Some characterizations of reversibility of quantum channels

Anna Jenčová

Mathematical Institute, Slovak Academy of Sciences, Bratislava, Slovakia

Rényi Institute, Budapest, 2023

Reversible (sufficient) quantum channels

Let $\mathcal{S}$ be a set of quantum states, Φ a quantum channel.

We say that Φ is reversible (sufficient) with respect to $\mathcal{S}$ if there exists some channel Ψ (recovery channel) such that

$$\Psi \circ \Phi(\rho) = \rho, \qquad \rho \in \mathcal{S}.$$

D. Petz, Commun. Math. Phys., 1986

D. Petz, The Quarterly J. of Math., 1988

The setting and assumptions

$B(\mathcal{H})$ - operators on a finite dimensional Hilbert space $\mathcal{H}$

- A set of states

$$\mathcal{S} \subset \{\rho \in B(\mathcal{H}),\ \rho \geq 0,\ \mathrm{Tr}\ \rho = 1\}$$

- A channel $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$, completely positive and trace preserving

Assumptions:

There is a faithful (full rank) state $\sigma \in \mathcal{S}$, its image $\Phi(\sigma) \in B(\mathcal{K})$ is also faithful.

Preservation of the relative entropy

The relative entropy: for states ρ, σ

$$D(\rho\|\sigma) = \begin{cases} \mathrm{Tr}\left[\rho(\log(\rho) - \log(\sigma))\right], & \mathrm{supp}(\rho) \leq \mathrm{supp}(\sigma) \\ \\ \infty, & \text{otherwise.} \end{cases}$$

- Data processing inequality: for a channel Φ

$$D(\Phi(\rho)\|\Phi(\sigma)) \leq D(\rho\|\sigma),$$

- If $D(\rho\|\sigma) < \infty$, then reversibility is equivalent to

$$D(\Phi(\rho)\|\Phi(\sigma)) = D(\rho\|\sigma), \qquad \forall \rho \in \mathcal{S}.$$

D. Petz, Commun. Math. Phys., 1986

Universal recovery map

The Petz dual of Φ with respect to σ

$$\Phi_\sigma(\cdot) = \sigma^{1/2}\Phi^*(\Phi(\sigma)^{-1/2} \cdot \Phi(\sigma)^{-1/2})\sigma^{1/2}$$

- Φ_σ is a channel $B(\mathcal{K}) \to B(\mathcal{H})$ such that

$$\Phi_\sigma \circ \Phi(\sigma) = \sigma$$

- Φ is reversible with respect to $\mathcal{S}$ if and only if

$$\Phi_\sigma \circ \Phi(\rho) = \rho, \qquad \forall \rho \in \mathcal{S}$$

D. Petz, The Quarterly J. of Math., 1988

Algebraic conditions for reversibility

Semigroup of channels preserving $\mathcal{S}$

Let us consider the set of channels

$$\mathcal{C}_{\mathcal{S}} := \{\Theta : B(\mathcal{H}) \to B(\mathcal{H}),\ \Theta(\rho) = \rho,\ \forall \rho \in \mathcal{S}\}$$

- convex and compact semigroup (closed under composition)
- has a faithful fixed state: $\sigma \in \mathcal{S}$.

By the mean ergodic theorem, there is some $\mathcal{E}_{\mathcal{S}} \in \mathcal{C}_{\mathcal{S}}$ such that

$$\mathcal{E}_{\mathcal{S}} \circ \Theta = \Theta \circ \mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{S}}, \qquad \forall \Theta \in \mathcal{C}_{\mathcal{S}}.$$

B. Kümmerer, R. Nagel, Acta Sci. Math., 1979

We see that such $\mathcal{E}_{\mathcal{S}}$ is unique and

$$\mathcal{E}_{\mathcal{S}}^2 = \mathcal{E}_{\mathcal{S}}, \qquad \mathcal{E}_{\mathcal{S}}(\rho) = \rho, \quad \forall \rho \in \mathcal{S}.$$

The minimal sufficient subalgebra

The adjoint $\mathcal{E}_{\mathcal{S}}^*$ is a faithful conditional expectation

$$\Longrightarrow$$

its range is a subalgebra $\mathcal{M}_{\mathcal{S}} := \mathcal{E}_{\mathcal{S}}^*(B(\mathcal{H}))$.

$\mathcal{M}_{\mathcal{S}}$ is the minimal sufficient subalgebra with respect to $\mathcal{S}$:

- $\rho \mapsto \rho|_{\mathcal{M}_{\mathcal{S}}}$ is a channel reversible with respect to $\mathcal{S}$
- $\mathcal{M}_{\mathcal{S}}$ is contained in any subalgebra with this property.

The range of a conditional expectation

Let $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H})$ be such that $\mathcal{E}^*$ is a conditional expectation.

There is a decomposition $\mathcal{H} \equiv \oplus_n \mathcal{H}_n^L \otimes \mathcal{H}_n^R$ such that

$$\mathcal{E}^*(B(\mathcal{H})) \equiv \bigoplus_n B(\mathcal{H}_n^L) \otimes I_{\mathcal{H}_n^R}$$
$$\mathcal{E}(B(\mathcal{H})) \equiv \bigoplus_n B(\mathcal{H}_n^L) \otimes \omega_n$$

for some fixed states $\omega_n \in B(\mathcal{H}_n^R)$.

P. Hayden, R. Jósza, D. Petz, A. Winter, Commun. Math. Phys., 2004

The Koashi-Imoto decomposition

Applying this to $\mathcal{E}_{\mathcal{S}}$, we obtain

$$\mathcal{M}_{\mathcal{S}} \equiv \bigoplus_n B(\mathcal{H}_n^{\mathcal{S},L}) \otimes I_{\mathcal{H}_n^{\mathcal{S},R}}$$

$$\rho \equiv \bigoplus_n \mu_n(\rho)\rho_n \otimes \sigma_n, \qquad \forall \rho \in \mathcal{S},$$

- $\{\mu_n(\rho)\}$ is a probability distribution (classical part of $\mathcal{S}$)
- $\rho_n \in B(\mathcal{H}_n^{\mathcal{S},L})$ are states (depending on ρ)
- $\sigma_n \in B(\mathcal{H}_n^{\mathcal{S},R})$ are fixed states.

M. Koashi, N. Imoto, Phys. Rev. A, 2002
P. Hayden, R. Józsa, D. Petz, A. Winter, Commun. Math. Phys., 2004
A. Łuczak, Int. J. Theor. Phys., 2014
Y. Kuramochi, J. Math. Phys., 2018

Generators of $\mathcal{M}_{\mathcal{S}}$

The minimal sufficient subalgebra is generated by:

- Connes cocycles:

$$\rho^{it}\sigma^{-it}, \qquad \rho \in \mathcal{S},\ t \in \mathbb{R}.$$

- Radon Nikodym derivatives:

$$\sigma^{it}(\sigma^{-1/2}\rho\sigma^{-1/2})\sigma^{-it}, \qquad \rho \in \mathcal{S},\ t \in \mathbb{R}.$$

Reversible channels with respect to $\mathcal{S}$

Assume that Φ is reversible.

- Let Ψ be a recovery channel, then $\Psi \circ \Phi \in \mathcal{C}_{\mathcal{S}}$, so that
$$\mathcal{E}_{\mathcal{S}} \circ (\Psi \circ \Phi) = (\Psi \circ \Phi) \circ \mathcal{E}_{\mathcal{S}} = \mathcal{E}_{\mathcal{S}}.$$
- Note that $\mathcal{E}_{\mathcal{S}} \circ \Psi$ is also a recovery channel, so we may assume
$$\mathcal{E}_{\mathcal{S}} \circ \Psi = \Psi, \qquad \Psi \circ \Phi = \mathcal{E}_{\mathcal{S}}.$$
- We then have $\Phi \circ \Psi = \mathcal{E}_{\mathcal{S}_0}$, where
$$\mathcal{S}_0 := \{\Phi(\rho),\ \rho \in \mathcal{S}\}.$$

Reversible channels with respect to $\mathcal{S}$

A channel $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ is reversible with respect to $\mathcal{S}$ iff

$$\Phi^*|_{\mathcal{M}_{\mathcal{S}_0}} : \mathcal{M}_{\mathcal{S}_0} \xrightarrow{iso} \mathcal{M}_{\mathcal{S}}.$$

Equivalently, there is

- a decomposition $\mathcal{K} \equiv \oplus_n \mathcal{K}_n^L \otimes \mathcal{K}_n^R$
- unitaries $U_n : \mathcal{H}_n^{\mathcal{S},L} \to \mathcal{K}_n^L$
- channels $\Phi_n : B(\mathcal{H}_n^{\mathcal{S},R}) \to B(\mathcal{K}_n^R)$

such that

$$\Phi|_{B(\mathcal{H}_n^{\mathcal{S},L} \otimes \mathcal{H}_n^{\mathcal{S},R})} \equiv U_n^* \cdot U_n \otimes \Phi_n.$$

Reversible channels with respect to $\mathcal{S}$

Further conditions for reversibility: preserving the generators

- Connes cocycles

$$\Phi^*(\Phi(\rho)^{it}\Phi(\sigma)^{-it}) = \rho^{it}\sigma^{-it}, \qquad \forall \rho \in \mathcal{S},\ t \in \mathbb{R};$$

- Radon-Nikodym derivatives:

$$\Phi^*(\Phi(\sigma)^{-1/2}\Phi(\rho)\Phi(\sigma)^{-1/2}) = \sigma^{1/2}\rho\sigma^{1/2}, \qquad \forall \rho \in \mathcal{S};$$

- Petz dual

$$\Phi_\sigma \circ \Phi(\rho) = \rho, \qquad \forall \rho \in \mathcal{S}.$$

Conditions on $\mathcal{S}$

Given a channel Φ, what are the conditions for states in $\mathcal{S}$?

We fix a faithful state $\sigma \in \mathcal{S}$. Then we must have

$$\mathcal{S} \subset \mathrm{Fix}(\Phi_\sigma \circ \Phi) := \{\rho,\ \Phi_\sigma \circ \Phi(\rho) = \rho\}.$$

Put

$$\mathcal{F} := \lim_n \frac{1}{n} \sum_{k=1}^{n} (\Phi_\sigma \circ \Phi)^k,$$

then $\mathcal{F}^*$ is a conditional expectation and

$$\mathcal{F}(B(\mathcal{H})) = \mathrm{Fix}(\Phi_\sigma \circ \Phi).$$

Conditions on $\mathcal{S}$

There is

- a decomposition $\mathcal{H} \equiv \oplus_n \mathcal{H}_n^{\Phi,\sigma,L} \otimes \mathcal{H}_n^{\Phi,\sigma,R}$
- and states $\omega_n \in B(\mathcal{H}_n^{\Phi,\sigma,R})$

such that Φ is reversible with respect to $\mathcal{S}$ if and only if all $\rho \in \mathcal{S}$ have the form

$$\rho \equiv \bigoplus_n \lambda_n(\rho)\rho_n \otimes \omega_n$$

for some probability distribution $\{\lambda_n(\rho)\}$ and states $\rho_n \in B(\mathcal{H}^{\Phi,\sigma,L})$.

Reversibility by Rényi relative entropies

Preservation of standard Rényi relative entropies

The standard (Petz-type) Rényi relative entropies, $\alpha > 0$:

$$D_\alpha(\rho\|\sigma) = \begin{cases} \dfrac{1}{\alpha-1}\log \mathrm{Tr}\ \rho^\alpha\sigma^{1-\alpha} & \alpha \neq 1 \\ \mathrm{Tr}\ \rho(\log\rho - \log\sigma), & \alpha = 1. \end{cases}$$

- satisfy data processing inequality for $\alpha \in (0, 2]$.

Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$D_\alpha(\Phi(\rho)\|\Phi(\sigma)) = D_\alpha(\rho\|\sigma), \qquad \forall\rho \in \mathcal{S}$$

for some $\alpha \in (0, 2)$.

D. Petz, 1986, 1988
AJ, D. Petz, Commun. Math. Phys, 2006
F. Hiai, M. Mosonyi, D. Petz, C. Bény, Rev. Math. Phys. 2011
F. Hiai, M. Mosonyi, Rev. Math. Phys. 2017

Preservation of sandwiched Rényi relative entropies

The sandwiched (minimal) Rényi relative entropies, $\alpha > 0$:

$$\tilde{D}_\alpha(\rho\|\sigma) = \begin{cases} \frac{1}{\alpha-1}\log \mathrm{Tr}\,\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha & \alpha \neq 1 \\ \mathrm{Tr}\,\rho(\log\rho - \log\sigma), & \alpha = 1. \end{cases}$$

- satisfy data processing inequality for $\alpha \in [1/2, \infty]$

Φ is sufficient with respect to $\mathcal{S}$ if and only if

$$\tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma), \qquad \forall \rho \in \mathcal{S}$$

for some $\alpha \in (1/2, \infty)$.

Sandwiched Rényi relative entropy

We look at

$$\tilde{Q}_\alpha(\rho\|\sigma) := \mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha,$$

so that

$$\tilde{D}_\alpha(\rho\|\sigma) = \frac{1}{\alpha-1}\log \tilde{Q}_\alpha(\rho\|\sigma)$$

- For $\alpha > 1$: interpolation L_p-norms
- For $\alpha \in (1/2, 1)$: a variational formula, relation to case $\alpha > 1$
- The case $\alpha = 1$ (relative entropy): solved by Petz

An interpolation L_p-norm with respect to a state

Let us define a norm in $B(\mathcal{H})$, for $\alpha \geq 1$:

$$\|X\|_{\alpha,\sigma} = \left(\mathrm{Tr}\, |\sigma^{\frac{1-\alpha}{2\alpha}} X \sigma^{\frac{1-\alpha}{2\alpha}}|^{\alpha}\right)^{\frac{1}{\alpha}}$$

We have for any state ρ:

$$\tilde{Q}_{\alpha}(\rho\|\sigma) = \|\rho\|_{\alpha,\sigma}^{\alpha}$$

The norm can be obtained by complex interpolation between

$$\|X\|_{1,\sigma} = \mathrm{Tr}\, |X| = \|X\|_1, \qquad \|X\|_{\infty,\sigma} = \|\sigma^{-\frac{1}{2}} X \sigma^{-\frac{1}{2}}\|$$

S. Beigi, J. Math. Phys., 2013

Hadamard three lines theorem

For any function on $S = \{z \in \mathbb{C},\ \mathrm{Re}(z) \in [0,1]\}$,

$$f : S \to B(\mathcal{H}), \quad \text{continuous, analytic in } \mathrm{int}(S)$$

we have:

- for any $\alpha > 1$,

$$\|f(1/\alpha)\|_{\alpha,\sigma} \leq \max_{t\in\mathbb{R}} \|f(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f(1+it)\|_1$$

- If equality holds for some $\alpha > 1$, then it holds for all

Hadamard three lines theorem

For any $\rho \geq 0$ and α, we define a function

$$f_{\rho,\alpha}(z) = \|\rho\|_{\alpha,\sigma}^{1-z\alpha} \sigma^{\frac{1-z}{2}} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{z\alpha} \sigma^{\frac{1-z}{2}}, \qquad z \in S$$

- $f_{\rho,\alpha}(1/\alpha) = \rho$,
- The equality in Hadamard three lines theorem is attained:

$$\|f_{\rho,\alpha}(1/\alpha)\|_{\alpha,\sigma} = \max_{t\in\mathbb{R}} \|f_{\alpha,\sigma}(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f_{\alpha,\sigma}(1+it)\|_1$$

Positive trace preserving maps are contractions

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a positive trace preserving linear map:

- For $\alpha = 1$,

$$\|\Phi(X)\|_1 \leq \|X\|_1, \qquad X \in B(\mathcal{H})$$

- For $\alpha = \infty$,

$$\|\Phi(X)\|_{\infty,\Phi(\sigma)} = \|\Phi^*_\sigma(\sigma^{-1/2} X \sigma^{-1/2})\|_\infty \leq \|X\|_{\infty,\sigma}$$

- For $\alpha > 1$, by Riesz-Thorin (complex interpolation)

$$\|\Phi(X)\|_{\alpha,\Phi(\sigma)} \leq \|X\|_{\alpha,\sigma}, \qquad X \in B(\mathcal{H}).$$

S. Beigi, J. Math. Phys., 2013

The case $\alpha = 2$

Let $\alpha = 2$.

- $\|\cdot\|_{2,\sigma}$ is a Hilbert space norm, with the inner product

$$\langle X, Y \rangle_\sigma = \mathrm{Tr}\ X^* \sigma^{1/2} Y \sigma^{1/2}$$

- For a positive trace preserving map $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$,

$$\langle X, \Phi(Y) \rangle_{\Phi(\sigma)} = \langle \Phi_\sigma(X), Y \rangle_\sigma, \qquad X \in B(\mathcal{K}),\ Y \in B(\mathcal{H})$$

- Since Φ is a contraction,

$$\|\Phi(Y)\|_{2,\Phi(\sigma)} = \|Y\|_{2,\sigma} \iff \Phi_\sigma \circ \Phi(Y) = Y.$$

Preservation and reversibility

Let Φ be a channel and assume that for some $\alpha > 1$,

$$\|\Phi(\rho)\|_{\alpha,\Phi(\sigma)} = \|\rho\|_{\alpha,\sigma}\left(\iff \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)) = \tilde{D}_\alpha(\rho\|\sigma)\right)$$

For $\alpha = 2$, we get

$$\|\Phi(\rho)\|_{2,\Phi(\sigma)} = \|\rho\|_{2,\sigma} \iff \Phi_\sigma \circ \Phi(\rho) = \rho$$

so that Φ is reversible.

Preservation and reversibility

For $\alpha = \bar{\alpha} > 1$:

$$f(z) = f_{\rho,\bar{\alpha}}(z) = \|\rho\|_{\bar{\alpha},\sigma}^{1-z\bar{\alpha}} \sigma^{\frac{1-z}{2}} \left(\sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}} \rho \sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}} \right)^{z\bar{\alpha}} \sigma^{\frac{1-z}{2}}, \qquad z \in S$$

Then

$$\begin{aligned} \|\rho\|_{\bar{\alpha},\sigma} &= \|f(1/\bar{\alpha})\|_{\bar{\alpha},\sigma} = \|\Phi(f(1/\bar{\alpha}))\|_{\bar{\alpha},\Phi(\sigma)} \\ &\leq \max_{t\in\mathbb{R}} \|\Phi(f(it))\|_{\infty,\Phi(\sigma)} \max_{t\in\mathbb{R}} \|\Phi(f(1+it))\|_1 \\ &\leq \max_{t\in\mathbb{R}} \|f(it)\|_{\infty,\sigma} \max_{t\in\mathbb{R}} \|f(1+it)\|_1 = \|\rho\|_{\bar{\alpha},\sigma} \end{aligned}$$

We have equalities, for any $\alpha > 1$. This implies

$$\|\Phi(f(1/\alpha))\|_{\alpha,\Phi(\sigma)} = \|f(1/\alpha)\|_{\alpha,\sigma}, \qquad \alpha > 1.$$

Preservation and reversibility

We obtain

$$\|\Phi(\tau)\|_{2,\Phi(\sigma)} = \|\tau\|_{2,\sigma}, \text{ so that } \Phi_\sigma \circ \Phi(\tau) = \tau,$$

for

$$\tau := f(1/2) = \sigma^{\frac{1}{4}} \left(\sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}} \rho \sigma^{\frac{1-\bar{\alpha}}{2\bar{\alpha}}}\right)^{\frac{\bar{\alpha}}{2}} \sigma^{\frac{1}{4}}.$$

We know that $\Phi_\sigma \circ \Phi(\rho) = \rho$ iff ρ is of the form

$$\rho \equiv \bigoplus_n \rho_n \otimes \omega_n \qquad \text{(with fixed faithful states } \omega_n)$$

with respect to the decomposition

$$\mathcal{H} \equiv \oplus_n \mathcal{H}_n^{\Phi,\sigma,L} \otimes \mathcal{H}_n^{\Phi,\sigma,R}.$$

Since $\Phi_\sigma \circ \Phi(\sigma) = \sigma$ and $\Phi_\sigma \circ \Phi(\tau) = \tau$, this must be true.

A variational formula for $\alpha \in [1/2, 1)$

For $\alpha \in [1/2, 1)$, we have

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{X \in B(\mathcal{H})^{++}} \alpha \operatorname{Tr} \rho X + (1-\alpha) \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} X^{-1} \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{\frac{\alpha}{1-\alpha}}$$

With $\gamma := \frac{\alpha}{1-\alpha} > 1$, this can be written as

$$\tilde{Q}_\alpha(\rho\|\sigma) = \inf_{X \in B(\mathcal{H})^{++}} \alpha \operatorname{Tr} \rho X + (1-\alpha) \tilde{Q}_\gamma(\sigma^{1/2} X^{-1} \sigma^{1/2} \|\sigma).$$

If ρ is also faithful, attained at the unique element

$$\bar{X} = \sigma^{\frac{1}{2\gamma}} (\sigma^{\frac{1}{2\gamma}} \rho \sigma^{\frac{1}{2\gamma}})^{\alpha-1} \sigma^{\frac{1}{2\gamma}}.$$

R. L. Frank, E. H. Lieb, J. Math. Phys., 2013

F. Hiai, Quantum f-Divergences in von Neumann Algebras: Reversibility of Quantum Operations, 2021

Positive trace preserving maps

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a positive trace preserving map,

$$\tilde{Q}_\alpha(\rho\|\sigma) = \mathrm{Tr}\,\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha .$$

For $Y \in B(\mathcal{K})^{++}$, we have

$$\begin{aligned}\tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(Y)^{-1}\sigma^{1/2}\|\sigma) &\leq \tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(Y^{-1})\sigma^{1/2}\|\sigma)\\ &= \tilde{Q}_\gamma(\Phi_\sigma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2})\|\sigma)\\ &\leq \tilde{Q}_\gamma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma))\end{aligned}$$

We used the Choi inequality $\Phi^*(Y)^{-1} \leq \Phi^*(Y^{-1})$, definition of Φ_σ and monotonicity of $\tilde{Q}_\gamma$, $\gamma > 1$.

Positive trace preserving maps

We get, for $Y \in B(\mathcal{K})^{++}$,

$$\begin{aligned}\tilde{Q}_\alpha(\rho\|\sigma) &\le \alpha \,\mathrm{Tr}\ \rho\Phi^*(Y) + (1-\alpha)\tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(Y)^{-1}\sigma^{1/2}\|\sigma)\\ &\le \alpha \,\mathrm{Tr}\ \Phi(\rho)Y + (1-\alpha)\tilde{Q}_\gamma(\Phi(\sigma)^{1/2}Y^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma))\end{aligned}$$

Taking the inf,

$$\tilde{Q}_\alpha(\rho\|\sigma) \le \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)),$$

so that

$$\tilde{D}_\alpha(\rho\|\sigma) \ge \tilde{D}_\alpha(\Phi(\rho)\|\Phi(\sigma)).$$

Preservation and reversibility

Let $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a channel such that

$$\tilde{Q}_\alpha(\rho\|\sigma) = \tilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma)).$$

If ρ is faithful, then the infima in the variational formulas are attained at unique $\bar{X} \in B(\mathcal{H})^{++}$ resp. $\bar{Y} \in B(\mathcal{K})$ and

$$\bar{X} = \Phi^*(\bar{Y}).$$

We also infer that

$$\begin{aligned} \tilde{Q}_\gamma(\sigma^{1/2}\bar{X}^{-1}\sigma^{1/2}\|\sigma) &= \tilde{Q}_\gamma(\sigma^{1/2}\Phi^*(\bar{Y}^{-1})\sigma^{1/2}\|\sigma) \\ &= \tilde{Q}_\gamma(\Phi_\sigma(\Phi(\sigma)^{1/2}\bar{Y}^{-1}\Phi(\sigma)^{1/2})\|\sigma) \\ &= \tilde{Q}_\gamma(\Phi(\sigma)^{1/2}\bar{Y}^{-1}\Phi(\sigma)^{1/2}\|\Phi(\sigma)) \end{aligned}$$

Preservation and reversibility

Put

$$\mu = \sigma^{1/2}\bar{X}^{-1}\sigma^{1/2}, \qquad \nu = \Phi(\sigma)^{1/2}\bar{Y}^{-1}\Phi(\sigma)^{1/2}$$

Then

$$\Phi_\sigma(\nu) = \mu, \qquad \tilde{Q}_\gamma(\nu\|\Phi(\sigma)) = \tilde{Q}_\gamma(\mu\|\sigma) = \tilde{Q}_\gamma(\Phi_\sigma(\nu)\|\Phi_\sigma(\Phi(\sigma)))$$

By the results for $\gamma > 1$, $\Phi \circ \Phi_\sigma(\nu) = \nu$, so that

$$\Phi_\sigma \circ \Phi(\mu) = \Phi_\sigma \circ \Phi \circ \Phi_\sigma(\nu) = \Phi_\sigma(\nu) = \mu.$$

From

$$\mu = \sigma^{\frac{\gamma-1}{2\gamma}} \left(\sigma^{\frac{1}{2\gamma}}\rho\sigma^{\frac{1}{2\gamma}}\right)^{1-\alpha} \sigma^{\frac{\gamma-1}{2\gamma}},$$

we get $\Phi_\sigma \circ \Phi(\rho) = \rho$ as before.

Reversibility by hypothesis testing

Quantum hypothesis testing

Suppose $\rho, \sigma \in \mathcal{S}(\mathcal{H})$ are given, one of them is the true state:

- we test the hypothesis $H_0 = \sigma$ against $H_1 = \rho$
- a test: an effect $0 \leq T \leq I$,

$$\mathrm{Tr}\,[T\omega] - \text{probability of rejecting } H_0 \text{ in the state } \omega$$

- error probabilities:

$$\alpha(T) = \mathrm{Tr}\,[\sigma T], \qquad \beta(T) = \mathrm{Tr}\,[\rho(I - T)]$$

- Bayes error probabilities for $\lambda \in [0, 1]$:

$$P_e(\lambda, \sigma, \rho, T) := \lambda\alpha(T) + (1 - \lambda)\beta(T)$$

Quantum Neyman-Pearson lemma

Put $P_{s,\pm} := \mathrm{supp}((\rho - s\sigma)_\pm)$, $P_{s,0} := I - P_{s,+} - P_{s,-}$.

A test T is Bayes optimal for $\lambda \in (0,1)$ if and only if

$$T = P_{s,+} + X, \quad 0 \le X \le P_{s,0}, \qquad s = \frac{\lambda}{1-\lambda}$$

and then

$$\begin{aligned} P_e(\lambda,\sigma,\rho) &:= \min_{0 \le T \le I} P_e(\lambda,\sigma,\rho,T) \\ &= (1-\lambda)(1 - \mathrm{Tr}\,[(\rho - s\sigma)_+]) \\ &= (1-\lambda)(s - \mathrm{Tr}\,[(\rho - s\sigma)_-]) \\ &= \frac{1}{2}(1 - (1-\lambda)\|\rho - s\sigma\|_1). \end{aligned}$$

Data processing inequalities

We clearly have for any quantum channel Φ and $\lambda \in [0,1]$:

$$P_e(\lambda, \Phi(\sigma), \Phi(\rho)) \geq P_e(\lambda, \sigma, \rho),$$

or equivalently, for any $s \in \mathbb{R}$:

$$\|\Phi(\rho) - s\Phi(\sigma)\|_1 \leq \|\rho - s\sigma\|_1;$$
$$\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_+\right] \leq \mathrm{Tr}\left[(\rho - s\sigma)_+\right];$$
$$\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] \leq \mathrm{Tr}\left[(\rho - s\sigma)_-\right].$$

Equality in DPI

The following are equivalent:

- $P_e(\lambda, \Phi(\sigma), \Phi(\rho)) = P_e(\lambda, \sigma, \rho)$, $\lambda \in [0,1]$;
- $\|\Phi(\rho) - s\Phi(\sigma)\|_1 = \|\rho - s\sigma\|_1$, $s \in \mathbb{R}$;
- $\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_+\right] = \mathrm{Tr}\left[(\rho - s\sigma)_+\right]$, $s \in \mathbb{R}$;
- $\mathrm{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] = \mathrm{Tr}\left[(\rho - s\sigma)_-\right]$, $s \in \mathbb{R}$;
- $\Phi^*(Q_{s,+}) = P_{s,+}$, $s \in \mathbb{R}$;
- $\Phi^*(Q_{s,-}) = P_{s,-}$, $s \in \mathbb{R}$.

$$\left(\ Q_{s,\pm} = \mathrm{supp}((\Phi(\rho) - s\Phi(\sigma))_\pm)\right)$$

Can we get recoverability?

An integral formula for relative entropy

For any pair if states ρ, σ:

$$D(\rho\|\sigma) = \int_{-\infty}^{\infty} \frac{dt}{|t|(1-t)^2} \operatorname{Tr}\left[((1-t)\rho + t\sigma)_-\right]$$

For $\lambda \geq 0$ such that $\sigma \leq \rho \leq \lambda\sigma$:

$$D(\rho\|\sigma) = \int_0^{\lambda} \frac{ds}{s} \operatorname{Tr}\left[(\rho - s\sigma)_-\right] + \log(\lambda) + 1 - \lambda$$

If such λ does not exist, both sides are ∞.

P. Frenkel, arxiv:2208.12194

Reversibility via hypothesis testing

Let $\rho, \sigma \in B(\mathcal{H})$ be any states, $\Phi : B(\mathcal{H}) \to B(\mathcal{K})$ a channel.

Assume that

$$P_e(\lambda, \Phi(\sigma), \Phi(\rho)) = P_e(\lambda, \sigma, \rho), \lambda \in [0, 1]$$

Equivalently,

$$\mathrm{Tr}\,(\Phi(\rho) - s\Phi(\sigma))_- = \mathrm{Tr}\,(\rho - s\sigma)_-, s \in \mathbb{R},$$

the same is true with σ replaced by $\sigma_0 := \frac{1}{2}(\rho + \sigma)$.

Reversibility via hypothesis testing

We have

$$\rho \leq 2\sigma_0, \qquad \Phi(\rho) \leq 2\Phi(\sigma_0).$$

By the integral representation,

$$\begin{aligned} D(\rho\|\sigma_0) &= \int_0^2 \frac{ds}{s} \operatorname{Tr}\left[(\rho - s\sigma)_-\right] + \log(2) - 1 \\ &= \int_0^2 \frac{ds}{s} \operatorname{Tr}\left[(\Phi(\rho) - s\Phi(\sigma))_-\right] + \log(2) - 1 \\ &= D(\Phi(\rho)\|\Phi(\sigma_0)) \end{aligned}$$

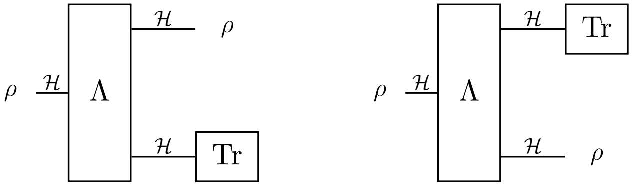
It follows that

$$\Phi_{\sigma_0} \circ \Phi(\rho) = \rho, \quad \Phi_{\sigma_0} \circ \Phi(\sigma) = \sigma.$$

Broadcasting and distinguishability

Broadcasting

A broadcasting channel $\Lambda : B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{H})$, ρ a state:



No-broadcasting: for $\Lambda_1 := \mathrm{Tr}_2 \circ \Lambda$, $\Lambda_2 := \mathrm{Tr}_1 \circ \Lambda$,

$\Lambda_1(\rho) = \Lambda_2(\rho) = \rho$ for all ρ is impossible

Restricted to a subset $\mathcal{S}$ of states:

Broadcasting is possible $\iff$ $\mathcal{S}$ is commutative.

Broadcasting and distinguishability

Let ρ, σ be states. Instead of broadcasting $\{\rho, \sigma\}$, we require

Both Λ_1 and Λ_2 preserve distinguishability of ρ, σ:

$$\begin{aligned} P_e(\lambda, \Lambda_1(\rho), \Lambda_1(\sigma)) &= P_e(\lambda, \Lambda_2(\rho), \Lambda_2(\sigma)) \\ &= P_e(\lambda, \rho, \sigma), \qquad \lambda \in [0,1]. \end{aligned}$$

Then Λ_1, Λ_2 are reversible with respect to $\{\rho, \sigma\}$. If Ψ_1, Ψ_2 are the recovery channels, then

$$(\Psi_1 \otimes \Psi_2) \circ \Lambda$$

is a broadcasting channel for $\{\rho, \sigma\}$. Hence ρ, σ must commute.

Broadcasting and distinguishability

If we assume $P_e(\lambda, \Lambda_1(\rho), \Lambda_1(\sigma)) = P_e(\lambda, \rho, \sigma)$, $\lambda \in [0,1]$:

- there is a channel $\Lambda' : B(\mathcal{H}) \to B(\mathcal{H} \otimes \mathcal{H})$ such that Λ'_1 preserves ρ and σ and $(\Lambda'_1)^*$ is a conditional expectation, while $\Lambda'_2 = \Lambda_2$.
- The ranges of $(\Lambda'_1)^*$ and Λ_2^* must commute
- Any test on the second part acts on the commutant $\mathcal{M}'_{\{\rho,\sigma\}}$
- $P_e(\lambda, \Lambda_2(\rho), \Lambda_2(\sigma)) \geq P_e(\lambda, \mu(\rho), \mu(\sigma))$, $\lambda \in [0,1]$,

  $\{\mu(\rho), \mu(\sigma)\}$ - the classical part of the Koashi-Imoto decomposition of $\{\rho, \sigma\}$.