Characterization of sufficient channels by a Rényi relative entropy

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1. Quantum versions of the Rényi relative entropy

In [10], it was suggested that the quantum generalization of the Rényi relative entropy should be

$$D_\alpha(\rho\|\sigma) = \begin{cases} D_\alpha^{(old)}(\rho,\sigma), & \alpha\in(0,1)\\ D_\alpha^{(new)}(\rho\|\sigma), & \alpha>1\end{cases}$$

for pairs of states ρ, σ on a finite dimensional Hilbert space. The "old" entropies

$$D_\alpha^{(old)}(\rho\|\sigma) = \frac{1}{\alpha-1}\log \mathrm{Tr}\,\rho^\alpha\sigma^{1-\alpha}$$

for $\alpha\in(0,1)$ have a direct operational interpretation as error exponents and cutoff rates in binary state discrimination [1, 9]. The "new", or "sandwiched" version [14, 11]

$$D_\alpha^{(new)}(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1}\log\mathrm{Tr}\,\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha, & \mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)\\ \infty, & \text{otherwise}\end{cases}$$

has similar applications in the strong converse domain, [10, 14]. These quantities have some important properties ([4, 12, 2, 3]):

(i) **positivity**: $D_\alpha(\rho,\sigma)\geq 0$ and equality holds if and only if $\rho=\sigma$.

(ii) **data processing inequality (DPI)**: If Φ is a channel (i.e. a completely positive trace preserving map), then

$$D_\alpha(\rho\|\sigma)\geq D_\alpha(\Phi(\rho)\|\Phi(\sigma))$$

2. Sufficient channels

Let Φ be a channel and let $\mathcal{S}$ be a set of states. We say that Φ is **sufficient** with respect to $\mathcal{S}$ if there is some channel Ψ, called a **recovery map**, such that

$$\Psi\circ\Phi(\rho)=\rho\qquad \rho\in\mathcal{S}.$$

Sufficient channels can be characterized by equality in the data processing inequality for a large class of information theoretic quantities, see e.g. [4, 5]. In particular, if $\mathcal{S}$ contains an invertible element σ, then Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$D_\alpha(\rho\|\sigma)=D_\alpha(\Phi(\rho)\|\Phi(\sigma)),\qquad \rho\in\mathcal{S}\tag{1}$$

for some $\alpha\in(0,1)$, [4, 7]. We show that this is true also for $\alpha>1$ and in infinite dimensions. Similarly to [2], we use an interpolating family of non-commutative L_p-spaces with respect to a state.

3. Noncommutative L_p-spaces and interpolation

Let $\mathcal{H}$ be a separable Hilbert space. For $p\leq 1$, let

$$\mathcal{L}_p(\mathcal{H})=\{X\in B(\mathcal{H}),\mathrm{Tr}\,|X|^p<\infty\}$$

be the Schatten class, with the norm $\|X\|_p=(\mathrm{Tr}\,|X|^p)^{1/p}$. Let

$$\mathcal{L}_\infty(\mathcal{H})=B(\mathcal{H}),$$

with $\|\cdot\|_\infty=\|\cdot\|$, the operator norm. Let also

$$\mathfrak{S}(\mathcal{H})=\{\rho\in\mathcal{L}_1(\mathcal{H})^+,\mathrm{Tr}\,\rho=1\}$$

be the set of normal states. Fix a faithful $\sigma\in\mathfrak{S}(\mathcal{H})$ and $p\in[1,\infty]$, $1/p+1/q=1$. The **noncommutative L_p-space with respect to** σ is defined as

$$L_p(\mathcal{H},\sigma):=\{\sigma^{1/2q}X\sigma^{1/2q},\ X\in\mathcal{L}_p(\mathcal{H})\},$$

with the norm

$$\|\sigma^{1/2q}X\sigma^{1/2q}\|_{p,\sigma}:=\|X\|_p.$$

Then $L_p(\mathcal{H},\sigma)\subseteq\mathcal{L}_1(\mathcal{H})=L_1(\mathcal{H},\sigma)$ and

$$L_p(\mathcal{H},\sigma)=C_{1/p}(L_\infty(\mathcal{H},\sigma),\mathcal{L}_1(\mathcal{H})),$$

where C_θ is given by complex interpolation [8, 15].
This has some consequences:

- For $1\leq p\leq p'\leq\infty$, $L_{p'}(\mathcal{H},\sigma)\subseteq L_p(\mathcal{H},\sigma)$ and
$$\|X\|_{p,\sigma}\leq\|X\|_{p',\sigma},\quad \forall X\in L_{p'}(\mathcal{H},\sigma).$$
- For $1\leq p<\infty$, $L_q(\mathcal{H},\sigma)$ is the dual space of $L_p(\mathcal{H},\sigma)$, with duality $\langle\cdot,\cdot\rangle: L_q(\mathcal{H},\sigma)\times L_p(\mathcal{H},\sigma)\to\mathbb{C}$ given by
$$\langle\,\sigma^{1/2p}X\sigma^{1/2p},\sigma^{1/2q}Y\sigma^{1/2q}\,\rangle:=\mathrm{Tr}\,XY,$$
for any $X\in\mathcal{L}_q(\mathcal{H})$, $Y\in\mathcal{L}_p(\mathcal{H})$.
- $L_2(\mathcal{H},\sigma)$ is a Hilbert space, with inner product
$$(S,T)\mapsto\langle\,S^*,T\,\rangle.$$

Let $\Phi:\mathcal{L}_1(\mathcal{H})\to\mathcal{L}_1(\mathcal{K})$ be a channel. Then

- for $1\leq p\leq\infty$, Φ restricts to a contraction $L_p(\mathcal{H},\sigma)\to L_p(\mathrm{supp}(\Phi(\sigma)),\Phi(\sigma))$.
- The equality
$$\langle\,\Phi_\sigma(X),\sigma^{1/2}Y\sigma^{1/2}\,\rangle=\langle\,X,\Phi(\sigma^{1/2}Y\sigma^{1/2})\,\rangle,$$
for all $X\in\mathcal{L}_1(\mathcal{K})$ and $Y\in B(\mathcal{H})$, defines a channel $\Phi_\sigma:\mathcal{L}_1(\mathcal{K})\to\mathcal{L}_1(\mathcal{H})$ - the **Petz recovery map**.

4. Quantum Rényi entropies and L_p-spaces

Let $\rho,\sigma\in\mathfrak{S}(\mathcal{H})$, $\alpha\in(0,1)$. Assume that $\tau\in\mathfrak{S}(\mathcal{H})$ is faithful. Then

$$\ell_\alpha:\omega\mapsto\tau^{(1-\alpha)/2}\omega^\alpha\tau^{(1-\alpha)/2}$$

defines a homeomorpism of the positive cone $\mathcal{L}_1(\mathcal{H})^+$ onto $L_{1/\alpha}(\mathcal{H},\tau)^+$. Put

$$D_\alpha(\rho\|\sigma):=\frac{1}{\alpha-1}\log\langle\,\ell_\alpha(\rho),\ell_{1-\alpha}(\sigma)\,\rangle$$

If $\mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)$, then we may put $\tau=\sigma$ and

$$D_\alpha(\rho\|\sigma)=\frac{1}{\alpha-1}\log\|\ell_\alpha(\rho)\|_1.$$

For $\alpha>1$, let $1/\alpha+1/\beta=1$. Put

$$L_\alpha(\sigma):=L_\alpha(\mathrm{supp}(\sigma),\sigma).$$

Then $\rho\in L_\alpha(\sigma)$ if $\mathrm{supp}(\rho)\subseteq\mathrm{supp}(\sigma)$ and there is some $\omega\in\mathcal{L}_1(\mathcal{H})^+$ such that

$$\rho=\sigma^{1/2\beta}\omega^{1/\alpha}\sigma^{1/2\beta}.$$

We have $\|\rho\|_{\alpha,\sigma}=(\mathrm{Tr}\,\omega)^{1/\alpha}$. Define

$$D_\alpha(\rho\|\sigma):=\begin{cases}\frac{\alpha}{\alpha-1}\log(\|\rho\|_{\alpha,\sigma}), & \text{if } \rho\in L_\alpha(\sigma)\\ \infty, & \text{otherwise}\end{cases}$$

If $\dim(\mathcal{H})<\infty$, $D_\alpha(\rho\|\sigma)$ coincides with the original definition. The following properties extend to infinite dimensional case:

(i) positivity

(ii) DPI

(iii) $\alpha\mapsto D_\alpha(\rho\|\sigma)$ is non-decreasing.

(iv) $(\rho,\sigma)\mapsto D_\alpha(\rho\|\sigma)$ is jointly lower semicontinuous.

5. Characterization of sufficient channels by factorization

We may assume that $\sigma\in\mathfrak{S}(\mathcal{H})$ is faithful.

Theorem 1. *[13] Let $\Phi:\mathcal{L}_1(\mathcal{H})\to\mathcal{L}_1(\mathcal{K})$ be a channel. Then Φ is sufficient with respect to $\{\rho,\sigma\}$ if and only if ρ is an invariant state of the channel $\Omega:=\Phi_\sigma\circ\Phi$.*

Since Ω possesses a faithful normal invariant state σ, the structure of its invariant states is known.

Theorem 2. *There is a unitary U and factorizations*

$$U\mathcal{H}=\bigoplus_n\mathcal{H}_n^L\otimes\mathcal{H}_n^R,\qquad \sigma=U^*\left(\bigoplus_n A_n^L\otimes\sigma_n^R\right)U,$$

where $A_n^L\in\mathcal{L}_1(\mathcal{H}_n^L)^+$ and $\sigma_n^R\in\mathfrak{S}(\mathcal{H}_n^R)$, such that for any $\rho\in\mathfrak{S}(\mathcal{H})$, Φ is sufficient with respect to $\{\sigma,\rho\}$ if and only if ρ can be factorized as

$$\rho=U^*\left(\bigoplus_n B_n^L\otimes\sigma_n^R\right)U$$

for some $B_n^L\in\mathcal{L}_1(\mathcal{H}_n^L)^+$.

6. Characterization of sufficient channels by D_α

Theorem 3. *Assume that $D_\alpha(\rho\|\sigma)<\infty$ for some $\alpha\in(0,1)\cup(1,\infty)$. Then the channel Φ is sufficient with respect to $\{\rho,\sigma\}$ if and only if*

$$D_\alpha(\Phi(\rho)\|\Phi(\sigma))=D_\alpha(\rho\|\sigma).$$

- For $\alpha\in(0,1)$, this was proved in [7].
- For $\alpha=2$, it is easy: equality in DPI implies:
$$\|\rho\|^2_{2,\sigma}=\|\Phi(\rho)\|^2_{2,\Phi(\sigma)}=\langle\,\rho,\Phi_\sigma\circ\Phi(\rho)\,\rangle\leq\|\rho\|^2_{2,\sigma}$$
so that $\rho=\Phi_\sigma\circ\Phi(\rho)$ by equality condition for Schwarz inequality.
- Assume the equality holds for some $1<\alpha<\infty$ and let
$$\rho=\sigma^{1/2\beta}\omega^{1/\alpha}\sigma^{1/2\beta}$$
for some $\omega\in\mathcal{L}_1(\mathcal{H})^+$. Let $S=\{z\in\mathbb{C},0\leq\Re(z)\leq 1\}$ and
$$\rho(z)=\|\omega\|_1^{(1/\alpha-z)}\sigma^{(1-z)/2}\omega^z\sigma^{(1-z)/2},\qquad z\in S$$
Then $z\mapsto\rho(z)$ is bounded and continuous on S, holomorphic in the interior, $\rho=\rho(1/\alpha)$ and we have
$$\|\Phi(\rho(\theta))\|_{1/\theta,\Phi(\sigma)}=\|\rho(\theta)\|_{1/\theta,\sigma},\qquad\forall\theta\in(0,1)$$
Let $\theta=1/2$ and put
$$\xi:=c\rho(1/2)=c_1\sigma^{1/4}\omega^{1/2}\sigma^{1/4}\in\mathfrak{S}(\mathcal{H}),$$
$c,c_1>0$ are normalization constants. Then
$$\|\Phi(\xi)\|_{2,\Phi(\sigma)}=\|\xi\|_{2,\sigma}$$
It follows that Φ is sufficient with respect to $\{\xi,\sigma\}$, which implies that ξ has a factorization as in Theorem 2. But then also ω, and, consequently, ρ has such a factorization. Hence Φ is sufficient with respect to $\{\rho,\sigma\}$.
- The converse is clear from the data processing inequality.

Remark 1. For $\dim(\mathcal{H})<\infty$, see [6]. Similar results can be obtained for pairs of normal states on arbitrary von Neumann algebras and normal unital completely positive maps.

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