Reversibility conditions for quantum operations

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MPI MIS Leipzig, May 22, 2012

Some basic references

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Introduction

A quantum system is represented by a C^* -algebra $\mathcal{A} \subseteq B(\mathcal{H})$:

- Observables $=$ self-adjoint operators: $\mathcal{A}^h$
- States of the system $=$ positive unital functionals: $\mathcal{S}(\mathcal{A})$

Suppose another system is represented by another algebra $\mathcal{B}$, with state space $\mathcal{S}(\mathcal{B})$.

A quantum operation is a map $T : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{B})$ representng a physical transformation of the states.

Introduction

Suppose $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$, $T : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{B})$ a quantum operation.

$\mathcal{S}$ can be seen as carrying some information, this is lost under T:

- in quantum statistics, $\mathcal{S}$ represents a statistical model, T a stochastic map (e.g. a restriction to a subalgebra)
- in error correction, $\mathcal{S}$ represents a code, T a noisy channel

Reversibility/sufficiency: the set $\mathcal{S}$ can be completely recovered by a quantum operation $S : \mathcal{S}(\mathcal{B}) \to \mathcal{S}(\mathcal{A})$.

The setting

- Let $\mathcal{A}$ be a finite dimensional C^*-algebra,

$$\mathcal{A} \equiv \bigoplus_{k=1}^{n} B(\mathcal{H}_k) \subseteq B(\mathcal{H}), \quad \dim(\mathcal{H}) < \infty$$

  Let Tr be the trace inherited from $B(\mathcal{H})$.

- $\mathcal{A}$ is a (finite dim) Hilbert space, with the Hilbert-Schmidt inner product

$$\langle a, b \rangle = \mathrm{Tr}\, a^* b, \qquad a, b \in \mathcal{A}$$

- State space = density operators:

$$\mathcal{S}(\mathcal{A}) = \{\rho \geq 0,\ \mathrm{Tr}\, \rho = 1\}$$

Quantum operations

- Quantum operations = completely positive trace preserving maps (channels) $T : \mathcal{A} \to \mathcal{B}$:

$$T(a) = \sum_i V_i^* a V_i, \;\; a \in \mathcal{A}, \quad \sum_i V_i V_i^* = I_{\mathcal{A}}$$

- The adjoint $T^* : \mathcal{B} \to \mathcal{A}$ with respect to the HS inner product,

$$\langle a, T^*(b)\rangle = \langle T(a), b\rangle, \qquad a, b \in \mathcal{A}$$

is completely positive and unital, $T^*(I_{\mathcal{B}}) = I_{\mathcal{A}}$.

Reversibility (sufficiency)

Let $T : \mathcal{A} \to \mathcal{B}$ be a quantum operation and let $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$ be a set of states.

Definition

We say that T is reversible (or sufficient) for $\mathcal{S}$ if there is a quantum operation $S : \mathcal{B} \to \mathcal{A}$, such that

$$S \circ T(\sigma) = \sigma, \qquad \sigma \in \mathcal{S}$$

Assumptions

We will suppose:

1. $T : \mathcal{A} \to \mathcal{B}$ is a quantum operation
2. $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$ and there is an invertible element $\rho \in \mathcal{S}$, such that $T(\rho)$ is invertible.

Let $(\mathcal{S}, T)$ be such that $\mathcal{S} \subset \mathcal{S}(\mathcal{A})$ and T a quantum operation. Then there is a projection $p \in \mathcal{A}$, such that

- $\mathcal{S}' = co(\mathcal{S}) \subseteq \mathcal{S}(p\mathcal{A}p)$,
- with $T' = T|_{p\mathcal{A}p}$, $(\mathcal{S}', T')$ satisfy 1. and 2.
- T is reversible for $\mathcal{S}$ if and only if T' is reversible for $\mathcal{S}'$.

The dual map

Let $T : \mathcal{A} \to \mathcal{B}$ be a quantum operation and $\rho \in \mathcal{S}(\mathcal{A})$ be such that ρ and $T(\rho)$ are invertible. The map $T^*_\rho : \mathcal{B} \to \mathcal{A}$

$$T^*_\rho(b) := \rho^{1/2} T^*(T(\rho)^{-1/2} b T(\rho)^{-1/2}) \rho^{1/2}, \; b \in \mathcal{B}$$

is called the dual of T with respect to ρ.

- T^*_ρ is a quantum operation
- $T^*_\rho \circ T(\rho) = \rho$

Relative entropy

Let $\sigma, \rho \in \mathcal{S}(\mathcal{A})$. Relative entropy is defined by

$$S(\sigma, \rho) = \begin{cases} \mathrm{Tr}\,\sigma(\log \sigma - \log \rho) & \text{if } \mathrm{supp}\,\sigma \leq \mathrm{supp}\,\rho \\ \infty & \text{otherwise} \end{cases}$$

- Monotonicity If $T : \mathcal{A} \to \mathcal{B}$ is a quantum operation, then

$$S(\sigma, \rho) \geq S(T(\sigma), T(\rho))$$

- If T is reversible, then

$$S(\sigma, \rho) = S(T(\sigma), T(\rho)), \qquad \sigma \in \mathcal{S}$$

A theorem of Petz (1986)

Theorem

Suppose both ρ and $T(\rho)$ are invertible. TFAE:

(i) *T is reversible for $\{\sigma, \rho\}$.*

(ii) $S(\sigma, \rho) = S(T(\sigma), T(\rho))$.

(iii) $T_\rho^* \circ T(\sigma) = \sigma$.

The dual map T_ρ^* plays the role of the recovery map S.

Corollary

Under assumptions 1. and 2., T is reversible for $\mathcal{S}$ if and only if T is reversible for all pairs $\{\sigma_1, \sigma_2\} \subset \mathcal{S}$.

Conditions for reversibility

There are several conditions characterizing reversibility:

1. Quantum f-divergences (relative entropy, Rényi relative entropy,...)
2. Radon-Nikodym derivative
3. Factorization of states in $\mathcal{S}$ (quantum factorization criterion)
4. Distinguishability measures obtained from hypothesis testing: L_1 - distance, quantum Chernoff bound, quantum Hoeffding bound
5. Quantum Fisher information and quantum χ^2-distance

Quantum f-divergence

Let $\sigma, \rho \in \mathcal{S}(\mathcal{A})$, ρ is invertible.

- $\mathcal{A}$ is a Hilbert space, with the HS inner product
- Relative modular operator: $\Delta_{\sigma,\rho} : \mathcal{A} \to \mathcal{A}$, positive,

$$\Delta_{\sigma,\rho}(a) = \sigma a \rho^{-1}$$

- Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function
- Quantum f-divergence:

$$S_f(\sigma, \rho) = \langle \rho^{1/2}, f(\Delta_{\sigma,\rho})(\rho^{1/2}) \rangle$$

Operator convex functions

- A function $f : \mathbb{R}^+ \to \mathbb{R}$ is operator convex if

$$f(tA + (1-t)B) \leq tf(A) + (1-t)f(B)$$

for $t \in [0,1]$, $A, B \in B(\mathcal{H})^+$, $\dim(\mathcal{H}) < \infty$

- Integral representation (Hiai, Mosonyi, Petz, Beny 2011):

$$f(x) = f(0) + \alpha x + \beta x^2 + \int_{(0,\infty)} \left(\frac{x}{1+t} - \frac{x}{x+t}\right) d\mu_f(t),$$

$\alpha \in \mathbb{R}$, $\beta \geq 0$ and μ_f is a non-negative measure on $(0,\infty)$ satisfying $\int (1+t)^{-2} d\mu_f(t) < \infty$.

Monotonicity and convexity

If f is operator convex, then S_f is non-increasing under quantum operations:

$$S_f(\sigma, \rho) \geq S_f(T(\sigma), T(\rho))$$

- S_f is jointly convex:

$$S_f(\sum_i p_i \sigma_i, \sum_i p_i \rho_i) \leq \sum_i p_i S(\sigma_i, \rho_i)$$

  where $p_i \geq 0$, $\sum p_i = 1$, $\sigma_i, \rho_i \in \mathcal{S}(\mathcal{A})$.
- $S_f(\sigma, \rho) \geq f(1)$

S_f is a distance (distinguishability) measure on $\mathcal{S}(\mathcal{A})$.

Examples

- Relative entropy,

$$f(x) = x \log x = \int_0^\infty \left(\frac{x}{1+t} - \frac{x}{x+t} \right) dt$$

- For $\alpha \in [0,1]$, $S_\alpha(\sigma, \rho) = 1 - \mathrm{Tr}\, \sigma^\alpha \rho^{1-\alpha}$, here

$$f_\alpha(x) = 1 - x^\alpha = 1 - x + \int_0^\infty \left(\frac{x}{1+t} - \frac{x}{x+t} \right) d\mu_\alpha(t)$$

$d\mu_\alpha(t) = \frac{\sin(\alpha\pi)}{\pi} t^{\alpha-1} dt.$

- Rényi relative entropy with $\alpha \in (0,1)$,

$$R_\alpha(\sigma, \rho) = \frac{1}{\alpha - 1} \log(1 - S_\alpha(\sigma, \rho))$$

- $S_f(\sigma, \rho) = \mathrm{Tr}\, \rho (L_\sigma + R_\rho)^{-1}(\rho)$, here

$$f(x) = \frac{1}{1+x}, \qquad \mu_f = \delta_{\{1\}}$$

Reversibility

Theorem

Under the conditions 1.and 2., TFAE:

(i) T *is reversible for* $\mathcal{S}$.

(ii) $S_f(\sigma, \rho) = S_f(T(\sigma), T(\rho))$ *for all* $\sigma \in \mathcal{S}$ *and all operator convex* f

(iii) *(Hiai, Mosonyi, Petz,Bény, 2011)*

$S_f(\sigma, \rho) = S_f(T(\sigma), T(\rho))$ *for all* $\sigma \in \mathcal{S}$ *and some operator convex* f *with* $|\operatorname{supp} \mu_f| \geq \dim(\mathcal{H})^2 + \dim(\mathcal{K})^2$.

(iv) $T_\rho^* \circ T(\sigma) = \sigma$ *for all* $\sigma \in \mathcal{S}$.

An example and a counterexample

- Since $\operatorname{supp}\mu_\alpha = (0,\infty)$ for $\alpha \in (0,1)$, T is reversible for $\mathcal{S}$ if and only if

$$\operatorname{Tr}\sigma^\alpha \rho^{1-\alpha} = \operatorname{Tr} T(\sigma)^\alpha T(\rho)^{1-\alpha}, \qquad \sigma \in \mathcal{S}$$

  for some $\alpha \in (0,1)$ (Petz 1988, Petz & AJ 2006).

- It can be shown that the equality

$$S_f(\sigma,\rho) = S_f(T(\sigma), T(\rho)), \qquad \sigma \in \mathcal{S}$$

  for the function

$$f(x) = \frac{1}{1+x}, \qquad \operatorname{supp}\mu_f = \{1\}$$

  does not imply reversibility of T.

A quantum Radon-Nikodym derivative

Let $\sigma, \rho \in \mathcal{S}(\mathcal{A})$, ρ invertible.

- The (commutant) Radon-Nikodym derivative

$$d = d(\sigma, \rho) := \rho^{-1/2} \sigma \rho^{-1/2}$$

- $d \geq 0$, $\|d\| = \inf\{\lambda > 0,\ \sigma \leq \lambda \rho\}$, $\|d\| = 1$ iff $\sigma = \rho$.
- For $T : \mathcal{A} \to \mathcal{B}$ a quantum operation, let $T_\rho := (T_\rho^*)^* : \mathcal{A} \to \mathcal{B}$ be the adjoint of T_ρ^*, so that

$$T_\rho(a) = T(\rho)^{-1/2} T(\rho^{1/2} a \rho^{1/2}) T(\rho)^{-1/2}.$$

Then

$$T_\rho(d(\sigma, \rho)) = d(T(\sigma), T(\rho))$$

Reversibility

Theorem

Under the conditions 1.and 2., let $T^* : \mathcal{B} \to \mathcal{A}$ be the adjoint of T. TFAE:

(i) *T is reversible for $\mathcal{S}$.*

(ii) $T^*(d(T(\sigma), T(\rho))) = d(\sigma, \rho)$, *$\sigma \in \mathcal{S}$.*

(iii) $\Phi(d(\sigma, \rho)) = d(\sigma, \rho)$, *$\sigma \in \mathcal{S}$, where $\Phi = T^* \circ T_\rho$.*

Note that $\Phi^* = T_\rho^* \circ T : \mathcal{A} \to \mathcal{A}$ is a quantum operation, such that $\Phi^*(\rho) = \rho$. Moreover, (iii) means that $d(\sigma, \rho)$ is a fixed point of Φ.

The fixed point algebra

Let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map such that Φ^* is a quantum operation. Suppose that $\Phi^*(\rho) = \rho$ for some invertible $\rho \in \mathcal{S}(\mathcal{A})$.

Denote $\mathcal{F}_\Phi = \{a \in \mathcal{A}, \Phi(a) = a\}$.

- $\mathcal{F}_\Phi$ is a subalgebra in $\mathcal{A}$.
- $\rho^{it}\mathcal{F}_\Phi\rho^{-it} \subseteq \mathcal{F}_\Phi$, $t \in \mathbb{R}$.
- ρ can be decomposed as

$$\rho = \rho^A \rho^B$$

where $\rho^A \in \mathcal{F}_\Phi^+$ and $\rho^B \in (\mathcal{F}_\Phi' \cap \mathcal{A})^+$.

An extension of $\mathcal{S}$

- Let T be reversible for $\mathcal{S}$, $\sigma \in \mathcal{S}$, $d = d(\sigma, \rho)$.
- Then $d \in \mathcal{F}_\Phi$.
- Hence if $\sigma_t := \rho^{it}\sigma\rho^{-it}$, then

$$d_t := d(\sigma_t, \rho) = \rho^{it} d \rho^{-it} \in \mathcal{F}_\Phi.$$

Corollary

T is reversible for $\mathcal{S}$ if and only if T is reversible for $\{\rho^{it}\mathcal{S}\rho^{-it},\ t \in \mathbb{R}\}$.

Factorization of states in $\mathcal{S}$

Theorem

Under the conditions 1.and 2., TFAE:

(i) *T is reversible for $\mathcal{S}$.*

(ii) *There is some $\rho^B \in (\mathcal{F}'_\Phi \cap \mathcal{A})^+$, such that for $\sigma \in \mathcal{S}$,*

$$\sigma = \sigma^A \rho^B$$

with $\sigma^A \in \mathcal{F}_\Phi$.

(iii) *There is some $\rho^B \in \mathcal{A}^+$, such that for all $\sigma \in \mathcal{S}$,*

$$\sigma = T^*(\sigma_0^A)\rho^B, \quad T(\sigma) = \sigma_0^A T(\rho^B),$$

where $\sigma_0^A \in \mathcal{B}^+$.

Quantum hypothesis testing

Let $\sigma, \rho \in \mathcal{S}(\mathcal{A})$, ρ invertible. Let $H_0 = \rho$, $H_1 = \sigma$.

- Any test of H_0 against H_1 is given by some $0 \leq M \leq I$ in $\mathcal{A}$.
- M corresponds to rejecting the H_0 hypothesis.
- error of 1. kind:
  rejecting true - probability $\alpha(M) = \mathrm{Tr}\, M\rho$.
- error of 2. kind:
  not rejecting false - probability $\beta(M) = \mathrm{Tr}\,(I - M)\sigma$.
- For some $\lambda \in [0, 1]$, minimize

$$\lambda\alpha(M) + (1 - \lambda)\beta(M)$$

to obtain the Bayes optimal test.

Quantum Neyman-Pearson lemma (Helstrom,Holevo)

Let $\lambda \in [0,1]$. The minimum Bayes error probability is

$$\begin{aligned} \Pi_\lambda &:= \inf_{0 \le M \le I} (\lambda \alpha(M) + (1-\lambda)\beta(M)) \\ &= \frac{1}{2}(1 - \|(1-\lambda)\sigma - \lambda\rho\|_1), \end{aligned}$$

where $\|a\|_1 = \mathrm{Tr}\,|a| = \mathrm{Tr}\,(a^*a)^{1/2}$ is the L_1-norm. It satisfies monotonicity

$$\|a\|_1 \ge \|T(a)\|_1, \qquad a \in \mathcal{A}$$

if T is a quantum operation.

Reversibility and L_1-norm

Let $\tilde{\mathcal{S}} = \bigcup\{\rho^{is}\mathcal{S}\rho^{-is}, s \in \mathbb{R}\}$ be the extended family.

Theorem

Suppose 1. and 2. hold. Then

(i) *T is reversible for $\mathcal{S}$ if and only if*

$$\|\sigma - t\rho\|_1 = \|T(\sigma) - tT(\rho)\|_1, \quad \sigma \in \tilde{\mathcal{S}}, \ t \geq 0$$

(ii) *If $\mathcal{B}$ is abelian, then T is reversible for $\mathcal{S}$ if and only if*

$$\|\sigma - t\rho\|_1 = \|T(\sigma) - tT(\rho)\|_1, \quad \sigma \in \mathcal{S}, \ t \geq 0$$

Moreover, in this case, all elements in $\mathcal{S}$ commute.

Hypothesis testing for n copies

Let $\sigma, \rho \in \mathcal{S}(\mathcal{A})$

- Let $n \in \mathbb{N}$, n copies $\sigma^{\otimes n}$, $\rho^{\otimes n} \in \mathcal{S}(\mathcal{A}^{\otimes n})$
- $H_0 = \rho^{\otimes n}$, $H_1 = \sigma^{\otimes n}$
- A test $0 \leq M_n \leq I$, $M_n \in \mathcal{A}^{\otimes n}$
- Minimum Bayes error probability

$$\Pi_{\lambda,n} = \frac{1}{2}(1 - \|(1-\lambda)\sigma^{\otimes n} - \lambda\rho^{\otimes n}\|_1)$$

- As $n \to \infty$, $\Pi_{\lambda,n} \to 0$ exponentially fast

Quantum Chernoff distance

Theorem

(Audenaert et al. (2008)) For all $\lambda \in [0,1]$.

$$\lim_n -\frac{1}{n}\log \Pi_{\lambda,n} = -\log\left(\inf_{0\le s\le 1} \mathrm{Tr}\,\sigma^s\rho^{1-s}\right) =: \xi_{QCB}(\sigma,\rho)$$

ξ_{QCB} is called the quantum Chernoff distance. Properties:

- non-increasing under quantum operations:

$$\xi_{QCB}(\sigma,\rho) \ge \xi_{QCB}(T(\sigma),T(\rho))$$

- $\xi_{QCB}(\sigma,\rho) = 0$ if and only if $\sigma = \rho$.
- not a quantum f-divergence

QCB and reversibility

Suppose $\sigma, \rho \in \mathcal{S}(\mathcal{A})$, ρ and $T(\rho)$ invertible.

- There is some $s_0 \in [0,1]$ such that

$$\inf \mathrm{Tr}\, T(\sigma)^s T(\rho)^{1-s} = \mathrm{Tr}\, T(\sigma)^{s_0} T(\rho)^{1-s_0}$$

- We have

$$\begin{aligned} \xi_{QCB}(T(\sigma), T(\rho)) &= -\log \mathrm{Tr}\, T(\sigma)^{s_0} T(\rho)^{1-s_0} \\ &\leq -\log \mathrm{Tr}\, \sigma^{s_0} \rho^{1-s_0} \leq \xi_{QCB}(\sigma, \rho) \end{aligned}$$

- Suppose $\xi_{QCB}(T(\sigma), T(\rho)) = \xi_{QCB}(\sigma, \rho)$, then

$$\mathrm{Tr}\, \sigma^{s_0} \rho^{1-s_0} = \mathrm{Tr}\, T(\sigma)^{s_0} T(\rho)^{1-s_0}$$

QCB and reversibility

- Suppose $s_0 \in (0, 1)$. Then the equality implies that T is reversible for $\{\sigma, \rho\}$.
- Suppose $s_0 = 0, 1$. Then if $T(\sigma)$ is invertible, $\xi_{QCB}(\sigma, \rho) = \xi_{QCB}(T(\sigma), T(\rho)) = 0$, hence $\sigma = \rho$.
- If σ is invertible, then T is reversible for $\{\sigma, \rho\}$ iff

$$\xi_{QCB}(\sigma, \rho) = \xi_{QCB}(T(\sigma), T(\rho)).$$

- There are examples with σ not invertible, where the equality holds while T is not reversible.

QCB and reversibility

Theorem

Under the conditions 1. and 2., TFAE:

(i) T *is reversible for* $\mathcal{S}$.

(ii) $\xi_{QCB}(T(\sigma), T(\rho)) = \xi_{QCB}(\sigma, \rho)$, *for all* $\sigma \in co(\mathcal{S})$.

(iii) $\|T^{\otimes n}(\sigma^{\otimes n}) - tT(\rho^{\otimes n})\|_1 = \|\sigma^{\otimes n} - t\rho^{\otimes n}\|_1$, $\sigma \in \mathcal{S}$, $t \geq 0$, $n \in \mathbb{N}$.

If all elements in $\mathcal{S}$ *are invertible, we may replace* $co(\mathcal{S})$ *by* $\mathcal{S}$ *in (ii).*

Quantum Hoeffding distance

Asymmetric setting:

- The error probability $\alpha(M)$ is bounded, $\alpha(M) \leq \epsilon$
- The error probability $\beta(M)$ is optimized under this constraint:

$$\beta_\epsilon := \inf\{\beta(M),\ 0 \leq M \leq I, \alpha(M) \leq \epsilon\}$$

n copies, $M_n \in \mathcal{A}^{\otimes n}$ a sequence of tests of $\rho^{\otimes n}$ against $\sigma^{\otimes n}$.

- We require that $\alpha(M_n)$ decays exponentially fast: $\alpha(M_n) \leq e^{-nr}$ for some $r > 0$.
- Let

$$\beta_{n,r} := \inf\{\beta(M_n),\ 0 \leq M_n \leq I, \alpha(M_n) \leq e^{-nr}\}$$

Quantum Hoeffding distance

Theorem

(Audenaert et al. (2008))

$$\lim_n -\frac{1}{n}\log \beta_{n,r} = \sup_{0\le s<1} \frac{-sr - \log \mathrm{Tr}\,\sigma^s\rho^{1-s}}{1-s} =: H_r(\sigma,\rho)$$

$H_r(\sigma,\rho)$ is the quantum Hoeffding distance. Suppose ρ and σ are both invertible. Then

- $H_r(\sigma,\rho) \ge H_r(T(\sigma), T(\rho))$ for a quantum operation T
- $H_0(\sigma,\rho) := \lim_{r\to 0+} H_r(\rho,\sigma) = S(\sigma,\rho)$
- The function $r \mapsto H_r(\sigma,\rho)$ is strictly convex and decreasing for $r \in [0, S(\rho,\sigma)]$
- $H_r(\sigma,\rho) = 0$ for all $r \ge S(\rho,\sigma)$

QHD and reversibility

If σ, ρ and $T(\rho)$ are invertible, then T is reversible for $\{\sigma, \rho\}$ iff $H_r(\sigma, \rho) = H_r(T(\sigma), T(\rho))$ for some $r \in [0, S(T(\rho), T(\sigma))]$.

Theorem

Suppose the conditions 1. and 2. hold. Let $X \subset \mathcal{S}$ be such that $\mathcal{S} \subseteq \overline{co}(X \cup \{\rho\})$ and $T(\rho) \notin \overline{T(X)}$. Then there is some $r_0 > 0$, such that TFAE:

(i) *T is reversible for $\mathcal{S}$.*

(ii) *$H_r(\sigma, \rho) = H_r(T(\sigma), T(\rho))$ for all $\sigma \in \mathcal{S}$ and $r \geq 0$*

(iii) *$H_r(\sigma, \rho) = H_r(T(\sigma), T(\rho))$ for all $\sigma \in co(\mathcal{S})$ and some $r \in [0, r_0]$.*

If all $\sigma \in \mathcal{S}$ are invertible, we may replace $co(\mathcal{S})$ by X in (iii) and $r_0 = \inf_{\sigma \in X} S(T(\rho), T(\sigma))$.

Quantum Fisher information

Let $\mathcal{D} = \mathcal{D}(\mathcal{A})$ be the set of invertible elements in $\mathcal{S}(\mathcal{A})$.

- $\mathcal{D}$ is a differentiable manifold.
- Tangent space at $\rho \in \mathcal{D}$: $\mathcal{T}_\rho = \{x = x^* \in \mathcal{A},\ \mathrm{Tr}\, x = 0\}$
- A Riemannian metric in $\mathcal{D}$: g_ρ is an inner product in $\mathcal{T}_\rho$, $\rho \mapsto g_\rho$ is smooth
- Let $T : \mathcal{A} \to \mathcal{B}$ be a quantum operation, then

$$T(\mathcal{D}) \subseteq \mathcal{D}(\mathcal{B}), \quad T(\mathcal{T}_\rho) \subseteq \mathcal{T}_{T(\rho)}$$

- Monotone metric if

$$\lambda_\rho(x,x) \geq \lambda_{T(\rho)}(T(x), T(x)), \qquad x \in \mathcal{T}_\rho,\ \rho \in \mathcal{D},$$

A quantum Fisher information is a monotone Riemannian metric on $\mathcal{D}$.

Characterization of the quantum Fisher information

A function $f : \mathbb{R}^+ \to \mathbb{R}$ is operator monotone if $f(A) \le f(B)$ whenever $0 \le A \le B$, $A, B \in B(\mathcal{K})$, $\dim(\mathcal{K}) < \infty$.

Theorem

(Petz (1996)) A Riemannian metric g on $\mathcal{D}$ is monotone if and only if

$$g_\rho(x, y) = \operatorname{Tr} x J_\rho^{-1}(y), \qquad x, y \in \mathcal{T}_\rho$$

where $J_\rho = f(L_\rho R_\rho^{-1}) R_\rho$ and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an operator monotone function, such that $f(t) = tf(t^{-1})$.

We denote the metric corresponding to f by g^f.

Examples

- The SLD (Bures) metric: $f(t) = \frac{1+t}{2}$

$$g_\rho^f(x,y) = 2\mathrm{Tr}\,(L_\rho + R_\rho)^{-1}(x)y$$

- The BKM metric: $f(t) = \frac{t-1}{\log t}$

$$g_\rho^f(x,y) = \int_0^\infty \mathrm{Tr}\, x(\rho+s)^{-1}y(\rho+s)^{-1}ds$$

- The RLD metric: $f(t) = \frac{2t}{1+t}$

$$g_\rho^f(x,y) = \frac{1}{2}\mathrm{Tr}\, x(L_\rho^{-1} + R_\rho^{-1})(y)$$

Integral representation (Hansen (2005))

Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be operator monotone. Then

$$\frac{1}{f(t)} = \int_0^\infty \frac{1}{s+t} d\nu_f(s)$$

where ν_f is a positive Borel measure with support in $[0, \infty)$, such that $\int (1+s^2)^{-1} d\nu_f(s) < \infty$, $\int s(1+s^2)^{-1} d\nu + f(s) < \infty$.

It follows that

$$g_\rho^f(x, y) = \int_0^\infty \mathrm{Tr}\, x(sR_\rho + L_\rho)^{-1}(y) d\nu_f(s)$$

Quantum χ^2-divergence

The quantum χ^2-divergence is given by

$$\chi^2_{1/f}(\sigma, \rho) = g^f_\rho(\sigma - \rho, \sigma - \rho)$$

where g^f is a quantum Fisher information. By monotonicity of g^f,

$$\chi^2_{1/f}(\sigma, \rho) \geq \chi^2_{1/f}(T(\sigma), T(\rho))$$

for a quantum operation T.

Fisher information and reversibility

Denote by $Lin_\rho(\mathcal{S})$ the real linear span of $\mathcal{S} - \rho$. Then $Lin_\rho(\mathcal{S}) \subseteq \mathcal{T}_\rho$.

Theorem

Under the conditions 1. and 2., TFAE:

(i) *T is reversible for $\mathcal{S}$.*

(ii) *$g^f_\rho(x,x) = g^f_{T(\rho)}(T(x), T(x))$ for all $x \in Lin_\rho(\mathcal{S})$ and all monotone metrics g^f.*

(iii) *$\chi^2_{1/f}(\sigma,\rho) = \chi^2_{1/f}(T(\sigma), T(\rho))$ for all χ^2 divergences.*

(iv) *Equality in (ii) holds for some f with $|\mathrm{supp}\,\nu_f| \geq \dim(\mathcal{H})^2 + \dim(\mathcal{K})^2$.*

(v) *Equality in (iii) holds for some f as in (iv).*

Some open problems

- Support conditions for μ_f and ν_f: can be relaxed?
  - Cannot be removed: the supports must contain more than 1 point.
- Can we have full equivalence for L_1-distance?
  - Up to now, only for extended family $\tilde{\mathcal{S}}$.
- Characterization by fidelity?
- Infinite dimensions: $\mathcal{A}$ a von Neumann algebra?
  - Some results for f divergences, relative entropy (can be infinite!), S_α (Petz, Petz & AJ)
  - Characterization by commutant Radon-Nikodym derivative holds.
  - Factorization holds for type I algebras.
  - L_1 -distance?? QCB, QHB- ??