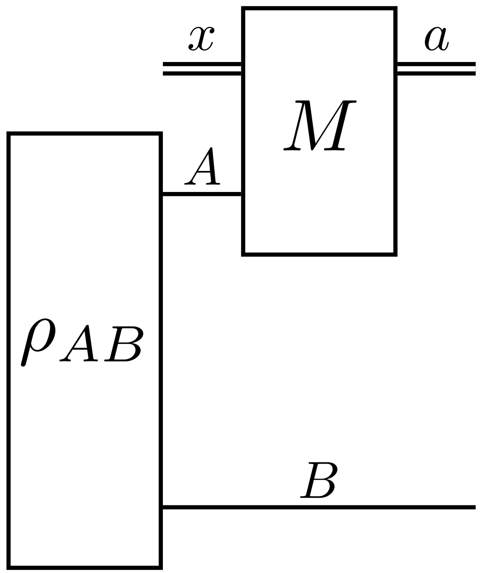
- Fisher information map

  H. Zhu, Sci. Rep. 2015; H. Zhu, M. Hayashi, L. Chen, PRL, 2016;

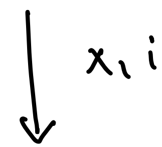
  T. Heinosaari, M.A. Jivulescu, I. Nechita, arXiv:2202.00725

- ...

Assemblages and quantum steering



Alice chooses a POVM from a given set $\{M_{i|x}\}_{x\in[n]}$



Bob obtains an assemblage of conditional states:

$$\{\rho_{i|x} := \mathrm{Tr}_A[(M_{i|x} \otimes I_B)(\rho_{AB})]\}$$

with the same average state:

$$\sum_i \rho_{i|x} = \rho_B, \qquad \forall x.$$

Assemblages and quantum steering

- In general, an assemblage is a set of ensembles with the same average state

$$\{\rho_{i|x}\}, \quad \rho_{i|x} \geq 0, \ \sum_i \rho_{i|x} = \rho \in \mathcal{S}, \quad x \in [n].$$

- The assemblage admits a LHS model if

$$\rho_{i|x} = \sum_\Lambda q_\lambda q(i|\lambda, x)\rho_\lambda$$

probabilities

states

conditional probabilities

- In the steering scenario: If no LHS model exists, then ρ_{AB} must be entangled and $\{M_{\cdot|x}\}$ must be incompatible.

H. M. Wiseman, S. J. Jones, and A. C. Doherty, PRL, 2007

Assemblages and POVMs

There is another connection between POVMs and assemblages:

$$\{M_{\cdot|x}\} \xrightarrow{\rho^{1/2} \cdot \rho^{1/2}} \{\rho_{\cdot|x}\}$$

$$\{M_{\cdot|x}\} \xleftarrow[\rho^{-1/2} \cdot \rho^{-1/2}]{} \{\rho_{\cdot|x}\}$$

- $\{M_{\cdot|x}\}$ is a set of measurements $\iff$ $\{\rho_{\cdot|x}\}$ is an assemblage with average state ρ,
- the measurements are compatible $\iff$ the assemblage admits a LHS.
- many results can be transferred from (in)compatibility to steering and back.

H. Y. Ku et al, Nature communications, 2022

Incompatibility and steering in GPTs

- General probabilistic theories: describing physical systems with probabilistic features
- The quantum state space $\mathcal{S}$ is replaced by a compact convex set K
    - states $\equiv$ elements of K
    - effects $\equiv$ affine maps $f : K \to [0, 1]$
    - measurements $\equiv$ collections $f_1, \ldots, f_k$ of effects, $\sum_i f_i = 1_K$.
- Analogous notions of compatibility and steering exist in GPTs.

A. Bluhm, AJ, I. Nechita, CMP, 2022; AJ, arXiv:2202.09109

The post-processing preorder on POVMs

Let $M = \{M_1, \ldots, M_k\}$, $N = \{N_1, \ldots, N_l\}$ be POVMs.

We write $M \leq N$ if M is a post-processing of N:

$$N_i = \sum_{j=1}^{l} p(i|j) N_j, \quad i = 1, \ldots, k,$$

for some conditional probabilities $p(i|j)$.

POVMs $\{M_{\cdot|x}\}_{x \in [N]}$ are compatible if and only if they have a common upper bound w. r. to $\leq$:

$$\exists \text{ a POVM } N, \qquad M_{\cdot|x} \leq N, \qquad \forall x.$$

The post-processing preorder on POVMs

- $\leq$ is a preorder on POVMs (reflexive, transitive)
- any preorder defines an equivalence relation:

$$M \sim N \quad \text{if} \quad M \leq N \quad \text{and} \quad N \leq M.$$

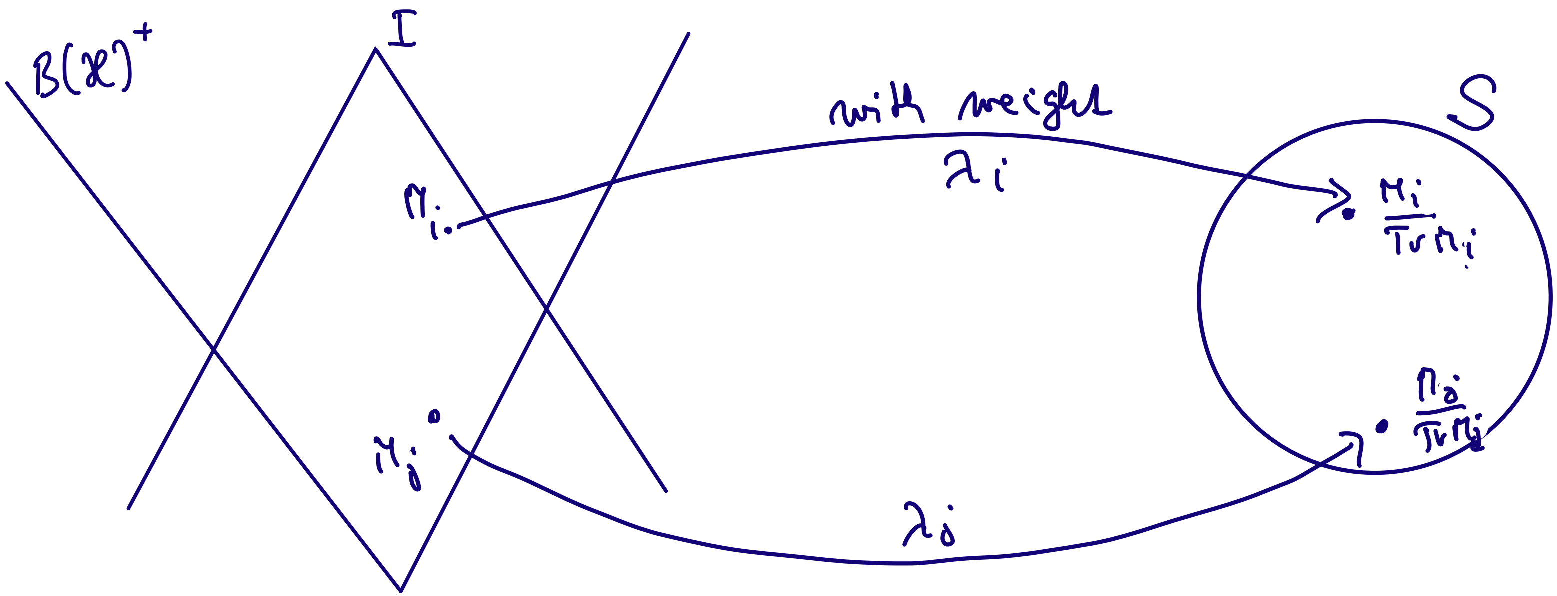
- $\leq$ becomes a partial order on the equivalence classes

The induced partial order on $\text{POVMs}|_{\sim}$ fully characterizes compatibility of measurements

A map on POVMs

Let $\mathcal{S} = \mathcal{S}(\mathcal{H})$ be the set of states. We define a map

$$\eta : \text{POVMs} \to \mathcal{P}(\mathcal{S}) \equiv \text{ probability measures over } \mathcal{S}$$



$$M_1, \ldots, M_k \mapsto \sum_{i=1}^{k} \lambda_i \delta_{\rho_i}, \quad \lambda_i := \frac{\mathrm{Tr}\,[M_i]}{d}, \; \rho_i := \frac{M_i}{\mathrm{Tr}\,[M_i]}.$$

- a simple probability measure (concentrated in finitely many points)

Properties of η

- η has range in $\mathcal{P}_\tau(\mathcal{S}) \equiv \nu \in \mathcal{P}(\mathcal{S})$ with barycenter $\bar{\mu} = \tau := \frac{1}{d} I$
- affine (with respect to a special convex structure):

  for POVMs $M_1, \ldots, M_k$ and $N_1, \ldots, N_l$,

  $$\eta(\lambda M_1, \ldots, \lambda M_k, (1-\lambda) N_1, \ldots, (1-\lambda) N_l) \\ = \lambda \eta(M_1, \ldots, M_k) + (1-\lambda) \eta(N_1, \ldots, N_l).$$

- surjective onto $\mathcal{P}_\tau(\mathcal{S})$ (if extended all POVMs on Borel subsets of $\mathcal{S}$):

  for any $\mu \in \mathcal{P}_\tau(\mathcal{S})$, there is a POVM

  $$\eta(M_\mu) = \mu, \qquad M_\mu(B) = d \int_B \rho d\mu, \;\; B \subset \mathcal{S}.$$

Post-processing preorder and Choquet order

Let M and N be POVMs. Then

- $M \sim N$ if and only if $\eta(M) = \eta(N)$;
- $M \leq N$ if and only if $\eta(M) \prec \eta(N)$, where $\prec$ is the Choquet order in $\mathcal{P}(\mathcal{S})$.

The Choquet order in $\mathcal{P}(\mathcal{S})$

Definition

Let $\nu, \mu \in \mathcal{P}(\mathcal{S})$. The Choquet order is defined as

$$\nu \prec \mu \quad \text{if} \quad \int f d\nu \leq \int f d\mu$$

for all continuous convex functions $f : \mathcal{S} \to \mathbb{R}$

A dual characterization

$$\nu \prec \mu \iff \text{if } \nu = \sum_i \lambda_i \nu_i, \text{ then } \mu = \sum_i \lambda_i \mu_i, \text{ with } \bar{\mu}_i = \bar{\nu}_i.$$

($\bar{\mu} = \int \rho d\mu(\rho)$ is the barycenter of μ.)

The Choquet order for simple measures

Let $\nu = \sum_{i=1}^{k} \lambda_i \delta_{\rho_i}$, $\mu \in \mathcal{P}(\mathcal{S})$.

- We may restrict to functions of the form

$$f_A(\rho) := \max_{1 \leq i \leq k} \mathrm{Tr}\,[A_i \rho], \quad A = (A_1, \ldots, A_k),\ A_1 = A_i^*.$$

- The condition becomes

$$\sum_{i=1}^{k} \lambda_i \,\mathrm{Tr}\,[A_i \rho] \leq \int f_A d\mu, \quad \forall A = (A_1, \ldots, A_k),\ A_i = A_i^*.$$

- If we assume $\bar{\nu} = \bar{\mu}$, we may restrict to $A = (A_1, \ldots, A_k)$ with $\sum_i A_i = 0$.

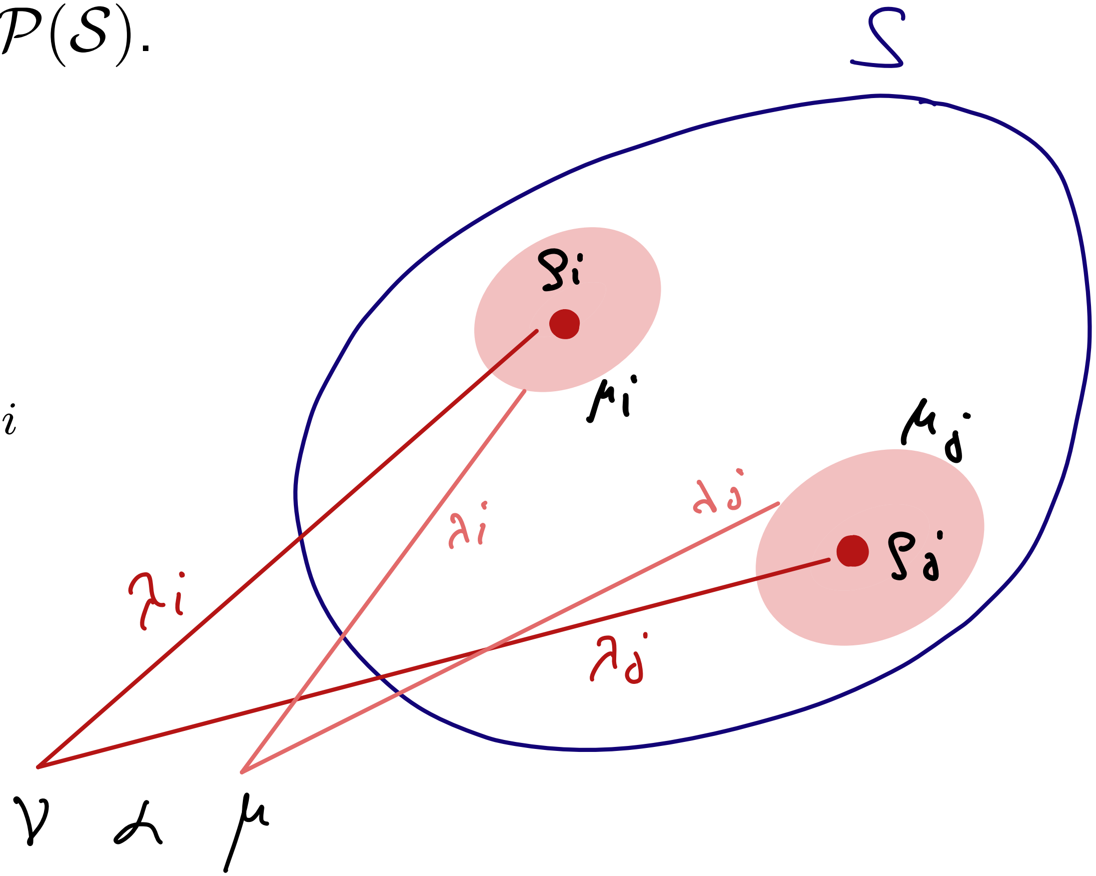
The Choquet order for simple measures

Let $\nu = \sum_{i=1}^{k} \lambda_i \delta_{\rho_i}$, $\mu \in \mathcal{P}(\mathcal{S})$.

Dual characterization:

$$\nu \prec \mu \iff \mu = \sum_i \lambda_i \mu_i$$

with $\bar{\mu}_i = \rho_i$.



$\implies$ μ is concentrated ‘closer’ to the set $P(\mathcal{H})$ of pure states

Minimal and maximal elements

- minimal elements with respect to $\prec$

$$\delta_\sigma, \qquad \sigma \in \mathcal{S}$$

- maximal elements with respect to $\prec$

  boundary measures, concentrated on the set $\mathcal{P}(\mathcal{H})$ of pure states

- every $\nu \in \mathcal{P}(\mathcal{S})$ is upper bounded by a boundary measure.

Post-processing preorder on POVMs

Equivalent conditions for $\{M_1, \ldots, M_k\} \leq \{N_1, \ldots, N_l\}$:

For all $f : \mathcal{S} \to \mathbb{R}$, continuous convex:

$$\sum_i \mathrm{Tr}\,[M_i] f\left(\frac{M_i}{\mathrm{Tr}\,[M_i]}\right) \leq \sum_j \mathrm{Tr}\,[N_j] f\left(\frac{N_j}{\mathrm{Tr}\,[N_j]}\right),$$

For all $A_1, \ldots, A_k \in B(\mathcal{H})^{sa}$, $\sum_i A_i = 0$:

$$\sum_{i=1}^{k} \mathrm{Tr}\,[A_i M_i] \leq \sum_j \max_i \mathrm{Tr}\,[A_i N_j]$$

Compatibility of POVMs

Let $\mathcal{M} = \{M_{\cdot|x}\}_{x \in X}$ be a set of POVMs.

- $\mathcal{M}$ is compatible if and only if there is some $\mu \in \mathcal{P}_\tau(\mathcal{S})$, such that
$$\eta(M_{\cdot|x}) \prec \mu$$
- M_μ is then a joint POVM
- μ can be assumed concentrated on pure states
- $\{M_{\cdot|x}\}$ are compatible if and only if any finite subset is compatible
- restrictions on μ by unitary invariance.

Incompatibility witnesses

Let $\mathcal{M} = \{M_{\cdot|x}\}_{x\in X}$ be a set of POVMs.

$\{M_{\cdot|x}\}$ are compatible if and only if

$$\sum_i \frac{\mathrm{Tr}\left[M_{i|x}\right]}{d} f\left(\frac{M_{i|x}}{\mathrm{Tr}\left[M_{i|x}\right]}\right) \leq \int f d\mu,$$

for some some $\mu \in \mathcal{P}_\tau(\mathcal{S})$ and all $x \in X$, $f : \mathcal{S} \to \mathbb{R}$ continuous, convex.

Incompatibility witnesses: functions f such that the inequalities are violated.

Linear incompatibility witnesses

$M = \{M_{1|x}, \ldots, M_{k_x|x}\}_{x\in[n]}$ a set of POVMs.

Let $A = \{A_{1|x}, \ldots, A_{k_x|x}\}_{x\in[n]}$, $\sum_i A_{i|x} = 0$, same shape as M.

If $\{M_{\cdot|x}\}$ are compatible, $\eta(M_{\cdot|x}) \prec \mu$, then

$$\frac{1}{d}\sum_{x,i} \mathrm{Tr}\left[A_{i|x}M_{i|x}\right] \leq \sum_x \int f_{A_{\cdot|x}} d\mu$$

$$\leq \sup_{\nu,\bar{\nu}=\tau} \int \sum_x f_{A_{\cdot|x}} d\nu = \widehat{\sum_x f_{A_{\cdot|x}}}(\tau),$$

$\hat{f} : \mathcal{S} \to \mathbb{R}$ is the upper envelope of $f : \mathcal{S} \to \mathbb{R}$:

$$\hat{f}(\rho) := \inf\{\mathrm{Tr}\,[B\rho],\ \mathrm{Tr}\,[B\,\cdot\,] \geq f\} = \sup_{\nu,\bar{\nu}=\rho} \int f d\nu, \qquad \rho \in \mathcal{S}.$$

Linear incompatibility witnesses

- Linear incompatibility witnesses of shape $\mathbf{k} = (k_1, \ldots, k_n)$:

$$\mathcal{W}_{\mathbf{k}} := \{\{A_{1|x}, \ldots, A_{k_x|x}\}_{x\in[n]}, \sum_i A_{i|x} = 0, \widehat{\sum_x f_{A_{\cdot|x}}}(\tau) \leq 1\}$$

- If $\{M_{1|x}, \ldots, M_{k_x|x}\}_{x\in[n]}$ are compatible, then

$$\sum_{x,i} \mathrm{Tr}\left[A_{i|x} M_{i|x}\right] \leq d, \qquad \forall A \in \mathcal{W}_{\mathbf{k}}$$

- $A \in \mathcal{W}_{\mathbf{k}}$ is sharp if the inequality is violated by some set of POVMs of shape $\mathbf{k}$ $\iff$

$$\sum_x \hat{f}_{A_{\cdot|x}}(\tau) > 1.$$

Linear incompatibility witnesses

- The set $\mathcal{W}_{\mathbf{k}}$ of linear witnesses is complete:

  $\{M_{1|x}, \dots M_{k_x|x}\}_{x\in[n]}$ compatible if and only if

  $$\widehat{\{M_{i|x}\}} := \sup_{\{A_{i|x}\}\in\mathcal{W}_{\mathbf{k}}} \sum_{x,i} \frac{1}{d} \operatorname{Tr}\left[A_{i|x} M_{i|x}\right] \leq 1,$$

- all the quantities for $\{M_{i|x}\}$ and $\{A_{i|x}\}$ are computable by SDP
- $\widehat{\{M_{i|x}\}}$ has an interpretation as a compatibility degree.

Another choice of f

Quadratic incompatibility witnesses:

For $C = C^*$, let $f_C(\rho) = (\mathrm{Tr}\,[C\rho])^2 = \langle\!\langle C|\rho\rangle\!\rangle\langle\!\langle\rho|C\rangle\!\rangle$.
Compatibility implies that

$$\sum_i \mathrm{Tr}\,[M_{i|x}] f_C\left(\frac{M_{i|x}}{\mathrm{Tr}\,[M_{i|x}]}\right) \leq d \int f_C d\mu, \quad \forall x, C$$

This can be rewritten as

$$\mathcal{G}(M_{\cdot|x}) := \sum_i \frac{|M_{i|x}\rangle\!\rangle\langle\!\langle M_{i|x}|}{\mathrm{Tr}\,[M_{i|x}]} \leq d \int |\rho\rangle\!\rangle\langle\!\langle\rho| d\mu =: H_\mu, \quad \forall x$$

H_μ is a superoperator, $\mathrm{Tr}\,[H_\mu] \leq d \implies$

$$g(\{M_{i|x}\}) := \inf\{\mathrm{Tr}\,[H],\ \mathcal{G}(M_{\cdot|x}) \leq H,\ \forall x\} \leq d.$$

H. Zhu, M. Hayashi, L. Chen, PRL, 2016

A compatibility degree (robustness)

For $\lambda \in [0,1]$ and a set $\{M_{i|x}\}$ of POVMs, put

$$M_{i|x}^{\lambda} = \lambda M_{i|x} + (1-\lambda)\frac{1}{k_x} I.$$

The compatibility degree:

$$s(\{M_{i|x}\}) := \sup\{\lambda \in [0,1],\ \{M_{\cdot|x}^{\lambda}\} \text{ are compatible}\}.$$

Since $\sum_{i,x} \mathrm{Tr}\,[M_{i|x}^{\lambda} A_{i|x}] = \lambda \sum_{i,x} \mathrm{Tr}\,[M_{i|x} A_{i|x}]$, we see that

$$s(\{M_{i|x}\}) = \min\{1, \widehat{\{M_{i|x}\}}^{-1}\}.$$

Compatibility degree for shape $\mathbf{k}$

Compatibility degree for all POVMs of shape $\mathbf{k}$:

$$s_{\mathbf{k}} := \sup\{\lambda \in [0,1],\ \{M^{\lambda}_{i|x}\} \text{ is compatible for all } \{M_{i|x}\} \text{ of shape } \mathbf{k}\}$$

$$= \inf_{\{M_{i|x}\}} s(\{M_{i|x}\}) = \min_{A \in \mathcal{W}_{\mathbf{k}}} \frac{\widehat{\sum_x f_{A_{\cdot|x}}}(\tau)}{\sum_x \hat{f}_{A \cdot|x}(\tau)}$$

Compatibility degree for k-outcome POVMs

Universal compatibility degree for k-outcome POVMs:

$$s_k := \sup\{\lambda \in [0,1],\ \{M^{\lambda}_{i|x}\} \text{ is compatible for any } \{M_{1|x}, \ldots, M_{k|x}\}_{x\in[n]},\ n \in \mathbb{N}\}.$$

Compatibility degree for k-outcome POVMs

For $0 < \lambda \leq s_k$:

- any finite subset of $\{\{M_i^\lambda\}, \{M_i\}$ is a k-outcome POVM$\}$ is compatible
- there is a boundary measure $\mu \in \mathcal{P}_\tau(\mathcal{S})$ such that

$$\eta(\{M_i^\lambda\}) \prec \mu, \qquad \forall \{M_i\}$$

- $\{\{M_i^\lambda\}\}$ is invariant under unitary conjugations $\implies$ we may assume that μ is the Haar measure over $\mathcal{P}(\mathcal{H})$.
- for all $A_1, \ldots, A_k$, $\sum_i A_i = 0$ and all POVMs $M_1, \ldots, M_k$:

$$\sum_i d^{-1} \operatorname{Tr}[A_i M_i^\lambda] = \lambda d^{-1} \sum_i \operatorname{Tr}[A_i M_i] \leq \int f_A d\mu$$

Compatibility degree for k-outcome POVMs

For $\lambda \leq s_k$:

- Taking supremum over POVMs $M_1, \dots, M_k$:

$$\lambda \hat{f}_A(\tau) \leq \int f_A d\mu, \quad f_A(\rho) = \max_i \mathrm{Tr}\,[A_i \rho].$$

We obtain

$$s_k = \inf_{\{A_i\}} \int_{P(\mathcal{H})} \max_i \langle\psi|A_i|\psi\rangle d\mu(|\psi\rangle\langle\psi|),$$

infimum over $A_1, \dots, A_k \in B(\mathcal{H})$, $\sum_i A_i = 0$, $\hat{f}_A(\tau) = 1$.

Compatibility of dichotomic POVMs

- Dichotomic POVMs:

$$\{M_x, I - M_x\}_{x\in[n]}, \qquad 0 \le M_x \le I.$$

- Linear incompatibility witnesses for dichotomic POVMs:

$$\{A_x, -A_x\}_{x\in[n]}, \quad f_{A_{\cdot|x}}(\rho) = |\operatorname{Tr}[A_x\rho]|, \ \rho \in \mathcal{S}.$$

- The inequality becomes

$$\sum_x \frac{1}{d}\operatorname{Tr}[A_x(2M_x - I)] \le 1, \quad \left(\sum_x \widehat{|\operatorname{Tr}[A_x\cdot]|}\right)(\tau) \le 1.$$

Compatibility of dichotomic POVMs

We identify

- dichotomic POVMs $\equiv \{F_x\}_{x\in[n]}$, $\|F_x\| \leq 1$
- dichotomic witnesses $\equiv \{A_x\}_{x\in[n]}$, $\left(\widehat{\sum_x |\operatorname{Tr}[A_x \cdot]|}\right)(\tau) \leq 1$
- the tuples $\{A_x\}_{x\in[n]}$, $\{F_x\}_{x\in[n]}$ with elements in $\mathbb{R}^n \otimes B(\mathcal{H})^{sa}$.

Dichotomic witnesses and compatibility of dichotomic POVMs are characterized by tensor crossnorms in $\mathbb{R}^n \otimes B(\mathcal{H})^{sa}$.

Tensor crossnorms

Let X, Y be Banach spaces. A norm $\|\cdot\|$ in $X \otimes Y$ is a tensor crossnorm if and only if for all $x \in X$, $y \in Y$, $\varphi \in X^*$, $\psi \in Y^*$,

$$\|x \otimes y\| \leq \|x\|_X \|y\|_Y, \qquad \|\varphi \otimes \psi\| \leq \|\varphi\|_{X^*} \|\psi\|_{Y^*}.$$

Minimal and maximal crossnorms: for $z \in X \otimes Y$,

- injective crossnorm

$$\|z\|_{\epsilon(X,Y)} = \sup\left\{ \langle \varphi \otimes \psi, z \rangle,\ \varphi \in X^*, \psi \in Y^*, \|\varphi\|_{X^*}^*, \|\psi\|_{Y^*}^* \leq 1 \right\}$$

- projective crossnorm

$$\|z\|_{\pi(X,Y)} = \inf\left\{ \sum_i \|x_i\|_X \|y_i\|_Y,\ z = \sum_i x_i \otimes y_i \right\}$$

Dichotomic witnesses and crossnorms

- Let $A = A^*$, $f_A(\rho) = |\operatorname{Tr}[A\rho]|$, then

$$\hat{f}_A(\tau) = d^{-1}\|A\|_1 \quad \text{(the norm in the Schatten class } S_1^d\text{)}$$

- For $\{A_x\}_{x\in[n]} \in \mathbb{R}^n \otimes B(\mathcal{H})^{sa}$,

$$\sum_x \hat{f}_{A_x}(\tau) = d^{-1}\sum_x \|A_x\|_1 = d^{-1}\|\{A_x\}\|_{\pi(\ell_1^n, S_1^d)}.$$

- Let us define

$$\|\{A_x\}\|_w := d\widehat{\sum_x f_{A_x}}(\tau),$$

this is a tensor crossnorm in $\ell_1^n \otimes S_1^d$.

Dichotomic witnesses and crossnorms

Let $\{A_x\} \in \mathbb{R}^n \otimes B(\mathcal{H})^{sa}$.

- $\{A_x\}$ is an incompatibility witness if and only if

$$\|\{A_x\}\|_w \leq d.$$

- $\{A_x\}$ is a strict incompatibility witness if and only if

$$\|\{A_x\}\|_{\pi(\ell_1^n, S_1^d)} > d.$$

Compatible dichotomic POVMs and crossnorms

Put $\| \cdot \|_c := \| \cdot \|_w^*$ - the dual norm.

- $\| \cdot \|_c$ is a tensor crossnorm in $\ell_\infty^n \otimes S_\infty^d$
- For $\{F_x\} \in \mathbb{R}^n \otimes B(\mathcal{H})^{sa}$, $F_x = 2M_x - I$ for some effects M_x if and only if

$$\|\{F_x\}\|_{\epsilon(\ell_\infty^n, S_\infty^d)} = \max_x \|F_x\| \leq 1.$$

- $F_x = 2M_x - I$ for some compatible effects M_x if and only if

$$\|\{F_x\}\|_c \leq 1.$$

Compatibility norms and matrix convex sets

The unit ball of $\|\cdot\|_{\pi(\ell_\infty^n, S_\infty^d)}$:

The matrix cube in dimension d:

$$\mathcal{C}_d = \{(F_1, \ldots, F_n),\ F_x = F_x^*,\ \|F_x\| \leq 1\}$$

$\bigcup_d \mathcal{C}_d$ - maximal matrix convex set over n-cube.

The unit ball of $\|\cdot\|_c$:

$$\mathcal{C}_d^c\{(F_1, \ldots, F_n),\ F_x = F_x^*,\ \|F_x\| \leq 1, \\ \exists V,\ V^*V = I, \{V^*F_xV\} \text{ commute}\}$$

$\bigcup_d \mathcal{C}_d^c$ - minimal matrix convex set over n-cube.

The compatibility degrees

Compatibility degree for n dichotomic measurements:

$$s_{2,n} = \min_{Z \in \mathbb{R}^n \otimes B(\mathcal{H})^{sa}} \frac{\|Z\|_{\epsilon(\ell_1^n, S_1^d)}}{\|Z\|_c} = \min_{Z \in \mathbb{R}^n \otimes B(\mathcal{H})^{sa}} \frac{\|Z\|_w}{\|Z\|_{\pi(\ell_\infty^n, S_\infty^d)}}.$$

Also obtained as inclusion constants for matrix convex sets

A. Bluhm, I. Nechita, JMP, 2018

The compatibility degrees

Universal compatibility degree for dichotomic measurements:

$$s_2 = \min_{\|A\|_1 = d} \int_{\mathcal{P}(\mathcal{H})} |\langle\psi|A|\psi\rangle d\mu(|\psi\rangle\langle\psi|)$$
$$= 4^{-n}\binom{2n}{n}, \quad n = \lfloor d/2 \rfloor.$$

First obtained using inclusion constant for minimal and maximal matrix convex sets. A. Bluhm, I. Nechita, Quantum, 2022

Compatibility of unbiased qubit effects

Unbiased qubit effects:

$$M_x = \frac{1}{2}(I + \vec{a}_x.\vec{\sigma}), \qquad \vec{a}_x \in \mathbb{R}^3, \ \|\vec{a}_x\|_2 \leq 1.$$

Then

$$\|\{2M_x - I\}\|_c = \|\{\vec{a}_x\}\|_{\pi(\ell_\infty^n, \ell_2^3)}$$

- $n = 2$: Busch compatibility condition

$$\|\{\vec{a}, \vec{b}\}\|_\pi = \frac{1}{2}(\|\vec{a} + \vec{b}\|_2 + \|\vec{a} - \vec{b}\|_2)$$

P. Busch, Phys. Rev. D, 1986

- general case:

$$\|\{\vec{a}_x\}\|_\pi = \max_{\substack{\|\sum_x t_x \vec{y}_x\|_2 \leq 1, \\ \forall t \in \{\pm 1\}^n}} \sum_x \langle \vec{a}_x, \vec{y}_x \rangle$$

Compatibility degrees for qubit effects

- Compatibility degree for n qubit effects

$$s_{2,n} = \min_{\{\vec{a}_x\}} \frac{\|\{\vec{a}_x\}\|_\epsilon}{\|\{\vec{a}_x\}\|_\pi} = \min_{\sum_x \|\vec{y}_x\|_2 \le 1} \max_{t \in \{\pm 1\}^n} \|\sum_x t_x \vec{y}_x\|_2$$

- the ϵ/π-ratio

- Solutions and bounds for some n

$$s_{2,2} = \frac{1}{\sqrt{2}}, \quad s_{2,3} = \frac{1}{\sqrt{3}}, \quad \frac{1}{\sqrt{3}} > s_{2,n} \ge \frac{1}{2} = \lim_{n \to \infty} s_{2,n}, \; n \ge 4.$$

- Universal compatibility degree for dichotomic qubit effects:

$$s_2 = \frac{1}{2} = \pi_1(\ell_2^3) \text{ 1-summing constant}$$