On the structure of higher order quantum maps

(arXiv:2411.09256)

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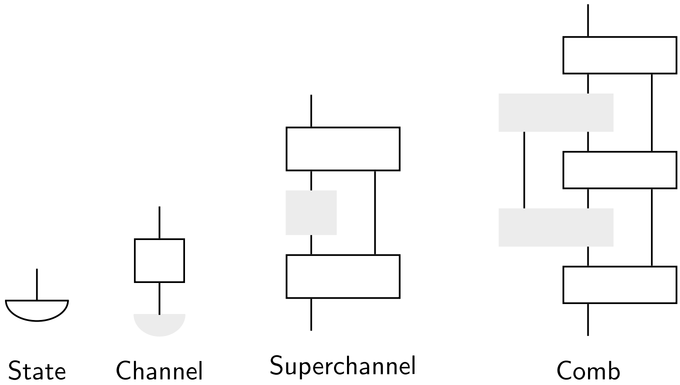
Higher order quantum maps

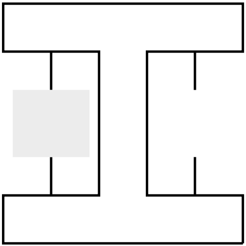
- hierarchy of "transformations between transformations"
- recursive construction:
  - starting from elementary objects (quantum states)
  - at each level, including all transformations between objects on lower levels
- admissibility: completely CP-preserving

Basic examples: states, channels and combs

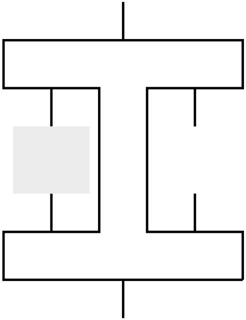


- Combs - Quantum circuits with holes
- definite causal order

Basic examples: process matrices



Process matrix 1



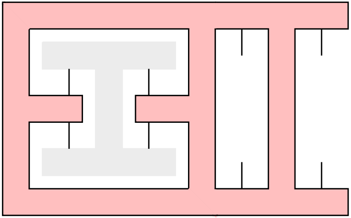
Process matrix 2

- indefinite causal order
- quantum switch

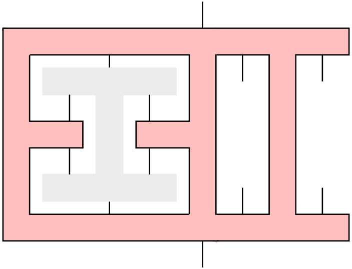
Oreshkov, Costa and Brukner, 2012
Chiribella et al., 2013

Basic examples: adaptors

Adaptors: transforming process matrices to process matrices



Adaptor 1



Adaptor 2

indefinite causal order

Types of HOM

Types of higher order maps:

- elementary types $\equiv$ quantum systems (states)
- for types x, y: $z = (x \to y) \equiv$ transformations x to y

Any type is thus constructed over a set of elementary types $A_1, \ldots, A_n$ - input and output systems of the maps

Examples:

- channels: $A_1 \to A_2$
- superchanels (2-combs): $(A_2 \to A_3) \to (A_1 \to A_4)$

Admissibility conditions - complete positivity

Bisio and Perinotti, 2019

Types of HOM

With the trivial system 1:

- dual: $\bar{x} \equiv (x \to 1)$
- tensor product: $x \otimes y \equiv \overline{(x \to \bar{y})}$

The hierarchy can be equivalently built using $\bar{\cdot}$ and $\otimes$:

$$(x \to y) = \overline{(x \otimes \bar{y})}.$$

Examples:

- no-signaling channels: $(A_1 \to A_2) \otimes (B_1 \to B_2)$
- process matrices $\overline{(A_1 \to A_2) \otimes (B_1 \to B_2)}$

Bisio and Perinotti, 2019

Choi operators of HOM types

Let $\mathcal{H} := \otimes_i \mathcal{H}_i$, $\mathcal{H}_1, \ldots, \mathcal{H}_n$ elementary types occurring in x

Choi operators of admissible (deterministic) maps of type x:

$$C \in B(\mathcal{H})^+$$
$$C \in S(x) \subseteq B_h(\mathcal{H}) \quad \text{a subspace}$$
$$\mathrm{tr}\,[C] = c_x, \quad c_x > 0.$$

Bisio and Perinotti, 2019

Combinatorial description of the subspaces

For $i = 1, \ldots, n$, we have an orthogonal decomposition

$$B_h(\mathcal{H}_i) = L_{i,0} \oplus L_{i,1}, \quad L_{i,0} := \mathbb{R} I_{\mathcal{H}_i}, \quad L_{i,1} := \{\mathrm{tr}\,[X] = 0\}.$$

For $s \in \{0,1\}^n$, define $L_s := \bigotimes_{i=1}^n L_{i,s_i}$, then

$$B_h(\mathcal{H}) = \bigoplus_{s \in \{0,1\}^n} L_s.$$

is an orthogonal decomposition labeled by binary strings.

Bisio and Perinotti, 2019

Combinatorial description of the subspaces

For any type x, there is a (unique) function $f : \{0,1\}^n \to \{0,1\}$, $f(00..0) = 1$, such that

$$S(x) = S_f := \sum_{s \in \{0,1\}} f(s) L_s, \qquad c_x = \Pi_{i \in I_f} \dim(\mathcal{H}_i)$$

where I_f is the set of indices of the input spaces.

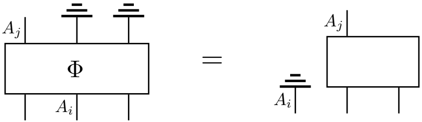
The inputs are characterized as

$$I_f := \{i \in [n] \colon f(e^i) = 0\}, \qquad e^i = 00..0\overset{i}{1}0..0$$

We put $O_f := [n] \setminus I_f$ the set of outputs.

Signaling structure and composition

No signaling from A_i to A_j $(i \not\rightsquigarrow_\Phi j)$:



We write $i \not\rightsquigarrow_x j$ if this holds for all maps of type x.

- Singnaling relations in x can be characterized by f
- Composition of HOM is restricted by signaling

Apadula, Bisio, Perinotti, 2024

Algebraic properties of functions and subspaces

Fix $n \in \mathbb{N}$ - the number of elementary types. Let

$$\mathcal{F}_n := \{f : \{0,1\}^n \to \{0,1\},\ f(00\dots00) = 1\}$$

- $\mathcal{F}_n$ is a Boolean algebra, with operations

$$\vee,\ \wedge,\ \top = 1_n,\ \bot = p_n := \chi_{00\dots0},\ f^* = 1 - f + p_n$$

- tensor product $\mathcal{F}_n \times \mathcal{F}_m \to \mathcal{F}_{m+n}$
- this structure is preserved by $f \mapsto S_f$:

$$S_{f\vee g} = S_f \vee S_g,\ S_{f\wedge g} = S_f \wedge S_g,\ S_{1_n} = B_h(\mathcal{H}),\ S_{p_n} = \mathbb{R}I_{\mathcal{H}},$$
$$S_{f^*} = S_f^{\perp} \vee \mathbb{R}I_{\mathcal{H}},\ S_{f\otimes g} = S_f \otimes S_g$$

- $\{S_f, f \in \mathcal{F}_n\}$ is a Boolean algebra.

The type functions

We define a subset of type functions $\mathcal{T}_n \subseteq \mathcal{F}_n$:

- $f \in \mathcal{F}_n$ such that $S_f = S(x)$ for some type x
- functions constructed from $1(s_1), \ldots, 1(s_n)$ by taking tensor products and complements, in any order
- smallest subsystem $\{\mathcal{T}_n\}_n$ of functions with $\mathcal{T}_1 = \mathcal{F}_1$, closed under complements, tensor products and permutations

We will investigate the structure of type functions and the related structure of HOM.

Example: channels

For $T \subseteq [n]$, put $p_T(s) := \Pi_{i \in T}(1 - s_i)$.

- $p_T \in \mathcal{T}_n$,
- $S_{p_T} = I_{\mathcal{H}_T} \otimes B_h(\mathcal{H}_{[n]\setminus T})$, $\mathcal{H}_T := \otimes_{i \in T} \mathcal{H}_i$,

  replacement channels $B(\mathcal{H}_T) \to B(\mathcal{H}_{[n]\setminus T})$
- $p_T^* = 1_n - p_T + p_n \in \mathcal{T}_n$,

  channels $B(\mathcal{H}_{[n]\setminus T}) \to B(\mathcal{H}_T)$

For any $f \in \mathcal{T}_n$, we have

$$p_{I_f} \leq f \leq p_{O_f}^*.$$

Subtype functions

Let $I \subseteq [n]$. Any $f \in \mathcal{F}_n$ satisfying

$$p_I \leq f \leq p^*_{[n]\setminus I}$$

is called a subtype function.

- convex subsets of channels containing all replacement channels
- for fixed I, subtype functions form a distributive lattice
- contain all type functions f, g with inputs in I, as well as $f \vee g$, $f \wedge g$ (intersections/affine combinations of HOM types)

The Möbius transform and $\mathcal{P}_f$

Any $f \in \mathcal{F}_n$ has a unique expression of the form

$$f = \sum_{T \subseteq [n]} \hat{f}_T p_T, \qquad \hat{f}_T \in \mathbb{Z}.$$

Let $\mathcal{P}_f$ be the poset $\{T \subseteq [n],\ \hat{f}_T \neq 0\}$, ordered by inclusion.

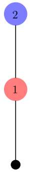
If $f \in \mathcal{T}_n$, then $\mathcal{P}_f$ is a graded poset with even rank and

$$f = \sum_{T \in \mathcal{P}_f} (-1)^{\rho(T)} p_T,$$

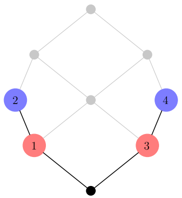
here ρ is the rank function of $\mathcal{P}_f$.

The labeled Hasse diagram

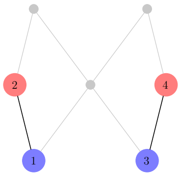
We can visualise $\mathcal{P}_f$ by its Hasse diagram, with vertices labeled by indices not present in any vertex below:



Channel



Non-signalling channel



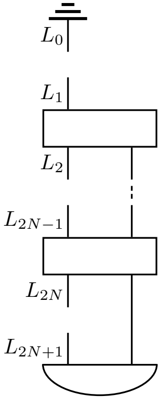
Process matrix 1

- Input indices $\equiv$ labels of elements with even rank (blue)
- Output indices (red)
- Black dot $\equiv \emptyset$.

Combs

Theorem

Let $f \in \mathcal{F}_n$. Then f is a type function describing combs if and only if $\mathcal{P}_f$ is a chain of even length.



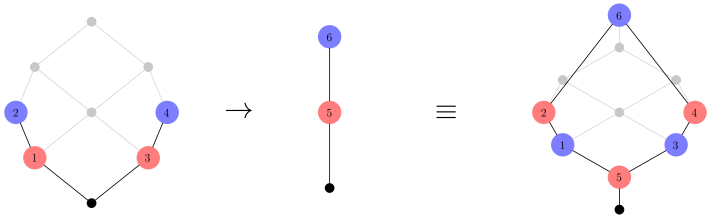
- $L_{2N+1} := [n] \setminus (\cup_j L_j)$
- If $L_0 = \emptyset$ or $L_{2N+1} = \emptyset$, the corresponding spaces are trivial.

Operations on f and $\mathcal{P}_f$

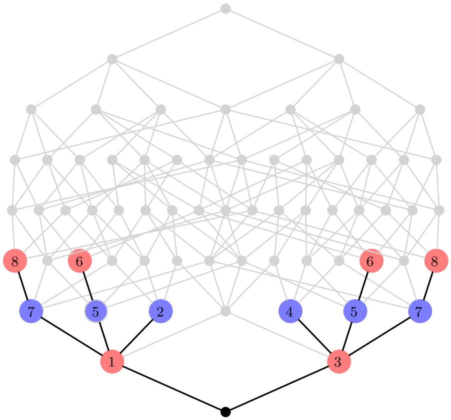
Let $f, g \in \mathcal{T}_n$. Then

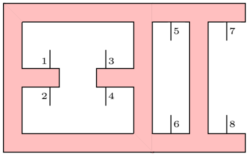
- $\mathcal{P}_{f \otimes g} = \mathcal{P}_f \times \mathcal{P}_g$ direct product of posets
- $\mathcal{P}_{f^*}$ is obtained from $\mathcal{P}_f$ by adding/removing $\emptyset$ and $[n]$.
- $(f \to g) = (f \otimes g^*)^*$.

Example: process matrix 2



Example: adaptor 1





The poset $\mathcal{P}_f^0$

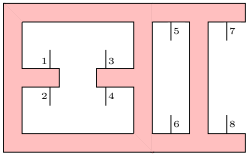
We define $\mathcal{P}_f^0$ as the subposet consisting only of labeled elements and the empty set (black dot) if present.

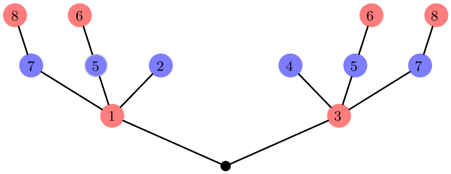
Theorem

The function $f \in \mathcal{T}_n$ is fully determined by the poset $\mathcal{P}_f^0$.

No longer a graded poset.

Example: adaptor 1





Signaling in $\mathcal{P}_f^0$

Theorem

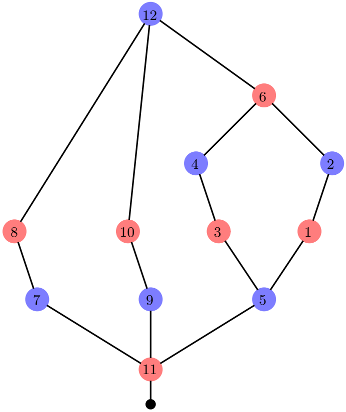
Let $f \in \mathcal{T}_n$, $i \in I_f$, $j \in O_f$. Then

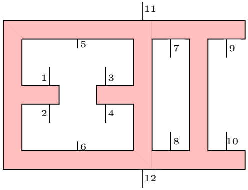
- If j is not a label, then $i \not\rightsquigarrow_f j$.
- if there are comparable vertices $S, T \in \mathcal{P}_f^0$ such that $i \in L_S$ and $j \in L_T$, then

$$i \not\rightsquigarrow_f j \iff S \leq T.$$

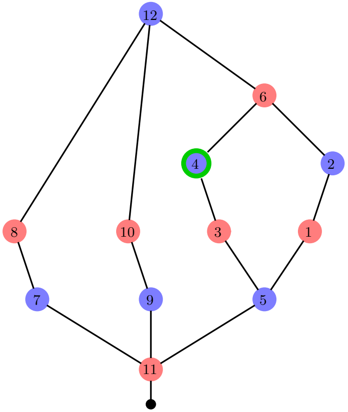
- otherwise, $i \not\rightsquigarrow_f j$ iff
  - $\exists\ S, T$ with $i \in L_S$ and $j \in L_T$, connected by a chain to the same minimal element in $\mathcal{P}_f^0$
  - each such connecting pair of chains has an intersection of even length.

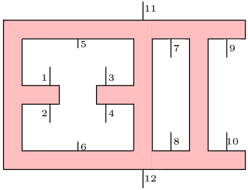
Example: adaptor 2





Example: adaptor 2, signaling





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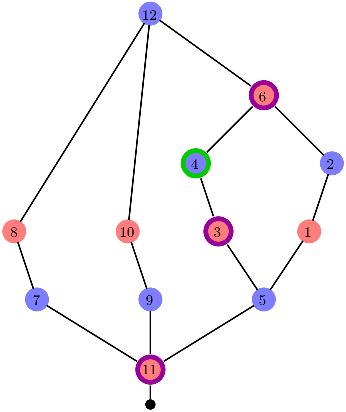

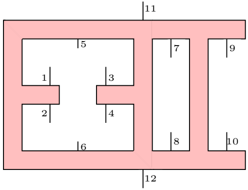

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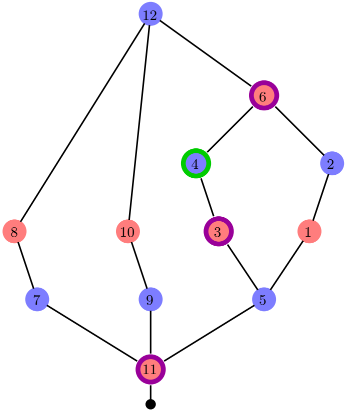
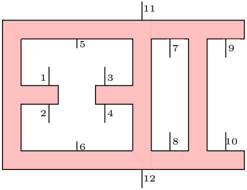
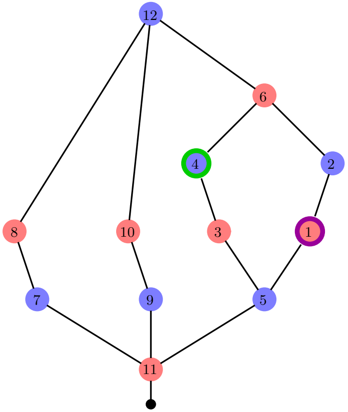
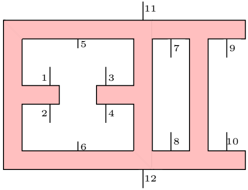




$$4 \not\rightsquigarrow 6,\ 4 \rightsquigarrow 3,\ 4 \rightsquigarrow 11$$

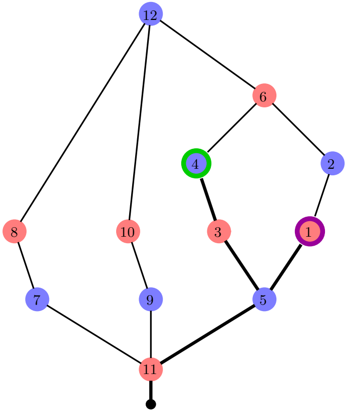
Example: adaptor 2, signaling

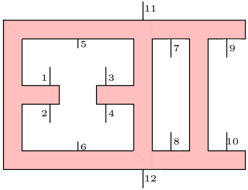




$$4 \not\rightsquigarrow 6,\ 4 \rightsquigarrow 3,\ 4 \rightsquigarrow 11$$

Example: adaptor 2, signaling

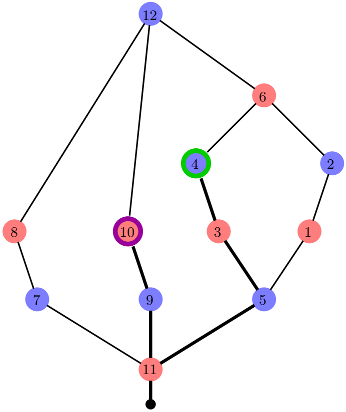


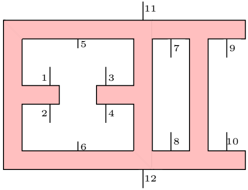


$4 \not\rightsquigarrow 6,\ 4 \rightsquigarrow 3,\ 4 \rightsquigarrow 11$

$4 \not\rightsquigarrow 1$

Example: adaptor 2, signaling

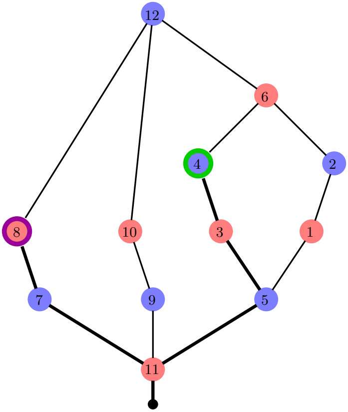


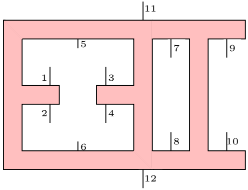


$4 \not\rightsquigarrow 6,\ 4 \rightsquigarrow 3,\ 4 \rightsquigarrow 11$

$4 \not\rightsquigarrow 1,\ 4 \rightsquigarrow 10$

Example: adaptor 2, signaling





$$4 \not\rightsquigarrow 6,\ 4 \rightsquigarrow 3,\ 4 \rightsquigarrow 11$$

$$4 \not\rightsquigarrow 1,\ 4 \rightsquigarrow 10,\ 4 \rightsquigarrow 8$$

The causal product

For $f \in \mathcal{F}_m, g \in \mathcal{F}_n$, we define causal products in $\mathcal{F}_{m+n}$ as

$$f_1 \lhd f_2 := f_1 \otimes 1_n + p_m \otimes (f_2 - 1_n)$$
$$f_2 \lhd f_1 := 1_m \otimes f_2 + (f_1 - 1_m) \otimes p_n.$$

- $S_{f \lhd g} = S_g \prec S_f$ - the 'prec' operator

  - Hoffreumon, Oreshkov, 2022

- for type functions:
  - $f \lhd g$ is a type function iff f or g is a chain
  - $p_{I_f \cup I_g} \leq f \lhd g \leq p^*_{O_f \cup O_g}$
- $\mathcal{P}_{f \lhd g}$ is (almost) the ordinal sum of $\mathcal{P}_f$ and $\mathcal{P}_g$
- connection of chains

Structure theorem/normal form[1]

Theorem

Let $f \in \mathcal{T}_n$. There are

- chains $\beta_1 \in \mathcal{T}_{n_1}, \ldots, \beta_k \in \mathcal{T}_{n_k}$
- permutations $\pi_{a,b}$ on $[k]$, $a \in A$, $b \in B$

such that

$$f = \bigvee_{a \in A} \bigwedge_{b \in B} \left(\beta_{\pi_{a,b}^{-1}(1)} \lhd \ldots \lhd \beta_{\pi_{a,b}^{-1}(k)}\right)$$

This form can be found from $\mathcal{P}_f^0$.

[1] Hoffreumon, Oreshkov, 2022

Conclusions/outlook

Conclusions

- Hom types – posets – labeled Hasse diagrams
- signaling structure and normal forms

Outlook

- Efficient ways to construct the normal form from $\mathcal{P}_f^0$ or $\mathcal{P}_f$
- Relate properties/constructions of postets to those of HOMs
- different types of signaling?
- admissible composition of HOMs