Quantum versions of the classical randomization criterion

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Classical statistical experiments and randomizations

- Statistical experiment:

$$\mathcal{E} = (X, \{p_1, \dots, p_n\})$$

  probability distributions over a finite set X
- Randomization of $\mathcal{E}$: let $\mu : X \times Y \to [0,1]$ a Markov kernel,

$$\mu(\mathcal{E}) := (Y, \{\mu(p_1), \dots, \mu(p_n)\})$$

- Suppose $\mathcal{F} = (Y, \{q_1, \dots, q_n\})$, is it a randomization of $\mathcal{E}$?
- How far is $\mathcal{F}$ from a randomization of $\mathcal{E}$?

$$\delta(\mathcal{E}, \mathcal{F}) = \inf_{\mu} \sup_{i} \|\mu(p_i) - q_i\|_1$$

- Le Cam distance

$$\Delta(\mathcal{E}, \mathcal{F}) = \max\{\delta(\mathcal{E}, \mathcal{F}), \delta(\mathcal{F}, \mathcal{E})\}$$

Statistical decision problems

Decision problem: $(\mathcal{E}, D, w)$,

- $\mathcal{E}$ is an experiment
- D is a finite set of decisions
- $w : \{1, \dots, n\} \times D \to \mathbb{R}^+$ loss function
- (D, w) - (classical) decision space

Decision rule: a Markov kernel $\mu : X \times D \to [0, 1]$,
$\mu(x, d)$ - probability of choosing d if x was observed

Risk of μ at i:

$$R_{\mathcal{E}}(i, w, \mu) = \sum_{x,d} p_i(x)\mu(x, d)w(i, d)$$

Deficiency of experiments

Let $\mathcal{E} = (X, \{p_1, \ldots, p_n\})$, $\mathcal{F} = (Y, \{q_1, \ldots, q_n\})$, $\epsilon \geq 0$.

$\mathcal{E}$ is ϵ-deficient w.r. to $\mathcal{F}$, $\mathcal{E} \succeq_\epsilon \mathcal{F}$, if:

for any (D, w) and any Markov kernel $\mu : Y \times D \to [0, 1]$
there is a Markov kernel $\nu : X \times D \to [0, 1]$ such that

$$R_\mathcal{E}(i, w, \nu) \leq R_\mathcal{F}(i, w, \mu) + \epsilon \max_d w_{i,d}, \quad i = 1, \ldots, n$$

or, equivalently,

$$\sum_i R_\mathcal{E}(i, w, \nu) \leq \sum_i R_\mathcal{F}(i, w, \mu) + \epsilon \|w\|,$$

where $\|w\| = \sum_i \max_d w_{i,d}$.

The classical randomization criterion

Theorem (Blackwell, 1951)

$\mathcal{E} \succeq_0 \mathcal{F}$ *if and only if* $\mathcal{F}$ *is a randomization of* $\mathcal{E}$ *(*$\delta(\mathcal{E},\mathcal{F}) = 0$*).*

Theorem (Törgersen 1970)

$\mathcal{E} \succeq_\epsilon \mathcal{F}$ *if and only if* $\delta(\mathcal{E},\mathcal{F}) \leq 2\epsilon$*: there is a Markov kernel* $\lambda : X \times Y \to [0,1]$*, such that*

$$\|\lambda(p_i) - q_i\|_1 \leq 2\epsilon, \quad i = 1, \ldots, n$$

Quantum experiments and randomizations

Quantum statistical experiment

$$\mathcal{E} = (H, \{\rho_1, \dots, \rho_n\}),$$

$\dim(H) < \infty$, ρ_i are states (density matrices):

$$\rho_i \in \mathfrak{S}(H) = \{\rho \in B(H)^+, \operatorname{Tr} \rho = 1\}$$

Randomization: $\alpha : \mathfrak{S}(H) \to \mathfrak{S}(K)$ affine map,

$$\mathcal{E} \mapsto \alpha(\mathcal{E}) = \{\alpha(\rho_1), \dots, \alpha(\rho_n)\}$$

- α extends to a positive trace preserving map $B(H) \to B(K)$.
- α is required to be completely positive:

$\alpha \otimes id_L : B(H \otimes L) \to B(K \otimes L)-$ is positive for all $\dim(L) < \infty$

Quantum experiments and decision problems

Decision problem: $(\mathcal{E}, D, w)$,

- $\mathcal{E} = (H, \{\rho_1, \dots, \rho_n\})$
- (D, w) a classical decision space

Decision rule is a measurement (POVM)

$$\{M_d, d \in D\},\ M_d \in B(H)^+,\ \sum_d M_d = I$$

probability of choosing d at i: $\mathrm{Tr}\, M_d \rho_i$

Risk:

$$R_{\mathcal{E}}(i, w, M) = \sum_d w_{i,d} \mathrm{Tr}\, M_d \rho_i$$

Classical deficiency of quantum experiments

Let $\mathcal{E} = (H, \{\rho_1, \ldots, \rho_n\})$, $\mathcal{F} = (K, \{\sigma_1, \ldots, \sigma_n\})$, $\epsilon \geq 0$.

$\mathcal{E}$ is classically ϵ-deficient w.r. to $\mathcal{F}$, $\mathcal{E} \succeq_\epsilon^c \mathcal{F}$, if:

for any (D, w) and any POVM $\{M_d, d \in D\} \subset B(K)^+$
there is a POVM $\{N_d, d \in D\} \subset B(H)^+$ such that

$$\sum_\theta R_{\mathcal{E}}(i, w, N) \leq \sum_i R_{\mathcal{F}}(i, w, M) + \epsilon \|w\|,$$

where $\|w\| = \sum_i \max_d w_{i,d}$.

Quantum randomization criterion

We may define

$$\delta(\mathcal{E},\mathcal{F}) = \inf_{\alpha} \sup_{i} \|\alpha(\rho_i) - \sigma_i\|_1$$

inf taken over all channels. It is easy to see that

$$\delta(\mathcal{E},\mathcal{F}) \leq 2\epsilon \implies \mathcal{E} \succeq_{\epsilon}^{c} \mathcal{F}$$

The opposite is not true, even for positive α (Matsumoto)

Some special cases

The equivalence holds for

- abelian $\mathcal{F}$ (all σ_i commute)
- pairs of qubit states: $\dim(H) = \dim(K) = 2$, $n = 2$, $\epsilon = 0$
  an equivalent condition is

$$\|\rho_1 - t\rho_2\|_1 \geq \|\sigma_1 - t\sigma_2\|_1, \quad t \geq 0$$

  not true for $\dim > 2$ or $n > 2$

  (Alberti and Uhlmann, 1981)

Quantum Blackwell theorem

Theorem (Buscemi, 2012)

$\delta(\mathcal{E}, \mathcal{F}) = 0$ *if and only if* $\mathcal{E} \otimes \mathcal{E}_0 \succeq_0^c \mathcal{F} \otimes \mathcal{E}_0$, *where*

$$\mathcal{E} \otimes \mathcal{E}_0 = (H \otimes K, \{\rho_i \otimes \tau_j, i = 1, \ldots, n, j = 1, \ldots, d_K^2\})$$

$\mathrm{span}(\{\tau_j, j = 1, \ldots, d_K^2\}) = B(K)$.

Quantum decision spaces

Quantum decision space is a pair (D, W), where

- D is a Hilbert space, $\dim(D) < \infty$
- $W : \{1, \ldots, n\} \to B(D)^+$ loss function
- (D, W) is classical if W has commutative range

Let $\mathcal{E} = (H, \{\rho_i, \ldots, \rho_n\})$.
Quantum decision rule is a channel $\phi : B(H) \to B(D)$
Risk:

$$R_{\mathcal{E}}(i, W, \phi) = \mathrm{Tr}\, W_i \phi(\rho_i)$$

Quantum deficiency and randomization criterion

$\mathcal{E}$ is quantum ϵ-deficient w.r. to $\mathcal{F}$, $\mathcal{E} \succeq_\epsilon \mathcal{F}$, if:

for every quantum (D, W) and any channel $\phi : B(K) \to B(D)$, there is a channel $\psi : B(H) \to B(D)$ such that

$$\sum_i R_{\mathcal{E}}(i, W, \psi) \leq \sum_i R_{\mathcal{F}}(i, W, \phi) + \epsilon \|W\|,$$

$\|W\| = \sum_i \|W_i\|$.

Theorem (Matsumoto 2010)

$\mathcal{E} \succeq_\epsilon \mathcal{F}$ *if and only if*

$$\delta(\mathcal{E}, \mathcal{F}) = \inf_\alpha \max_i \|\alpha(\rho_i) - \sigma_i\|_1 \leq 2\epsilon,$$

Possible extensions

- positivity requirements for randomizations (positive, k-positive, entanglement-breaking,...)
- positivity requirements for decision rules
- decision problems for quantum operations: channels, combs, other protocols

Positive cones, bases and norms

- $(\mathcal{V}, P)$ real finite dim. ordered vector space,
- $(\mathcal{V}^*, P^*)$ - dual space with dual cone

$$P^* = \{p^* \in \mathcal{V}^*, \langle\, p^*, p \,\rangle \geq 0, \forall p \in P\}$$

- base of P: $S \subset P$ compact convex and such that for any $0 \neq p \in P$, $p = tb$, $t > 0$, $p \in S$ uniquely
- all bases of P have the form

$$S_e = \{p \in P, \langle\, e, p \,\rangle = 1\}$$

for some $e \in int(P^*)$

Base norm and order unit norm

Let $S = S_e$ be a base of P

- base norm

$$\|v\|_S = \inf\{\lambda + \mu, v = \lambda s_1 - \mu s_2, \lambda, \mu \geq 0, s_1, s_2 \in S\}$$

- order unit norm

$$\|f\|_S^* = \|f\|_e = \inf\{\lambda > 0, \lambda e \pm f \in P^*\}$$

Base sections and norms

- base section: $B = \mathcal{T} \cap S$, S a base of P, $\mathcal{T} \subseteq \mathcal{V}$ a subspace, $\mathcal{T} \cap int(P) \neq \emptyset$
- dual section: $\tilde{B} = \{\tilde{b} \in P^*, \langle \tilde{b}, b \rangle = 1, \ \forall b \in B\}$
- $\tilde{B}$ is a base section in P^*, $\tilde{\tilde{B}} = B$

We define

$$\mathcal{O}_B = \{p_1 - p_2, \ p_1, p_2 \in P, \ p_1 + p_2 \in B\}$$

Theorem

$\mathcal{O}_B$ is the unit ball of a norm $\|\cdot\|_B$ in $\mathcal{V}$. The dual norm is $\|\cdot\|_{\tilde{B}}$.

Properties

- If $B = S$ a base of P, then $\| \cdot \|_B$ is the base norm.
- If $B = \{b\}$, then $b \in int(P)$ and $\| \cdot \|_B = \| \cdot \|_b$ is the order unit norm.
- In general,

$$\|x\|_B = \inf_{b \in B \cap int(P)} \|x\|_b = \sup_{\substack{B \subseteq S \\ S \text{ is a base of } P}} \|x\|_S.$$

- If $x \in P$, $\|x\|_B = \sup_{\tilde{b} \in \tilde{B}} \langle \tilde{b}, x \rangle$

Discrimination of elements of base sections

- Suppose $b_1, b_2 \in B$ are given and $b \in \{b_1, b_2\}$ with prior probabilities λ, $1 - \lambda$.
- test (measurement): $m : B \to [0, 1]$ affine maps,

$$m : b \mapsto \mathrm{Prob}\{b_2 \text{ is chosen }\}$$

- given by $m \in P^*$, such that $\tilde{b} - m \in P^*$ for some $\tilde{b} \in \tilde{B}$ and

$$m(b) = \langle\, m, b \,\rangle$$

- Bayes error probability:

$$E(m) = \lambda \langle\, m, b_1 \,\rangle + (1 - \lambda) \langle\, \tilde{b} - m, b_2 \,\rangle$$

- minimum Bayes error probability

$$\min_m E(m) = \frac{1}{2}(1 - \|\lambda b_1 - (1 - \lambda) b_2\|_B)$$

Decision problems for elements of base sections

Let $\mathcal{E} = \{b_1, \ldots, b_n\} \subset B$, $\lambda_1, \ldots, \lambda_n$ prior probabilities

- (D, w) (classical) decision space
- decision function (measurement): affine map $m : B \to P(D)$

$$m : b \mapsto \{\text{probabilities of choosing } d, d \in D\}$$

- $m_d \in P^*$, $\sum_d m_d \in \tilde{B}$
- Bayes risk:

$$E(m) = \sum_i \lambda_i \sum_d w_{i,d} \langle\, m_d, b_i \,\rangle$$

- minimum Bayes risk

$$\min_m E(m) = \|w\| - \max_m \langle\, m, \bar{b}_{w'} \,\rangle = \|w\| - \|\bar{b}_{w'}\|_{B_D}$$

$B_D = \{(b, \ldots, b), b \in B\}$ is a base section in $P^{|D|}$,
$\bar{b}_{w'} \in P^{|D|}$.

States

- $\mathcal{V} = B_h(H) = \{X = X^* \in B(H)\} = \mathcal{V}^*$, $\langle X, Y \rangle = \mathrm{Tr}\, XY$
- $P = B(H)^+ = P^*$
- $B = \mathfrak{S}(H) = S_I$,
- $\|X\|_B = \|X\|_1 = \mathrm{Tr}\,|X|$
- $\|X\|_I = \|X\|$
- $\|\cdot\|_{B_D} = \|\cdot\|^\diamond$

The space of linear maps

- $\mathcal{V} = \mathcal{L}(H, K) = \{\phi : B(H) \to B(K),\ \phi(A^*) = \phi(A)^*\}$
- Define $s : \mathcal{L}(H, H) \to \mathbb{R}$,

$$s(\phi) = \sum_{i,j} \langle\, e_i, \phi(|e_i\,\rangle\langle\, e_j|)e_j\,\rangle, \qquad \phi \in \mathcal{L}(H, H)$$

- $\mathcal{V}^* = \mathcal{L}(K, H)$,

$$\langle\, \psi, \phi\,\rangle = s(\psi \circ \phi) = s(\phi \circ \psi), \quad \phi \in \mathcal{L}(H, K),\ \psi \in \mathcal{L}(K, H)$$

$$\langle\, \alpha \circ \psi, \phi\,\rangle = \langle\, \psi, \phi \circ \alpha\,\rangle$$

Quantum channels

- $P = CP(H, K)$ completely positive maps
- $\mathcal{C}(H, K)$ the set of channels is a base section in P
- the dual section is the set

$$\mathcal{S}(K, H) = \{B \mapsto (\mathrm{Tr}\, B)\sigma, \sigma \in \mathfrak{S}(H)\}$$

- measurements: (H_0, ρ, M), $\rho \in \mathfrak{S}(H \otimes H_0)$, M a POVM in $B(K \otimes H_0)$

$$m_d(\phi) = \mathrm{Tr}\, M_d(\phi \otimes id)(\rho)$$

- the base section norm

$$\|\phi\|_\diamond = \sup_{\rho \in \mathfrak{S}(H \otimes H)} \|(\phi \otimes id)(\rho)\|_1$$

- the dual norm $\|\cdot\|^\diamond$

Optimality of tests with max. entangled input state

Choi matrix: $C(\phi) = (\phi \otimes id)(|\psi_H\rangle\langle\psi_H|)$,
$|\psi_H\rangle$ maximally entangled state

Theorem (AJ, 2013)

Let $\Phi_1, \Phi_2 \in \mathcal{C}(H, K)$ and consider the symmetric hypothesis testing problem. Then there exists an optimal test $(H, |\psi_H\rangle\langle\psi_H|, M)$ with max. entangled input state if and only if

$$\mathrm{Tr}_K |C(\Phi_1) - C(\Phi_2)| \propto I$$

Quantum experiments and randomizations II

Let $\mathcal{E} = (H, \{\rho_1, \ldots, \rho_n\})$.

- cq-channel $\Phi^{cq}_{\mathcal{E}} : B(\mathbb{C}^n) \to B(H)$,

$$A \mapsto \sum_i A_{ii} \rho_i$$

- randomization: $\alpha \in \mathcal{C}(H, K)$,

$$\Phi^{cq}_{\alpha(\mathcal{E})} = \alpha \circ \Phi^{cq}_{\mathcal{E}}$$

- if $\mathcal{E}' = (H, \{\rho'_1, \ldots, \rho'_n\})$,

$$\|\Phi^{cq}_{\mathcal{E}} - \Phi^{cq}_{\mathcal{E}'}\|_\diamond = \sup_i \|\rho_i - \rho'_i\|_1$$

Quantum channels and post-processigs

- Let $\Phi \in \mathcal{C}(H, K)$
- post-processing: $\alpha \circ \Phi$, $\alpha \in \mathcal{C}(K, K')$
- for $\Psi \in \mathcal{C}(H, K')$,

$$\delta_{post}(\Phi, \Psi) := \inf_{\alpha} \|\alpha \circ \Phi - \Psi\|_{\diamond}$$

- post-processing Le Cam distance

$$\Delta_{post}(\Phi, \Psi) = \max\{\delta_{post}(\Phi, \Psi), \delta_{post}(\Psi, \Phi)\}$$

- for experiments

$$\delta_{post}(\Phi^{cq}_{\mathcal{E}}, \Phi^{cq}_{\mathcal{F}}) = \delta(\mathcal{E}, \mathcal{F})$$

Post-processing decision problems

Let $\Phi \in \mathcal{C}(H, K)$

- D a Hilbert space
- decision rule: post-processing $\psi \circ \Phi$, $\psi \in \mathcal{C}(K, D)$
- loss function: affine map $\mathcal{C}(H, \mathcal{D}) \to \mathbb{R}^+ \equiv \Gamma \in CP(D, H)$,
- (D, Γ) post-processing decision space, classical if Γ is a cq-map
- risk

$$\langle \psi \circ \Phi, \Gamma \rangle = \langle \psi, \Gamma \circ \Phi \rangle$$

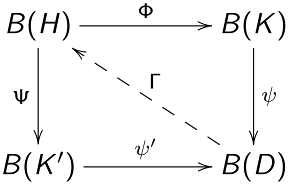
Post-processing deficiency

Let $\Phi \in \mathcal{C}(H, K)$, $\Psi \in \mathcal{C}(H, K')$, $\epsilon \geq 0$.

Definition

Post-processing deficiency: $\Phi \succeq_\epsilon \Psi$ if for every D, loss function $\Gamma \in CP(D, H)$ and $\psi' \in \mathcal{C}(K', D)$, there is some $\psi \in \mathcal{C}(K, D)$, such that

$$\langle \psi \circ \Phi, \Gamma \rangle \leq \langle \psi' \circ \Psi, \Gamma \rangle + \epsilon \|\Gamma\|^\diamond$$



Post-processing randomization theorem

Theorem

Let $\Phi \in \mathcal{C}(H, K)$, $\Psi \in \mathcal{C}(H, K')$, $\epsilon \geq 0$. The following are equivalent.

- $\Phi \succeq_\epsilon \Psi$
- *For every $\Gamma \in CP(K', H)$,*

$$\|\Psi \circ \Gamma\|^\diamond \leq \|\Phi \circ \Gamma\|^\diamond + \epsilon \|\Gamma\|^\diamond$$

- *There is a channel $\alpha \in \mathcal{C}(K, K')$ such that*

$$\|\alpha \circ \Phi - \Psi\|_\diamond \leq 2\epsilon$$

Quantum statistical experiments

Theorem

Let $\mathcal{E} = (H, \{\rho_1, \ldots, \rho_n\})$, $\mathcal{F} = (K, \{\sigma_1, \ldots, \sigma_n\})$. The following are equivalent.

- $\mathcal{E} \succeq_{\epsilon,\mathcal{P}} \mathcal{F}$
- *for any $W_1, \ldots, W_n \in B(K)^+$*

$$\|\Phi_{W,\mathcal{F}}\|_{\mathcal{P}}^{\diamond} \leq \|\Phi_{W,\mathcal{E}}\|_{\mathcal{P}}^{\diamond} + \epsilon\|W\|,$$

  where $\Phi_{W,\mathcal{E}}(A) = \sum_\theta \mathrm{Tr}\,(W_i A)\rho_i$, $\|W\| = \sum_i \|W_i\|$.
- *there is some $\alpha \in \mathcal{C}_{\mathcal{P}}(H, K)$ such that*

$$\sup_i \|\alpha(\rho_i) - \sigma_i\|_1 \leq 2\epsilon$$

Classical post-processing deficiency

Classical post-processing deficiency $\Phi \succeq_{\epsilon,cl} \Psi$: Restrict to classical decision spaces (Γ is a cq-map)

Theorem

The following are equivalent.

- $\Phi \succeq_{\epsilon,cl} \Psi$
- *for any finite sequence* $W_1, \ldots, W_n \subset B(H)^+$,

$$\|\Phi^{cq}_{\Phi(W)}\|^{\diamond} \leq \|\Phi^{cq}_{\Psi(W)}\|^{\diamond} + \epsilon \|\Phi^{cq}_{W}\|^{\diamond}$$

- *For any POVM* $M_1, \ldots, M_n \subset B(K)$ *there is a POVM* $N_1, \ldots, N_n \subset B(H)$ *such that*

$$\|N \circ \Phi - M \circ \Psi\|_{\diamond} \leq 2\epsilon$$

Classical and quantum post-processing deficiency

Theorem

Let $\Phi \in \mathcal{C}(H, K)$*,* $\Psi \in \mathcal{C}(H, K')$*.*

- *For any* $\epsilon \geq 0$*,* $\Phi \succeq_\epsilon \Psi$ *if and only if*

$$\Phi \otimes id_{K'} \succeq_{\epsilon, cl} \Psi \otimes id_{K'}$$

- *Let* $\xi \in \mathcal{C}(H_0, K')$ *be surjective. Then* $\Phi \succeq_0 \Psi$ *if and only if*

$$\Phi \otimes \xi \succeq_{0, cl} \Psi \otimes \xi$$

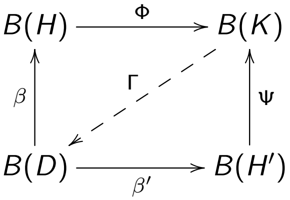
Pre-processing deficiency

pre-processings: $\Phi \circ \beta$, for some channel β.
Let $\Phi \in \mathcal{C}(H, K)$, $\Psi \in \mathcal{C}(H', K)$.

Definition

Pre-processing deficiency: $\Phi \succeq^{\epsilon} \Psi$ if for every D, loss function $\Gamma \in CP(K, D)$ and $\beta' \in \mathcal{C}(D, H)$, there is some $\beta \in \mathcal{C}(D, H')$, such that

$$\langle \Phi \circ \beta, \Gamma \rangle \leq \langle \Psi \circ \beta', \Gamma \rangle + \epsilon \|\Gamma\|^{\diamond}$$



Pre-processing randomization theorem

Theorem

Let $\Phi \in \mathcal{C}(H, K)$*,* $\Psi \in \mathcal{C}(H', K)$*,* $\epsilon \geq 0$*. The following are equivalent.*

- $\Phi \succeq^{\epsilon} \Psi$
- *For every* $\Gamma \in CP(K, H')$*,*

$$\|\Gamma \circ \Psi\|^{\diamond} \leq \|\Gamma \circ \Phi\|^{\diamond} + \epsilon \|\Gamma\|^{\diamond}$$

- *There is a channel* $\beta \in \mathcal{C}(H', H)$ *such that*

$$\|\Phi \circ \beta - \Psi\|_{\diamond} \leq 2\epsilon$$

Classical pre-processing deficiency

- (D, Γ) is classical if Γ is a qc-map.
- classical pre-processing deficiency: $\Phi \succeq^{\epsilon,cl} \Psi$
- $\mathcal{E} \succeq^{\epsilon,cl} \mathcal{F}$ iff $\mathcal{E} \supseteq_\epsilon \mathcal{F}$:

$$\sup_{\sigma \in \mathfrak{S}(H')} \inf_{\rho \in \mathfrak{S}(H)} \|\Psi(\sigma) - \Phi(\rho)\|_1 \leq 2\epsilon$$

Theorem

- $\Phi \succeq^0 \Psi$ *iff* $\Phi \otimes \xi \supseteq_0 \Psi \otimes \xi$ *for some injective* $\xi \in \mathcal{C}(H', K_0)$
- $\Phi \succeq^\epsilon \Psi$ *implies* $\Phi \otimes \xi \supseteq_\epsilon \Psi \otimes \xi$ *for any channel* ξ
- $\Phi \otimes id \supseteq_\epsilon \Psi \otimes id$ *implies* $\Phi \succeq^{\epsilon'} \Psi$*, for* $\epsilon' = \epsilon + \frac{1}{2}\sqrt{\epsilon}$*.*

Cleannes of POVMs (Buscemi et al. 2005)

Let $M = (M_1, \dots, M_m)$ be a POVM on H.

- A qc-channel

$$\Phi_M^{qc} : A \mapsto \sum_i \mathrm{Tr}\,(M_i A)|\,i\,\rangle\langle\, i|$$

- $\Phi_M^{qc} \circ \beta = \Phi_{\beta^*(M)}^{qc}$
- Let $N = (N_1, \dots, N_n)$ be a POVM on H, $\Phi_M^{qc} \succeq_0 \Phi_N^{qc}$ iff M is post-processing cleaner than N
- Let $N = (N_1, \dots, N_m)$ be a POVM on H', then $\Phi_M^{qc} \succeq^0 \Phi_N^{qc}$ iff M is pre-processing cleaner than N.

Randomization criterion for POVMs

Theorem

Let $M_1, \ldots, M_n \subset B(H)$, $N_1, \ldots, N_n \subset B(H')$. The following are equivalent.

- $M \succeq^{\epsilon} N$
- *For every $W_1, \ldots, W_n \subset B(H')^+$,*

$$\|\Phi_{M,W}\|^{\diamond} \leq \|\Phi_{N,W}\|^{\diamond} + \epsilon \|\Phi_W^{cq}\|^{\diamond}$$

- *there is some $\beta \in \mathcal{C}(H', H)$ such that*

$$\|\Phi^{qc}_{\beta^*(M)} - \Phi^{qc}_N\|_{\diamond} \leq 2\epsilon$$