Non-classical features in general probabilistic theories

Anna Jenčová

Mathematical Institute, Slovak Academy of Sciences
Bratislava, Slovakia
jencova@mat.savba.sk

In a class of general probabilistic theories (GPT), we characterize incompatibility of measurements, steering and Bell nonlocality as particular forms of entanglement. See [4] for details.

1. GPT: Basic definitions and assumptions

The basic ingredients in a GPT are states (preparations) and effects (yes/no experiments), assigning probabilities to each state [2]. We generally assume

- **convexity**:
  - a state space is a (compact) convex subset K of a finite dimensional vector space
  - effects: affine maps $K \to [0,1]$
  - channels (devices): affine maps $K \to K'$
- **local tomography**:
  - composite systems have tensor product structure: the compositie state space $K_1\tilde{\otimes}K_2$ satisfies

$$K_1 \otimes K_2 \subseteq K_1\tilde{\otimes}K_2 \subseteq K_1\hat{\otimes}K_2,$$

  - $K_1 \otimes K_2$ is the set of **separable states**
  - $K_1\hat{\otimes}K_2$ is the maximal tensor product
  - **entangled states** are elements of $K_1\tilde{\otimes}K_2 \setminus K_1 \otimes K_2$

Some notation:

- $E(K)$ = the set of effects
- $A(K)$ = all affine maps $K \to \mathbb{R}$
- $A(K)^+$ = the convex cone of positive maps in $A(K)$
- $V(K) := \cup_{\lambda>0}\lambda K \simeq$ positive functionals on $A(K)$

2. Examples

2.1 Classical state space

An n-dimensional simplex Δ_n. If K is a state space, a channel $f: K \to \Delta_n$ represents a **measurement** with values in $\{\omega_0, \omega_1, \dots, \omega_n\}$:

$$f(x) = \sum_i f_i(x)\delta_i, \qquad f_i \in E(K)$$

mapping $x \in K$ to outcome probabilities $f_0(x), \dots, f_n(x)$.

2.2 Quantum state space

The set

$$\mathfrak{S}(\mathcal{H}) := \{\rho \in B(\mathcal{H})^+, \mathrm{Tr}\,\rho = 1\}$$

of density operators on a Hilbert space, $\dim(\mathcal{H}) < \infty$.

2.3 Semiclassical state space

A product of simplices

$$\mathsf{S} = \mathsf{S}_{l_1,\dots,l_k} := \Delta_{l_1-1} \times \dots \times \Delta_{l_k-1}.$$

This state space will be useful below. Elements of S can be interpreted as conditional probabilities.

Observation: Each projection $\mathsf{m}^i : \mathsf{S} \to \Delta_{l_i-1}$ is a measurement. The corresponding effects m^i_j generate extreme the rays of $A(\mathsf{S})^+$.

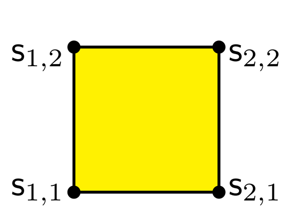
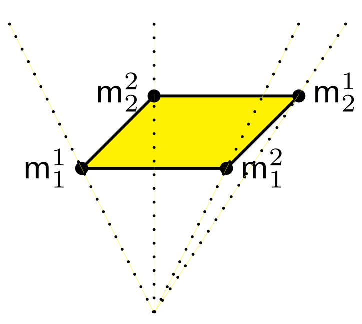


Figure 1: $\mathsf{S}_{2,2}$



$A(\mathsf{S}_{2,2})^+$

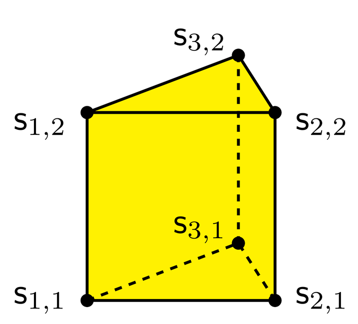
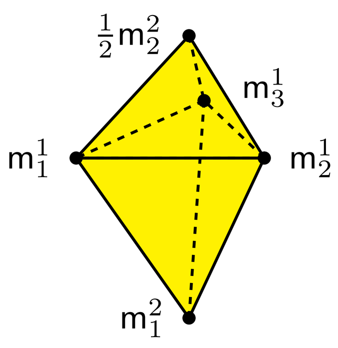


Figure 2: $\mathsf{S}_{3,2}$



a base of $A(\mathsf{S}_{3,2})^+$

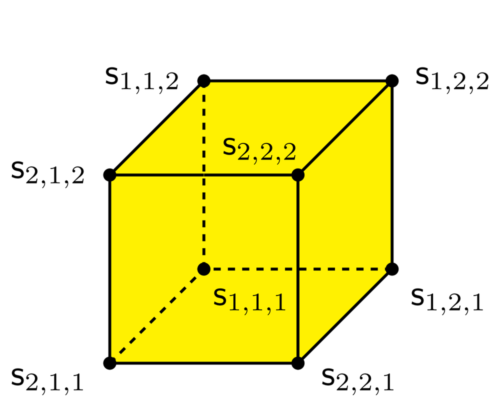
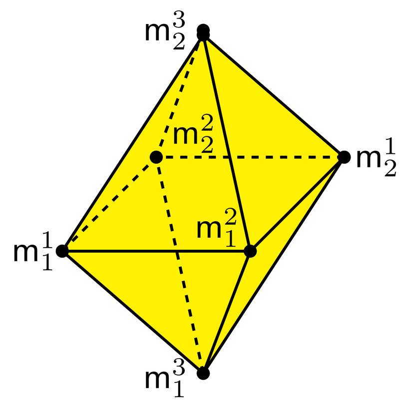


Figure 3: $\mathsf{S}_{2,2,2}$



a base of $A(\mathsf{S}_{2,2,2})^+$

3. Incompatible measurements

Let $\mathcal{F} = \{f^1, \dots, f^k\}$ be a collection of measurements $f^i : K \to \Delta_{l_i-1}$. We say that $\mathcal{F}$ is **compatible** if all f^i are marginals of a single **joint measurement** $g: K \to \Delta_{\Pi_i l_i - 1}$:

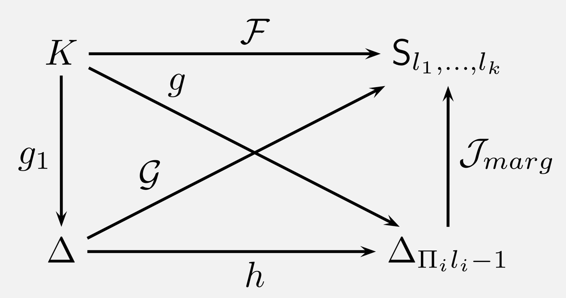
$$f^i_j = \sum_{n_1,\dots,n_{i-1},n_{i+1},\dots,k} g_{n_1,\dots,n_{i-1},j,n_{i+1},\dots,n_k}.$$

Observation: $\mathcal{F}$ defines a channel

$$\mathcal{F} = f^1 \times \dots \times f^k : K \to \mathsf{S}_{l_1,\dots,l_k}.$$

Theorem

$\mathcal{F}$ is compatible if and only if it is entanglement breaking:



3.1 Incompatibility witnesses

Let $x_i \in K$ be a basis of $A(K)^*$, $e_i \in A(K)$ a dual basis. For any linear map $\Phi : A(K)^* \to A(K)^*$, put

$$\mathrm{Tr}\,(\Phi) = \sum_i \langle e_i, \Phi(x_i)\rangle.$$

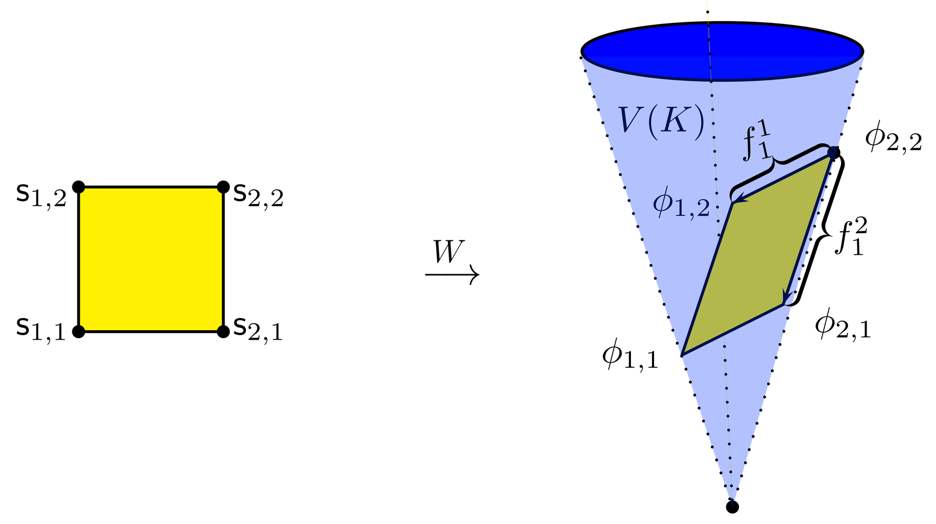
Theorem

$\mathcal{F}$ is incompatible if and only if there is an **incompatibility witness**: an affine map $W: \mathsf{S} \to V(K)$ such that

$$\mathrm{Tr}\,(W \circ \mathcal{F}) < 0.$$

Example 1. Let $\mathcal{F} = \{f^1, f^2\}$. A witness W maps $\mathsf{S}_{2,2}$ onto a parallelogram with vertices $\phi_{i,j} = W(\mathsf{s}_{i,j}) \in V(K)$ and

$$\mathrm{Tr}\,(W \circ \mathcal{F}) = \langle 1_K, \phi_{2,2}\rangle + \langle f^1_1, \phi_{1,2} - \phi_{2,2}\rangle + \langle f^2_1, \phi_{2,1} - \phi_{2,2}\rangle.$$



W is a witness iff

$$\|\phi_{1,2} - \phi_{2,2}\|_K + \|\phi_{2,1} - \phi_{2,2}\|_K > \langle 1_K, \phi\rangle,$$

$\|\cdot\|_K$ is the base norm.

3.2 Incompatibility degree

We can use witnesses to quantify incompatibility:

$$ID_{\mathcal{C}}(\mathcal{F}) := \max_{W\in\mathcal{C}} -\mathrm{Tr}\,(W \circ \mathcal{F}),$$

Putting

$$\mathcal{C} = \{W, \mathrm{Tr}\,(W \circ \mathcal{T}_{p,q}) = 1\},$$

where $\mathcal{T}_{p,q} : K \ni x \mapsto (p,q) \in \mathsf{S}_{2,2}$, we obtain known incompatibility degrees [3]. Note that $ID_{p,q}(\mathcal{F}) \leq 1/k$.

Theorem (maximal incompatibility)

Let $\mathcal{F}$ be a pair of two-outcome measurements. Then the incompatibility degree $ID_{p,q}(\mathcal{F}) = 1/2$ if and only if $\mathcal{F}$ is a **retraction**.

4. Steering in GPT

Similarly to quantum steering [6], we have the following scenario: A and B share an unknown state $y_{AB} \in K_A\tilde{\otimes}K_B$. A applies measurements from the collection $\mathcal{F}$, B knows the post-measurement states

$$y_{j|i} = (f^i_j \otimes id)(y_{AB}),$$

but not $\mathcal{F}$. The conditional states form an **assemblage**:

$$\{y_{j|i} \in K_B, \ \sum_j y_{j|i} = y_B, \forall i\}$$

We say that an assemblage **certifies steering** if it does not admit a local hidden state model

$$y_{j|i} = \sum_\lambda p(\lambda)p(j|i,\lambda)x_\lambda, \qquad x_\lambda \in K_B$$

Observation: All assemblages can be identified with elements of $\mathsf{S}\hat{\otimes}K_B$. The assemblage $\{y_{j|i}\}$ is obtained as $(\mathcal{F} \otimes id)(y_{AB})$.

Theorem

An assemblage $(\mathcal{F} \otimes id)(y_{AB})$ certifies steering iff it is entangled.

4.1 Steering witnesses

Any y_{AB} defines an affine map $T_y : A(K_B)^+ \to V(K_A)$.

Theorem

The assemblage $(\mathcal{F} \otimes id)(y_{AB})$ certifies steering iff there is a **steering witness** $W_{steer} : \mathsf{S} \to A(K_B)^+$, such that

$$\mathrm{Tr}\,(W_{steer} \circ \mathcal{F} \circ T_y) = \mathrm{Tr}\,(T_y \circ W_{steer} \circ \mathcal{F}) < 0.$$

Note that $T_y \circ W_{steer}$ must be an incompatibility witness.

5. Bell nonlocality

This time, both parties apply measurements from collections $\mathcal{F}_A$ and $\mathcal{F}_B$ to their respective systems and report the results. The state y_{AB} is **Bell nonlocal** if the obtained conditional probabilities cannot be explained by a local hidden variable model

$$p(a,b|i_A,i_B) = \sum_\lambda p(\lambda)p(a|i_A,\lambda)p(b|i_B,\lambda).$$

Observation: This is a particular case of steering: put $y'_{AB} := (id \otimes \mathcal{F}_B)(y_{AB}) \in K_A\hat{\otimes}\mathsf{S}_B$, then $(\mathcal{F}_A \otimes id)(y'_{AB}) = (\mathcal{F}_A \otimes \mathcal{F}_B)(y) \in \mathsf{S}_A\hat{\otimes}\mathsf{S}_B$ and y_{AB} is Bell nonlocal iff $(\mathcal{F}_A \otimes \mathcal{F}_B)(y) \notin \mathsf{S}_A \otimes \mathsf{S}_B$.

5.1 Bell witnesses and Bell's inequalities

A **Bell witness** is a steering witness in this special case: $W_{Bell} : \mathsf{S}_A \to A(\mathsf{S}_B)^+$. Bell's inequalities have the form

$$\mathrm{Tr}\,(T_{y'} \circ W_{Bell} \circ \mathcal{F}_A) = \mathrm{Tr}\,(T_y \circ \mathcal{F}^*_B \circ W_{Bell} \circ \mathcal{F}_A) \geq 0$$

Note that $W = T_y \circ \mathcal{F}^*_B \circ W_{Bell}$ must be an incompatibility witness for violation of this inequality.

Example 2. For CHSH, W_{Bell} is an affine isomorphism $V(\mathsf{S}_{2,2}) \to A(\mathsf{S}_{2,2})^+$. Therefore, CHSH violation is equivalent to incompatibility if also $V(K_A) \simeq A(K_B)^+$ (cf. [7, 1]).

In general, $V(\mathsf{S}_A) \not\simeq A(\mathsf{S}_B)^+$, (see Fig. 2 and 3). This might explain the fact that some incompatibility cannot be detected by violation of Bell's inequalities, [5].

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