Rényi relative entropies and noncommutative L_p-spaces

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Divergences in quantum information theory

A divergence is a measure of statistical "dissimilarity" of two (quantum) states

Operational significance

- relation to performance of some procedures in information theoretic tasks

Important properties

- strict positivity: $D(\rho\|\sigma) \geq 0$ and $D(\rho\|\sigma) = 0$ iff $\rho = \sigma$
- data processing inequality (DPI): for any channel T

$$D(\rho\|\sigma) \geq D(T(\rho)\|T(\sigma))$$

Channels and DPI

A channel T:

- transforms states to states: $\sigma \mapsto \sigma'$
- positive trace preserving linear map
- usually complete positivity is assumed:

$$T \otimes id_n \text{ is positive for all } n$$

- weaker positivity assumptions: 2-positive, Schwarz inequality:

$$T^*(a^*a) \geq T^*(a^*)T^*(a), \qquad \forall a.$$

DPI means that channels decrease distinguishability of states

Classical Rényi relative α-entropies

For p, q probability measures over a finite set X: (Rényi, 1961)

$$D_\alpha(p\|q) := \frac{1}{\alpha - 1} \log \sum_x p(x)^\alpha q(x)^{1-\alpha}, \quad 0 < \alpha \neq 1$$

- unique family of divergences satisfying a set of postulates
- fundamental quantities appearing in many information - theoretic tasks
- relative entropy as a limit:

$$\lim_{\alpha \to 1} D_\alpha(p\|q) = S(p\|q) := \sum_x p(x) \log\left(\frac{p(x)}{q(x)}\right).$$

Extensions for density matrices

Relative entropy: (Umegaki, 1962)

$$S(\rho\|\sigma) = \mathrm{Tr}\left[\rho(\log(\rho) - \log(\sigma))\right]$$

Standard Rényi relative entropies: (Petz, 1984)

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log \mathrm{Tr}\left[\rho^\alpha \sigma^{1-\alpha}\right], \quad 0 < \alpha \neq 1$$

- interpretation in hypothesis testing
- relative entropy as a limit

$$\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$$

- DPI holds for $\alpha \in (0, 2]$

Extensions for density matrices

Sandwiched Rényi relative entropies: (Müller-Lennert et al.,2013; Wilde et al., 2014)

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right],\ 0 < \alpha \neq 1$$

- interpretation in hypothesis testing
- relative entropy as a limit

$$\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho\|\sigma) = S(\rho\|\sigma)$$

- DPI holds for $\alpha \in [1/2, 1) \cup (1, \infty]$

Extensions to normal states of a von Neumann algebra

- Araki relative entropy: (Araki, 1976)
- Standard Rényi relative entropies (quasi-entropies, standard f-divergences): (Petz, 1985)
- Sandwiched Rényi relative entropies: (Berta et al., 2018; AJ 2018,2021)
- Other versions: other f-divergences (maximal, measured): (Hiai, 2021)

Outline

- Sandwiched Rényi relative entropies from noncommutative L_p-spaces

- Positive channels and data processing inequality

- Equality in DPI and reversibility of channels

Sandwiched Rényi relative entropies from noncommutative L_p-spaces

A general setting

- $\mathcal{M}$ a von Neumann algebra, $\mathcal{M}_*$ predual
- a $*$-representation $\pi : \mathcal{M} \to B(\mathcal{H})$
- ρ, σ normal states on $\mathcal{M}$ or $\rho, \sigma \in \mathcal{M}_*^+$ (σ faithful)
- $\boldsymbol{\rho} \in \mathcal{H}$ a vector representative of ρ:

$$\rho(a) = \langle\, \boldsymbol{\rho}, \pi(a)\boldsymbol{\rho}\, \rangle.$$

- the spatial derivative: $\Delta(\boldsymbol{\rho}/\sigma)$

The standard Rényi relative entropy

In this setting

- for $0 < \alpha \neq 1$:

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log\langle\, \boldsymbol{\rho}, \Delta(\boldsymbol{\rho}/\sigma)^{\alpha-1}\boldsymbol{\rho}\,\rangle$$

- Araki relative entropy as a limit value $\alpha \to 1$:

$$\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = S(\rho\|\sigma) = \langle\, \boldsymbol{\rho}, \log(\Delta(\boldsymbol{\rho}/\sigma)\boldsymbol{\rho}\,\rangle$$

- DPI holds for $\alpha \in (0, 2]$.

Sandwiched Rényi relative entropy

For $\alpha \in [1/2, 1) \cup (1, \infty]$: (Berta, Scholz & Tomamichel, 2018)

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \tilde{Q}_\alpha(\rho\|\sigma), \quad \tilde{Q}_\alpha(\rho\|\sigma) = (\|\boldsymbol{\rho}\|_{2\alpha,\sigma}^{(BST)})^{2\alpha},$$

with the Araki-Masuda L_p-norm (with respect to $\mathcal{M}'$):

$$\|\boldsymbol{\rho}\|_{p,\sigma}^{(BST)} = \begin{cases} \sup\limits_{\boldsymbol{\omega} \in \mathcal{H},\ \|\boldsymbol{\omega}\|=1} \|\Delta(\boldsymbol{\omega}/\sigma)^{1/2-1/p} \boldsymbol{\rho}\| & \text{if } 2 \leq p \leq \infty \\ \inf\limits_{\substack{\boldsymbol{\omega} \in \mathcal{H}, \|\boldsymbol{\omega}\|=1, \\ s(\rho') \leq s(\omega')}} \|\Delta(\boldsymbol{\omega}/\sigma)^{1/2-1/p} \boldsymbol{\rho}\| & \text{if } 1 \leq p < 2. \end{cases}$$

Sandwiched Rényi relative entropy

Properties:

Well defined (not depending on π or $\boldsymbol{\rho}$)

In finite dimensions:

$$\tilde{Q}_\alpha(\rho\|\sigma) = \mathrm{Tr}\left[\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right]$$

Relation to the standard Rényi relative entropies:

$$\tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma)$$

DPI for completely positive channels

Sandwiched Rényi relative entropy

Limit values:

Araki relative entropy:

$$\lim_{\alpha \to 1} \tilde{D}_\alpha(\rho \| \sigma) = S(\rho \| \sigma)$$

Max-relative entropy:

$$\lim_{\alpha \to \infty} \tilde{D}_\alpha(\rho \| \sigma) = D_{max}(\rho \| \sigma) = \log \inf\{\lambda > 0,\ \rho \leq \lambda \sigma\}$$

Uhlmann fidelity:

$$\lim_{\alpha \to 1/2} \tilde{D}_\alpha(\rho \| \sigma) = -\log F(\rho \| \sigma).$$

Haagerup L_p-spaces and a standard form

Haagerup L_p-spaces: $L_p(\mathcal{M})$, $1 \leq p \leq \infty$, the norm: $\|\cdot\|_p$

- $\mathcal{M} \simeq L_\infty(\mathcal{M})$,
- $\mathcal{M}_* \simeq L_1(\mathcal{M})$:

$$\psi \mapsto h_\psi, \qquad \mathrm{Tr}\,[h_\psi] = \psi(1),$$

- $L_2(\mathcal{M})$ a Hilbert space

$$(\xi, \eta) = \mathrm{Tr}\,[\xi^* \eta], \qquad \xi, \eta \in L_2(\mathcal{M})$$

Standard form: $(\lambda, L_2(\mathcal{M}), J, L_2(\mathcal{M})^+)$

$$\lambda(x)\xi = x\xi, \quad J\xi = \xi^*, \qquad x \in \mathcal{M},\ \xi \in L_2(\mathcal{M}).$$

$h_\rho^{1/2}$ - (unique) vector representative of $\rho \in \mathcal{M}_*^+$ in $L_2(\mathcal{M})^+$.

The Kosaki L_p-spaces

$\sigma \in \mathcal{M}_*^+$ (faithful), $\eta \in [0,1]$:

- a continuous embedding

$$\mathcal{M} \to \mathcal{M}^\eta \subseteq L_1(\mathcal{M}), \quad x \mapsto h_\sigma^\eta x h_\sigma^{(1-\eta)}$$

- interpolation spaces: for $1 \le p \le \infty$

$$L_p^\eta(\mathcal{M},\sigma) := C_{1/p}(\mathcal{M}^\eta, L_1(\mathcal{M})) \subseteq L_1(\mathcal{M})$$

- for $1/p + 1/q = 1$, the map

$$i_{p,\sigma}^\eta : k \mapsto h_\sigma^{\eta q} k h_\sigma^{(1-\eta)q}$$

is an isometric isomorphism of $L_p(\mathcal{M})$ onto $L_p^\eta(\mathcal{M},\sigma)$.

The Kosaki L_p-spaces

We will use:

The right L_p-spaces ($\eta = 1$):

$$L_p^R(\mathcal{M}, \sigma) = \{h_\sigma^{1/q} k, \ k \in L_p(\mathcal{M})\},$$

$$\text{the norm: } \|h_\sigma^{1/q} k\|_{p,\sigma}^R = \|k\|_p$$

The symmetric L_p-spaces ($\eta = 1/2$):

$$L_p(\mathcal{M}, \sigma) = \{h_\sigma^{1/2q} k h_\sigma^{1/2q}, \ k \in L_p(\mathcal{M})\},$$

$$\text{the norm: } \|h_\sigma^{1/2q} k h_\sigma^{1/2q}\|_{p,\sigma} = \|k\|_p$$

An expression via the symmetric L_p-spaces, $\alpha > 1$

Theorem

Let $\alpha > 1$, $\sigma, \rho \in \mathcal{M}_*^+$. Then $\tilde{Q}_\alpha(\rho\|\sigma) = \|h_\rho\|_{\alpha,\sigma}^\alpha$.

- finite if and only if for some $\mu \in \mathcal{M}_*^+$,

$$h_\rho = h_\sigma^{\frac{\alpha-1}{2\alpha}} h_\mu^{\frac{1}{\alpha}} h_\sigma^{\frac{\alpha-1}{2\alpha}}, \ \|h_\rho\|_{\alpha,\sigma}^\alpha = \mu(1)$$

- we put $\mu_\alpha(\rho\|\sigma) := \mu$, formally

$$h_\mu = (h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho h_\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha, \qquad \tilde{Q}_\alpha(\rho\|\sigma) = \mathrm{Tr}\,[h_\mu].$$

An expression for $\alpha \in [1/2, 1)$

Theorem

Let $\alpha \in [1/2, 1)$, $\sigma, \rho \in \mathcal{M}_*^+$. Then

$$\tilde{Q}_\alpha(\rho\|\sigma) = \|h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2}\|_{2\alpha}^{2\alpha}.$$

- always finite: $h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2} \in L_{2\alpha}(\mathcal{M})$
- we have $|h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2}| = h_\mu^{1/2\alpha}$ for some $\mu \in \mathcal{M}_+^*$, formally

$$h_\mu = (h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho h_\sigma^{\frac{1-\alpha}{2\alpha}})^\alpha, \qquad \tilde{Q}_\alpha(\rho\|\sigma) = \mathrm{Tr}\,[h_\mu]$$

- we put $\mu_\alpha(\rho\|\sigma) := \mu$

An expression via Kosaki right L_p-spaces

We use an embedding: $L_2(\mathcal{M}) \to L_1(\mathcal{M})$,

$$\xi \mapsto h_\sigma^{1/2}\xi, \qquad \xi \in L_2(\mathcal{M})$$

and the Kosaki right L_p-spaces $L_p^R(\mathcal{M}, \sigma)$:

Theorem

Let $\rho, \sigma \in \mathcal{M}_*^+$, $\alpha \in [1/2, 1) \cup (1, \infty]$. Then

$$\tilde{Q}_\alpha(\rho\|\sigma) = (\|h_\sigma^{1/2} h_\rho^{1/2}\|_{2\alpha,\sigma}^R)^{2\alpha}.$$

A variational expression

For $\alpha > 1$, $\gamma = \frac{\alpha}{\alpha - 1}$

$$\tilde{Q}_\alpha(\rho\|\sigma) = \sup_{x\in\mathcal{M}^+} \alpha\rho(x) - (\alpha-1)\mathrm{Tr}\left[(h_\sigma^{\frac{\alpha-1}{2\alpha}} x h_\sigma^{\frac{\alpha-1}{2\alpha}})^{\frac{\alpha}{\alpha-1}}\right]$$

$$= \sup_{x\in\mathcal{M}^+} \alpha\rho(x) - (\alpha-1)\|h_\sigma^{1/2} x h_\sigma^{1/2}\|_{\gamma,\sigma}^{\gamma}$$

In finite dimensions: (Frank & Lieb, 2013)

A variational expression

For $\alpha \in (1/2, 1)$, $\gamma = \frac{\alpha}{1-\alpha} > 1$: (Hiai, 2021)

$$\begin{aligned}
\tilde{Q}_\alpha(\rho\|\sigma) &= \inf_{x \in \mathcal{M}^{++}} \alpha\rho(x) + (1-\alpha)\mathrm{Tr}\left[\left(h_\sigma^{\frac{1-\alpha}{2\alpha}} x^{-1} h_\sigma^{\frac{1-\alpha}{2\alpha}}\right)^{\frac{\alpha}{1-\alpha}}\right] \\
&= \inf_{x \in \mathcal{M}^{++}} \alpha\rho(x) + (1-\alpha)\| h_\sigma^{1/2} x^{-1} h_\sigma^{1/2})\|_{\gamma,\sigma}^{\gamma}
\end{aligned}$$

In finite dimensions: (Frank & Lieb, 2013)

Some further properties of $\tilde{D}_\alpha$

Strict positivity: if $\rho(1) = \sigma(1)$

$$\tilde{D}_\alpha(\rho\|\sigma) \geq 0 \text{ with equality if and only if } \rho = \sigma$$

Strict monotonicity:

$$\alpha \mapsto \tilde{D}_\alpha(\rho\|\sigma) \text{ strictly increasing if } \rho \neq \sigma$$

Relations to standard Rényi relative entropies

$$D_{2-1/\alpha}(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma)$$

$\swarrow$ Hölder

$\searrow$ Hadamard 3 lines

Some further properties of $\tilde{D}_\alpha$

Joint lower semicontinuity (in the $L_1(\mathcal{M})$-norm topology)

Order relations:

$$
\begin{aligned}
\sigma_1 \leq \sigma_2 \quad & \Longrightarrow \tilde{D}_\alpha(\rho\|\sigma_1) \geq \tilde{D}_\alpha(\rho\|\sigma_2) \\
\rho_1 \leq \rho_2,\ \alpha > 1 \quad & \Longrightarrow \tilde{D}_\alpha(\rho_1\|\sigma) \leq \tilde{D}_\alpha(\rho_2\|\sigma) \\
\rho_1 \leq \rho_2,\ \alpha \in [1/2, 1) & \Longrightarrow \tilde{D}_\alpha(\rho_1\|\sigma) \geq \tilde{D}_\alpha(\rho_2\|\sigma)
\end{aligned}
$$

DPI for positive channels

Positive channels and data processing inequality

The Petz dual

$T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ positive trace preserving map

$T^* : \mathcal{N} \to \mathcal{M}$ (adjoint) positive, unital, normal map

Petz dual: (Petz, 1988) $T^*_\sigma : \mathcal{M} \to \mathcal{N}$, defined by

$$T(h_\sigma^{1/2} x h_\sigma^{1/2}) = h_{T(\sigma)}^{1/2} T^*_\sigma(x) h_{T(\sigma)}^{1/2}, \quad x \in \mathcal{M}$$

Properties:

- positive, normal, unital map
- n-positive if an only if T is n-positive

Petz recovery map:
$T_\sigma : L_1(\mathcal{N}) \to L_1(\mathcal{M})$, the preadjoint of T^*_σ

The Petz dual for T_σ is T^*.

The Petz dual

$$\begin{tikzcd}
L_1(\mathcal{M}) \arrow[rr, shift left, "T"] & & L_1(\mathcal{N}) \arrow[ll, shift left, "T_\sigma"] \\
L_\infty(\mathcal{M}, \sigma) \arrow[u, phantom, sloped, "\subseteq"] & & L_\infty(\mathcal{N}, T(\sigma)) \arrow[u, phantom, sloped, "\subseteq"] \\
\mathcal{M} \arrow[u, "i^{1/2}_{\infty,\sigma}"] \arrow[rr, shift left, "T_\sigma^*"] & & \mathcal{N} \arrow[u, "i^{1/2}_{\infty,T(\sigma)}"'] \arrow[ll, shift left, "T^*"]
\end{tikzcd}$$

In particular, $T(L_\infty(\mathcal{M}, \sigma)) \subseteq L_\infty(\mathcal{N}, T(\sigma))$ and

$$\|T(h_\sigma^{1/2} x h_\sigma^{1/2})\|_{\infty, T(\sigma)} = \|T_\sigma^*(x)\| \leq \|x\| = \|h_\sigma^{1/2} x h_\sigma^{1/2}\|_{\infty,\sigma}, \; x \in \mathcal{M}$$

> T defines a contraction $L_\infty(\mathcal{M}, \sigma) \to L_\infty(\mathcal{N}, T(\sigma))$.

DPI for positive channels, $\alpha > 1$

Proposition

T is a contraction $L_p(\mathcal{M}, \sigma) \to L_p(\mathcal{N}, T(\sigma))$, for all $1 \leq p \leq \infty$.

- True for $p = 1$: $\|Th\|_1 \leq \|h\|_1$ for all $h \in L_1(\mathcal{M})$.
- True for $p = \infty$.
- True for all p: Riesz-Thorin theorem.

DPI for positive channels, $\alpha \in [1/2, 1)$

From the variational formula: for any $y \in \mathcal{N}^{++}$, $\gamma = \frac{\alpha}{1-\alpha} \geq 1$:

$$
\begin{aligned}
\tilde{Q}_\alpha(\rho\|\sigma) &= \inf_{x \in \mathcal{M}^{++}} \alpha\rho(x) + (1-\alpha)\|h_\sigma^{1/2} x^{-1} h_\sigma^{1/2}\|_{\gamma,\sigma}^\gamma \\
&\leq \alpha\rho(T^*(y)) + (1-\alpha)\|h_\sigma^{1/2} T^*(y)^{-1} h_\sigma^{1/2}\|_{\gamma,\sigma}^\gamma \\
\text{(Choi inequality)} \quad &\leq \alpha T(\rho)(y) + (1-\alpha)\|h_\sigma^{1/2} T^*(y^{-1}) h_\sigma^{1/2}\|_{\gamma,\sigma}^\gamma \\
\text{(Petz dual)} \quad &= \alpha T(\rho)(y) + (1-\alpha)\| T_\sigma \left(h_{T(\sigma)}^{1/2} y^{-1} h_{T(\sigma)}^{1/2} \right) \|_{\gamma,\sigma}^\gamma \\
(T_\sigma \text{ a contraction}) \quad &\leq \alpha T(\rho)(y) + (1-\alpha)\|h_{T(\sigma)}^{1/2} y^{-1} h_{T(\sigma)}^{1/2}\|_{\gamma,T(\sigma)}^\gamma
\end{aligned}
$$

Taking inf over $y \in \mathcal{N}^{++}$: $\tilde{Q}_\alpha(\rho\|\sigma) \leq \tilde{Q}_\alpha(T(\rho)\|T(\sigma))$.

DPI for positive channels

Theorem

Let $\rho, \sigma \in \mathcal{M}_*^+$, $\alpha \in [1/2, 1) \cup (1, \infty]$. Then

$$\tilde{D}_\alpha(T(\rho) \| T(\sigma)) \leq \tilde{D}_\alpha(\rho \| \sigma)$$

for any positive trace preserving map $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$.

Taking the limit $\alpha \to 1$:

$$S(T(\rho) \| T(\sigma)) \leq S(\rho \| \sigma).^{a}$$

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[a] For $B(\mathcal{H})$: (Müller-Hermes & Reeb, 2017)

Equality in DPI and reversibility of channels

Reversibility of channels

A channel $T : L_1(\mathcal{M}) \to L_1(\mathcal{N})$ is reversible with respect to $\{\rho, \sigma\}$ if there is a recovery map: a channel $T' : L_1(\mathcal{N}) \to L_1(\mathcal{M})$ such that

$$T'T(\rho) = \rho, \qquad T'T(\sigma) = \sigma.$$

We from now on assume that a channel is a 2-positive trace preserving map.

Reversibility of channels

Reversibility problem:

Let D be a divergence (satisfies DPI), then reversibility implies

$$D(T(\rho)\|T(\sigma)) = D(\rho\|\sigma).$$

Is the converse true?

$$D(T(\rho)\|T(\sigma)) = D(\rho\|\sigma) < \infty \overset{?}{\Longrightarrow} T \text{ is reversible.}$$

Reversibility of channels

Theorem (Petz, 1986; 1988)

Assume that $S(\rho\|\sigma) < \infty$. Then a channel T is reversible with respect to $\{\rho, \sigma\}$ if and only if

$$S(\rho\|\sigma) = S(T(\rho)\|T(\sigma)).$$

Other divergences: (Hiai & Mosonyi, 2017; Hiai, 2021)

- true also for D_α, $\alpha \in (0,1) \cup (1,2)$
- not true for D_2 or $\tilde{D}_{1/2}$.

An application: SSA and Markov states

For $\mathcal{M} = B(\mathcal{H}_{ABC})$, $\mathcal{H}_{ABC}$ separable:

Strong subadditivity of entropy (SSA): (Lieb & Ruskai, 1973)

$$S(\omega_{ABC}) + S(\omega_B) \leq S(\omega_{AB} + S(\omega_{BC})$$

Equivalenty,

$$S(\omega_{AB} \| \omega_A \otimes \omega_B) \leq S(\omega_{ABC} \| \omega_A \otimes \omega_{BC}) \quad \text{(DPI for } T = \mathrm{Tr}\, _C)$$

Equality in SSA and short Markov chains: (Hayden et al., 2004)

$$\text{equality in SSA} \iff \omega_{ABC} = (id_A \otimes T')(\omega_{AB})$$

for some $T' : B \to BC$.

Universal recovery map

Note that T_σ is a channel and we always have $T_\sigma T(\sigma) = \sigma$.

Theorem (Petz, 1988)

T is reversible with respect to $\{\rho, \sigma\}$ if and only if

$$T_\sigma T(\rho) = \rho.$$

Mean ergodic theorem (Kümmerer & Nagel, 1979)

Let $E : \mathcal{M} \to \mathcal{M}$ be the conditional expectation onto the set of fixed points of $(T_\sigma T)^*$. Then

$$T_\sigma T(\rho) = \rho \iff E_*(\rho) = \rho.$$

Reversibility problem for $\tilde{D}_\alpha$, $\alpha > 1$

The problem can be reformulated as follows:

Let $p > 1$ and assume that $h_\rho \in L_p(\mathcal{M}, \sigma)$. Then

$$\|Th_\rho\|_{p,T(\sigma)} = \|h_\rho\|_{p,\sigma} \overset{?}{\iff} T_\sigma T h_\rho = h_\rho \iff E_* h_\rho = h_\rho.$$

Remarks:

- $\Longleftarrow$ holds by the fact that T_σ is a contraction.
- T_σ is the adjoint of T with respect to the duality of $L_p(\mathcal{M}, \sigma)$ and $L_q(\mathcal{M}, \sigma)$, $1/p + 1/q = 1$.
- $\iff$ holds for $p = 2$, since $L_2(\mathcal{M}, \sigma)$ is a Hilbert space.

Conditional expectations and L_p-spaces

Let $E : \mathcal{M} \to \mathcal{M}$ be a conditional expectation with range $\mathcal{M}_0$.
Then: (Junge & Xu, 2003)

- we may identify $L_p(\mathcal{M}_0) \subseteq L_p(\mathcal{M})$,
- E extends to a contractive projection E_p on $L_p(\mathcal{M})$ with range $L_p(\mathcal{M}_0)$
- for $h \in L_p(\mathcal{M}_0)$, $k \in L_q(\mathcal{M})$, $l \in L_r(\mathcal{M}_0)$, $p^{-1} + q^{-1} + r^{-1} = s^{-1} \leq 1$

$$E_s(hkl) = hE_q(k)l.$$

- If $E_*(\sigma) = \sigma$, then $E_*(L_p^\eta(\mathcal{M}, \sigma)) \simeq L_p^\eta(\mathcal{M}_0, \sigma|_{\mathcal{M}_0})$.

Reversibility problem for $\tilde{D}_\alpha$, $\alpha > 1$

Let $\mu = \mu_p(\rho\|\sigma) \in \mathcal{M}_*^+$: $h_\rho = h_\sigma^{1/2q} h_\mu^{1/p} h_\sigma^{1/2q}$ and put

$$h_t := h_\sigma^{(1-t)/2} h_\mu^t h_\sigma^{(1-t)/2} \in L_1(\mathcal{M})^+, \qquad t \in [0,1].$$

For $t \in (0,1)$, consider the equalities:

$$\|Th_t\|_{1/t,T(\sigma)} = \|h_t\|_{1/t,\sigma} \overset{?}{\iff} E_*(h_t) = h_t.$$

- Any of the equalities holds for some $t \in (0,1) \iff$ it holds for all $t \in [0,1]$
- the equalities are equivalent for $t = 1/2$

$\implies$ the equivalence holds for all $t \in (0,1)$.

Reversibility problem for $\tilde{D}_\alpha$, $\alpha \in (1/2, 1)$

Here $\mu = \mu_\alpha(\rho\|\sigma)$ is given by: $h_\mu = |h_\sigma^{\frac{1-\alpha}{2\alpha}} h_\rho^{1/2}|^{2\alpha}$

Using extended conditional expectations:

$$E_* h_\rho = h_\rho \iff E_* h_\mu = h_\mu.$$

We can show that

$$E_* h_\mu = h_\mu \iff \tilde{Q}_\alpha(T(\rho)\|T(\sigma)) = \tilde{Q}_\alpha(\rho\|\sigma)$$

using the variational formula for $\tilde{Q}_\alpha$.

Reversibility problem for $\tilde{D}_\alpha$, $\alpha \in (1/2, 1)$

If $\mu\sigma \le \rho \le \lambda\sigma$, we use

Lemma (Hiai, 2021)

Assume that $\mu\sigma \le \rho \le \lambda\sigma$ for some $\lambda, \mu > 0$, $\alpha \in (1/2, 1)$. Then

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{x \in \mathcal{M}^{++}} \alpha\rho(x) + (1-\alpha)\|h_\sigma^{1/2} x^{-1} h_\sigma^{1/2}\|_{\gamma,\sigma}^\gamma$$

is attained at a unique $x \in \mathcal{M}^{++}$ such that

$$h_\sigma^{1/2} x^{-1} h_\sigma^{1/2} = h_\sigma^{1/2\gamma^*} h_\mu^{1/\gamma} h_\sigma^{1/2\gamma^*}, \quad 1/\gamma + 1/\gamma^* = 1$$

In general: limit arguments and uniform convexity of $L_p(\mathcal{M})$.

Reversibility problem for $\tilde{D}_\alpha$

Theorem

Let $\alpha \in (1/2, 1) \cup (1, \infty)$. Assume that $\tilde{D}_\alpha(\rho\|\sigma) < \infty$. Then T is reversible with respect to $\{\rho, \sigma\}$ if and only if

$$\tilde{D}_\alpha(T(\rho)\|T(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma).$$

Thank you.