Non-classical features in general probabilistic theories

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1. Introduction

General probabilistic theories (GPTs) describe physical models, including classical and quantum systems, in the framework of compact convex sets and their tensor products. GPTs share many non-classical features observed in quantum mechanics. We show how some of these features arise from the structure of the tensor products, which yelds their geometric characterization and explains some known relations between them, [1, 2, 4]. See [3] for details.

2. Basic definitions

- **state space**: compact convex subset K of a finite dimensional vector space. Set of states of a physical system.
- **effects**: affine maps $K \to [0,1]$, yes/no experiments.

> $E(K) \equiv$ all effects on K
> $A(K) \equiv$ all affine maps $K \to \mathbb{R}$
> $A(K)^+ \equiv$ the convex cone of positive maps in $A(K)$
> $V(K) := \cup_{\lambda>0}\lambda K \simeq$ positive functionals on $A(K)$.

A **classical state space** is an $n-1$-dimensional simplex S_n.
A **quantum state space** is the set

$$\mathfrak{S}(\mathcal{H}) := \{\rho \in B(\mathcal{H})^+, \mathrm{Tr}\,\rho = 1\}, \qquad \dim(\mathcal{H}) < \infty$$

of density operators on a Hilbert space, $A(\mathfrak{S}(\mathcal{H})) = B_h(\mathcal{H})$ (self-adjoint operators), $V(\mathfrak{S}(\mathcal{H})) \simeq A(\mathfrak{S}(\mathcal{H}))^+ = B(\mathcal{H})^+$.

- **channels**: affine maps $K \to K'$, physical devices.
- **measurements**: channels into the simplex $\mathcal{P}(\Omega)$ of probability measures on a finite set Ω. Given by collections $m_i \in E(K)$, $\sum_i m_i = 1$, where $m_i(x) = m(x)(\omega_i)$, $\omega_i \in \Omega$.
- **composite systems**: the state space $K_1\tilde{\otimes}K_2$ satisfies $K_1 \otimes K_2 \subseteq K_1\tilde{\otimes}K_2 \subseteq K_1\widehat{\otimes}K_2$, where

$$K_1 \otimes K_2 := \{\sum_j \lambda_j x_j^1 \otimes x_j^2,\ x_j^i \in K_i, \lambda_j \ge 0, \sum_j \lambda_j = 1\}$$

$$K_1\widehat{\otimes}K_2 := \{\varphi \in V(K_1) \otimes_{max} V(K_2),\ \langle \varphi, 1_{K_1\otimes K_2}\rangle = 1\}$$

- **separable states**: elements of $K_1 \otimes K_2$. All other states of the composite system $K_1\tilde{\otimes}K_2$ are **entangled**.

3. Semiclassical state spaces

A semiclassical state space is a product of simplices

$$\mathsf{S} = \mathsf{S}_{l_1,\dots,l_k} := \mathsf{S}_{l_1} \times \dots \times \mathsf{S}_{l_k}.$$

This is a convex polytope, with vertices

$$\mathsf{s}_{n_1,\dots,n_k} := (\epsilon^1_{n_1}, \dots, \epsilon^k_{n_k}), \quad n_i \in \{1,\dots,l_i\},$$

where ϵ^i_j are the vertices of S_{l_i}.

> **Observation**: Elements of S can be interpreted as conditional probabilities. Each projection $\mathsf{m}^i : \mathsf{S} \to \mathsf{S}_{l_i}$, $i = 1,\dots,k$ is a measurement. The extreme rays of $A(\mathsf{S})^+$ are generated by the corresponding effects m^i_j.

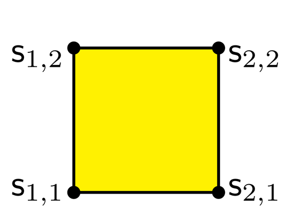
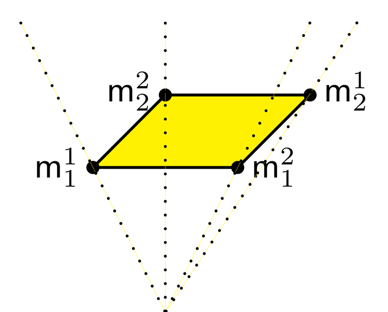


Figure 1: $\mathsf{S}_{2,2}$



$A(\mathsf{S}_{2,2})^+$

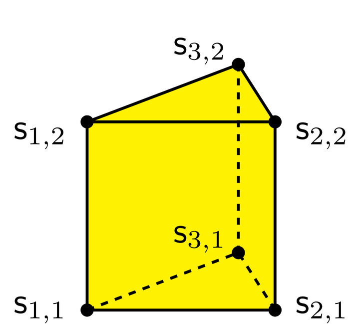
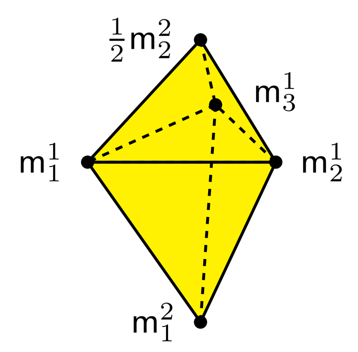


Figure 2: $\mathsf{S}_{3,2}$



a base of $A(\mathsf{S}_{3,2})^+$

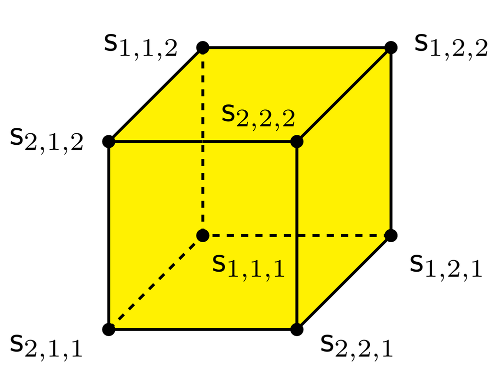
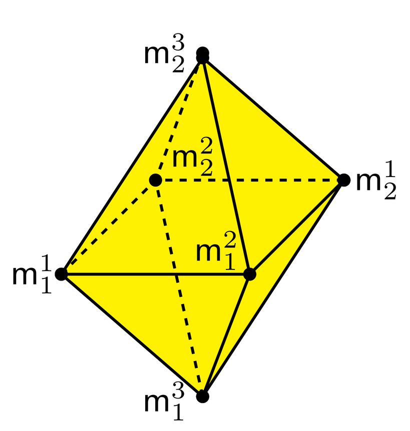


Figure 3: $\mathsf{S}_{2,2,2}$



a base of $A(\mathsf{S}_{2,2,2})^+$

4. Incompatible measurements

Let $\mathcal{M}_{l_1,\dots,l_k}(K)$ be the set of collections of measurements $M = \{f^1,\dots,f^k\}$, $f^i : K \to \mathsf{S}_{l_i}$. We say that M is **compatible** if there exists a **joint measurement** $g : K \to \mathsf{S}_{\Pi_i l_i}$, such that all f^i are its marginals:

$$f^i_j = \sum_{n_1,\dots,n_{i-1},n_{i+1},\dots,k} g_{n_1,\dots,n_{i-1},j,n_{i+1},\dots,n_k}.$$

> **Observation**: $M \in \mathcal{M}_{l_1,\dots,l_k}(K)$ corresponds to a channel $T_M : K \to \mathsf{S}_{l_1,\dots,l_k}$. There is some $\varphi_M \in V(\mathsf{S}) \otimes_{max} A(K)^+$, such that $f^i_j(x) = \langle T_M(x), \mathsf{m}^i_j\rangle = \langle \varphi_M, x \otimes \mathsf{m}^i_j\rangle$.

> **Theorem**
> M is compatible if and only if T_M is entanglement breaking if and only if $\varphi_M \in V(\mathsf{S}) \otimes_{min} A(K)^+$.

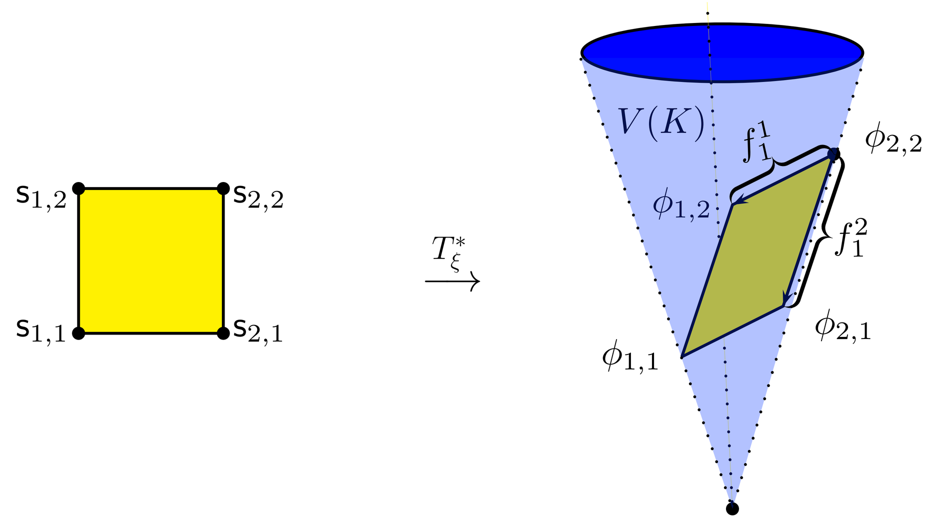
4.1 Incompatibility witnesses

M is incompatible if and only if there is an **incompatibility witness**: $\xi \in A(\mathsf{S})^+ \otimes_{max} V(K)$ such that $\langle \varphi_M, \xi\rangle < 0$.

> **A geometric description of witnesses**: for any ξ, there is a convex polytope in $V(K)$, which is the image of S under an affine map T^*_ξ. For a vertex $\mathsf{s}_{n_1,\dots,n_k}$, with adjacent edges $\mathsf{s}^i_j = \mathsf{s}_{n_1,\dots,j,\dots,n_k} - \mathsf{s}_{n_1,\dots,n_k}$, $j \ne n_i$ we have
>
> $$\langle \varphi_M, \xi\rangle = \langle 1_K, T^*_\xi(\mathsf{s}_{n_1,\dots,n_k})\rangle + \sum_i \sum_{j\ne n_i} \langle f^i_j, T^*_\xi(\mathsf{s}^i_j)\rangle.$$

Example 1. Any $(2,2)$-witness is a a parallelogram with vertices $\phi_{n_1,n_2} = T^*_\xi(\mathsf{s}_{n_1,n_2})$, satisfying $\phi_{1,1} + \phi_{2,2} = \phi_{1,2} + \phi_{2,1} =: \phi$. For the vertex $\phi_{2,2}$, we have

$$\langle \varphi_M, \xi\rangle = \langle 1_K, \phi_{2,2}\rangle + \langle f^1_1, \phi_{1,2} - \phi_{2,2}\rangle + \langle f^2_1, \phi_{2,1} - \phi_{2,2}\rangle$$



and ξ is a witness iff

$$\ell(\xi) := \|\phi_{1,2} - \phi_{2,2}\|_K + \|\phi_{2,1} - \phi_{2,2}\|_K > \langle 1_K, \phi\rangle,$$

$\|\cdot\|_K$ is the base norm. Similarly for k two-outcome measurements (the witnesses are $k \ge k'$-parallelepipeds).

4.2 Incompatibility degree

Let $M_u = \{\tau_1,\dots,\tau_k\}$ be a collection of **coin-tosses**: $\tau_i(x) \equiv \mu_i$, μ_i is the uniform distribution over Ω_i. The **incompatibility degree** of a collection M is defined as [2]

$$ID_u(M) := \inf\{0 < \lambda < 1,\ (1-\lambda)M + \lambda M_u \text{ is compatible}\}.$$

> **Proposition**
> Let $s_u(M) := \sup\{-\langle \varphi_M, \xi\rangle, \xi \text{ a witness }, \langle \varphi_{M_u}, \xi\rangle = 1\}$. Then
>
> $$ID_u(M) = \begin{cases} \frac{s_u(M)}{1+s_u(M)} & \text{if } s_u(M) \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

> **Maximal incompatibility degree** for $M \in \mathcal{M}_{2,\dots,2}(K)$:
>
> $$\max_{M\in\mathcal{M}_{2,\dots,2}(K)} ID_u(M) = 1 - \frac{1}{R}, \qquad R = \sup_\xi\{\ell(\xi)\}.$$
>
> The supremum in R is taken over all $k \ge k'$-parallelepipeds in $V(K)$ such that $\phi_{1,\dots,1} + \phi_{2,\dots,2} \in K$, $\ell(\xi)$ is the sum of $\|\cdot\|_K$-lenghts of edges adjacent to a vertex. In the quantum $(2,2)$-case, $R = \sqrt{2}$.

5. Steering

Let $y \in K\tilde{\otimes}K'$. A measurement $f : K \to \mathsf{S}_l$ maps y to a set of conditional states ϕ_j, $\sum_j \phi_j = y_{K'}$. Conditional states for a collection $M \in \mathcal{M}_{l_1,\dots,l_k}(K)$ form an **assemblage**

$$\{\phi^i_j \in V(K'), \sum_j \phi^i_j = \phi = y_{K'} \in K', \forall i\}.$$

An assemblage **does not certify steering** [5] if there is a (finite) set Λ of "classical messages", $p \in \mathcal{P}(\Lambda)$, $x_\lambda \in K'$ and conditional probabilities $p(j|i,\lambda)$ such that

$$\phi^i_j = \sum_\lambda p_\lambda p(j|i,\lambda)x_\lambda, \qquad i = 1,\dots,k,\ j = 1,\dots,l_i.$$

> **Observation**: $(T_M \otimes id)(y) \in \mathsf{S}\widehat{\otimes}K'$. There is a 1-1 correspondence between assemblages and elements of $\mathsf{S}\widehat{\otimes}K'$. An assemblage defines a set $\{\mu^i\}$ of probability measures on K' with a common barycenter ϕ.

> **Theorem**
> Let $\beta \in \mathsf{S}\widehat{\otimes}K'$. The following are equivalent.
> 1. β certifies steering.
> 2. β is entangled.
> 3. $\{\mu^i\}$ is not dominated in Choquet order.

Immediate consequences:

- Separable states are unsteerable.
- If M is compatible, T_M is entanglement breaking and $(T_M \otimes id)(y)$ does not certify steering.
- **Steering witnesses** can be described in the same way as incompatibility witnesses, replacing $V(K)$ by $A(K)^+$.

6. Bell's inequalities

> **Observation**: $\mathsf{S}\widehat{\otimes}\mathsf{S}' \simeq$ the set of conditional probabilities satisfying the **no-signaling** condition. Separable elements form the **local polytope**. For $M \in \mathcal{M}_{l_1,\dots,l_k}(K)$, $M' \in \mathcal{M}_{l'_1,\dots,l'_{k'}}(K')$ and $y \in K\tilde{\otimes}K'$, $(T_M \otimes T_{M'})(y) \in \mathsf{S}\widehat{\otimes}\mathsf{S}'$ describes the corresponding conditional probabilities.

Immediate consequences:

- M or M' compatible, or y separable, implies locality.
- **Bell's inequalities**: locality is determined by inequalities

$$\langle \mu, (T_M \otimes T_{M'})(y)\rangle = \langle (T^*_M \otimes T^*_{M'})(\mu), y\rangle \ge 0,$$

where $\mu \in A(\mathsf{S})^+ \otimes_{max} A(\mathsf{S}')^+$ is an extremal steering witness - a **Bell witness**.

Example 2. Any $(2,2),(2,2)$-Bell witness has the form

$$\mu_{i,j,k} = 1_\mathsf{S} \otimes \mathsf{m}^i_j + \mathsf{m}^k_1 \otimes (\mathsf{m}^{i'}_j - \mathsf{m}^i_j) + \mathsf{m}^{k'}_1 \otimes (\mathsf{m}^i_{j'} - \mathsf{m}^i_j)$$

where $i, i', j, j', k, k' = 1, 2$, $i \ne i'$, $j \ne j'$, $k \ne k'$. This gives the CHSH inequality.

6.1 Bell's inequalities and incompatibility

Let M be incompatible, μ a Bell witness. We have

$$\langle \mu, (T_M \otimes T_{M'})(y)\rangle = \langle \xi, \varphi_M\rangle,$$

where ξ is a witness, $\langle \xi, \varphi_{M_u}\rangle = 2$ and

$$T^*_\xi = T_y \circ T^*_{M'} \circ T^*_\mu. \tag{1}$$

It follows that maximal violation of Bell's inequality satisfies

$$\sup_{K'} \sup_{y\in K\tilde{\otimes}K'} \sup_{M'} -\langle \mu, (T_M \otimes T_{M'})(y)\rangle \le \frac{s_u(M)}{2}.$$

For CHSH, $T^*_{\mu_{i,j,k}} : V(\mathsf{S}_{2,2}) \to A(\mathsf{S}_{2,2})^+$ is an affine isomorphism. If also $A(K')^+ \simeq V(K)$, any witness has the form (1) and equality is attained (cf. [1]). But it is known that in quantum case Bell's inequalities cannot detect some forms of incompatibility [4].

> **Observation**: Unless $\mathsf{S} = \mathsf{S}' = \mathsf{S}_{2,2}$, $A(\mathsf{S})^* \not\simeq A(\mathsf{S}')$, so that there might be incompatibility witnesses not of the form (1).

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