Algebraic L-theory I

Tibor Macko

Universität Bonn
www.math.uni-bonn.de/macko

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Outline for the 4 talks

Main points

1. The L-groups via chain complexes
2. The surgery obstruction via chain complexes
3. The algebraic surgery exact sequence (the assembly map)
4. Identify the TOP-GSES with the ASES

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4. Identify the TOP-GSES with the ASES

Therefore

info about the assembly $\rightsquigarrow$ info about the structure set

Corollary

The Farrell-Jones conjecture implies the Borel conjecture.

Motivation

Recall

Starting with a degree one normal map we have the surgery obstruction

$$(f, b)\colon M \to X \quad \rightsquigarrow \quad \theta(f, b) \in L_n(\mathbb{Z}[\pi_1(X)], w)$$

such that $\theta(f, b) = 0$ if and only if (f, b) is normally cobordant to a homotopy equivalence (when $n \geq 5$).

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Problems

1. When $n \equiv 0$ (2) then $\theta(f,b)$ is a quadratic form
   When $n \equiv 1$ (2) then $\theta(f,b)$ is a quadratic formation;
2. Need to do surgery below the middle dimension to read off $\theta(f,b)$.

Motivation II

Aim

Rectify these deficiencies using Ranicki's algebraic theory of surgery.

Idea

Use chain complexes - motivated by Quillen's approach to algebraic K-theory.

Question

- How to define a symmetric bilinear form on a chain complex?
- How to define a quadratic form on a chain complex?

Forms on modules I

Slogan

A symmetric bilinear form is a fixed point.
A quadratic form is an orbit.

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A bilinear form is

$$\varphi \in \mathrm{Hom}_R(P, P^*) \cong (P \otimes_R P)^* \ni \lambda$$

We have an involution on forms

$$T \colon \mathrm{Hom}_R(P, P^*) \to \mathrm{Hom}_R(P, P^*) \quad T \colon (P \otimes_R P)^* \to (P \otimes_R P)^*$$

via

$$T(\varphi) = \varphi^* \circ \mathrm{ev} \quad T(\lambda)(x, y) = \overline{\lambda(y, x)}$$

Forms on modules II

Let $\varepsilon = \pm 1$.

An ε-symmetric bilinear form is

$$\varphi \in \ker(1 - \varepsilon T) = \mathrm{Hom}_R(P, P^*)^{\mathbb{Z}_2} = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\mathbb{Z}, \mathrm{Hom}_R(P, P^*))$$

An ε-quadratic form is

$$\psi \in \mathrm{coker}(1 - \varepsilon T) = \mathrm{Hom}_R(P, P^*)_{\mathbb{Z}_2} = \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}_2]} \mathrm{Hom}_R(P, P^*)$$

We have the symmetrization map

$$1 + \varepsilon T \colon \mathrm{Hom}_R(P, P^*)_{\mathbb{Z}_2} \to \mathrm{Hom}_R(P, P^*)^{\mathbb{Z}_2}$$

Forms versus co-forms

Let P be a f.g. free R-module.

In this case we could work with

$$P \otimes_R P \cong \mathrm{Hom}_R(P^*, P)$$

instead of

$$(P \otimes_R P)^* \cong \mathrm{Hom}_R(P, P^*).$$

The former is more convenient when working with chain complexes.

Structured chain complexes I

Let C, D be bounded chain complexes of f.g. free R-modules

The Hom-complex $\mathrm{Hom}_R(C,D)$ is a chain complex

$$\mathrm{Hom}_R(C,D)_n = \bigoplus_{q-p=n} \mathrm{Hom}_R(C_p, D_q)$$

$$d_{\mathrm{Hom}}(f) = d_D f - (-1)^n f d_C$$

The tensor product $C \otimes_R D$ is a chain complex

$$(C \otimes_R D)_n = \bigoplus_{p+q=n} C_p \otimes_R D_q$$

$$d_{\otimes}(x \otimes y) = d_C(x) \otimes y + (-1)^{|x|} x \otimes d_D(y)$$

We have the slant isomorphism

$$-\backslash - \colon (C \otimes_R D) \to \mathrm{Hom}_R(C^{-*}, D)$$

$$(x \otimes y) \mapsto (f \mapsto (-1)^{pq+p} \overline{f(x)} \cdot y)$$

Structured chain complexes II

A "form" on a chain complex C is

$$\omega \in (C \otimes_R C) \cong \mathrm{Hom}_R(C^{-*}, C)$$

The switch isomorphism is defined by

$$\begin{aligned} C \otimes_R D &\to D \otimes_R C \\ x \otimes y &\mapsto (-1)^{|x|\cdot|y|} y \otimes x. \end{aligned}$$

It defines an involution on forms

$$\begin{aligned} C \otimes_R C &\to C \otimes_R C \\ x \otimes y &\mapsto (-1)^{|x|\cdot|y|} y \otimes x. \end{aligned}$$

Structured chain complexes III

We need a homotopy invariant notion!

Homotopy invariant structures

fixed points $\rightsquigarrow$ homotopy fixed points
orbits $\rightsquigarrow$ homotopy orbits

Structured chain complexes III

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Homotopy invariant structures

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The standard $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$:

$$W := \cdots \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \longrightarrow 0$$

The standard periodic $\mathbb{Z}[\mathbb{Z}_2]$-resolution of $\mathbb{Z}$:

$$\widehat{W} := \cdots \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1+T} \mathbb{Z}[\mathbb{Z}_2] \xrightarrow{1-T} \cdots$$

Structured chain complexes IV

Notation

$$W^{\%}(C) := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C \otimes_R C)$$
$$W_{\%}(C) := W \otimes_{\mathbb{Z}[\mathbb{Z}_2]} (C \otimes_R C)$$
$$\widehat{W}^{\%}(C) := \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\widehat{W}, C \otimes_R C)$$

Definition

An n-dimensional symmetric structure on C is a cycle $\varphi \in W^{\%}(C)_n$.
An n-dimensional quadratic structure on C is a cycle $\psi \in W_{\%}(C)_n$.
An n-dimensional hyperquadratic structure on C is a cycle $\theta \in \widehat{W}^{\%}(C)_n$.

Structured chain complexes V

A cycle $\varphi \in W^{\%}(C)_n$ is a collection $\{\varphi_s \colon C^{n+s-*} \to C | s \in \mathbb{N}\}$ s.t.

$$d_C \varphi_s - (-1)^{n+s} \varphi_s d_{C^{-*}} = (-1)^n (\varphi_{s-1} + (-1)^s T(\varphi)_{s-1})$$

Each φ_s is a chain homotopy between φ_{s-1} and $T(\varphi)_{s-1}$.

A cycle $\psi \in W_{\%}(C)_n$ is a collection $\{\psi_s \colon C^{n-s-*} \to C | s \in \mathbb{N}\}$ s.t.

$$d_C \psi_s - (-1)^{n-s} \psi_s d_{C^{-*}} = T(\psi)_{s+1} + (-1)^{s+1} \psi_{s+1}$$

Each ψ_{s-1} as a chain homotopy between ψ_s and $T(\psi)_s$.

A cycle $\theta \in \widehat{W}^{\%}(C)_n$ is a collection $\{\theta_s \colon C^{n+s-*} \to C | s \in \mathbb{Z}\}$ s.t.

$$d_C \theta_s - (-1)^{n+s} \theta_s d_{C^{-*}} = (-1)^n (\theta_{s-1} + (-1)^s T(\theta)_{s-1})$$

Each θ_s as a chain homotopy between θ_{s-1} and $T(\theta)_{s-1}$ for all $s \in \mathbb{Z}$.
Only finitely many of them are non-zero, since C is bounded.

The symmetric construction I

Let X be a topological space. Recall the Alexander-Whitney diagonal map:

$$\Delta_0 \colon C(X) \to C(X) \otimes C(X)$$

The method of acyclic models produces a sequence of maps of degree s

$$\Delta_s \colon C(X) \to C(X) \otimes C(X)$$

for $s \geq 0$ such that

$$d\Delta_{s+1} + (-1)^s \Delta_{s+1} d = (T + (-1)^{s+1})\Delta_s.$$

The map Δ_1 is used to show the graded-commutativity of the cup product, the maps Δ_s for all $s \geq 1$ are used to construct Steenrod squares.

The symmetric construction II

These can be put together to a chain map of degree 0 as follows

$$\begin{aligned} \Delta_X \colon W \otimes C(X) &\to C(X) \otimes C(X) \\ 1_s \otimes x &\mapsto \Delta_s(x) \end{aligned}$$

The symmetric construction map

$$\varphi_X \colon C(X) \to W^{\%}(C(X))$$

is defined to be the adjoint of Δ_X. For a cycle $c \in C_n(X)$ we have

$$\varphi_X(c)_0 = - \cap c \colon C^{n-*}(X) \to C(X).$$

Important: The symmetric construction is natural in X since it is obtained by the methdo of acyclic models.

The symmetric construction III

The equivariant version:

$$\Delta_{\widetilde{X}} \colon W \otimes C(\widetilde{X}) \to C(\widetilde{X}) \otimes C(\widetilde{X})$$

$$\varphi_{\widetilde{X}} \colon C(\widetilde{X}) \to W^{\%}(C(\widetilde{X})),$$

Apply $\mathbb{Z} \otimes_{\mathbb{Z}[\pi_1(X)]} -$ to get

$$\varphi_{\widetilde{X}} \colon C(X) \to W^{\%}(C(\widetilde{X})) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, C(\widetilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} C(\widetilde{X})).$$

Upshot

Slogan:

An n-dimensional symmetric structure on a bounded chain complex is like a space X together with a choice of an n-dimensional cycle in $C(X)$ and a choice of cap product on the chain level and a choice of all higher homotopies of between the higher cap products and their adjoints.

Remark

Examples of quadratic and hyperquadratic structures occurring in nature are relegated to later talks.

Symmetric Poincaré complexes

Define the evaluation chain map

$$\mathrm{ev}\colon W^{\%}(C) \to \mathrm{Hom}_R(C^{-*}, C)$$
$$\varphi = \{\varphi_s\}_{s \geq 0} \mapsto \varphi_0$$

Definition

An n-dimensional symmetric complex (C, φ) is called Poincaré if

$$\varphi_0\colon C^{n-*} \to C$$

is a chain homotopy equivalence.

Quadratic Poincaré complexes

Define the symmetrization chain map

$$1 + T \colon W_{\%}(C) \to W^{\%}(C)$$

$$\psi = \{\psi_s\}_{s \geq 0} \mapsto \varphi = \{\varphi_s\}_{s \geq 0} \text{ with } \begin{cases} \varphi_s = (1+T)\psi_0 & s = 0 \\ \varphi_s = 0 & s \geq 1 \end{cases}$$

Definition

An n-dimensional quadratic complex (C, ψ) is called Poincaré if

$$(1+T)\psi_0 \colon C^{n-*} \to C$$

is a chain homotopy equivalence.

Remark

An analogous notion for hyperquadratic structures is not defined.

The Q-groups

Often just a homology class of a structure is important.

Definition

$$Q^n(C) = H_n(W^{\%}(C)) = H^n(\mathbb{Z}_2, C \otimes_R C)$$
$$Q_n(C) = H_n(W_{\%}(C)) = H_n(\mathbb{Z}_2, C \otimes_R C)$$
$$\widehat{Q}^n(C) = H_n(\widehat{W}^{\%}(C)) = \widehat{H}^n(\mathbb{Z}_2, C \otimes_R C)$$

Relations between structures

Proposition

We have a long exact sequence of Q-groups

$$\ldots \longrightarrow Q_n(C) \xrightarrow{1+T} Q^n(C) \xrightarrow{J} \widehat{Q}^n(C) \xrightarrow{H} Q_{n-1}(C) \longrightarrow \ldots$$

Proof.

The degreewise split short exact sequence

$$0 \longrightarrow \Sigma^{-1} W^{-*} \longrightarrow \widehat{W} \longrightarrow W \longrightarrow 0$$

induces the short exact sequence

$$0 \longrightarrow W^{\%}(C) \longrightarrow \widehat{W}^{\%}(C) \longrightarrow \Sigma W_{\%}(C) \longrightarrow 0$$

whose long exact homology sequence is the desired one. □

Properties of structures I

- Functoriality
  $f\colon C \to D \quad \rightsquigarrow \quad f \otimes f\colon C \otimes_R C \to D \otimes_R D \quad \rightsquigarrow$

$$f^{\%}\colon W^{\%}(C) \to W^{\%}(D)$$
$$f_{\%}\colon W_{\%}(C) \to W_{\%}(D)$$
$$\widehat{f}^{\%}\colon \widehat{W}^{\%}(C) \to \widehat{W}^{\%}(D)$$

- (Not)-commuting with $\oplus$

$$W^{\%}(C \oplus D) \simeq W^{\%}(C) \oplus W^{\%}(D) \oplus (C \otimes D)$$
$$W_{\%}(C \oplus D) \simeq W_{\%}(C) \oplus W_{\%}(D) \oplus (C \otimes D)$$
$$\widehat{W}^{\%}(C \oplus D) \simeq \widehat{W}^{\%}(C) \oplus \widehat{W}^{\%}(D)$$

Properties of structures II

- Products

$$- \otimes - : W^{\%}(C) \otimes W^{\%}(D) \to W^{\%}(C \otimes D)$$
$$- \otimes - : \widehat{W}^{\%}(C) \otimes \widehat{W}^{\%}(D) \to \widehat{W}^{\%}(C \otimes D)$$
$$- \otimes - : W^{\%}(C) \otimes W_{\%}(D) \to W_{\%}(C \otimes D)$$

are obtained using the diagonal maps

$$\Delta \colon W \to W \otimes W \qquad 1_s \mapsto \sum_{r=0}^{\infty} 1_r \otimes T^r_{s-r}$$
$$\Delta \colon \widehat{W} \to \widehat{W} \widehat{\otimes} \widehat{W} \qquad 1_s \mapsto \sum_{r=-\infty}^{\infty} 1_r \otimes T^r_{s-r}$$
$$\Delta \colon W^{-*} \to W \otimes W^{-*} \qquad 1_s \mapsto \sum_{r=-\infty}^{0} 1_r \otimes T^r_{r-s}$$

via the formula

$$\varphi_C \otimes \varphi_D \mapsto (\varphi_C \otimes \varphi_D) \circ \Delta.$$

Properties of structures III

- A unit
  A 0-dimensional symmetric complex

$$(\bar{\mathbb{Z}}, \nu) \quad \text{with} \quad \nu = 1 \in (\bar{\mathbb{Z}} \otimes \bar{\mathbb{Z}})_0.$$

- Homotopy invariance
  Let I be the cellular chain complex of the 1-simplex Δ^1.
  Choose 1-chain $\omega \in W^{\%}(I)$ such that $d_{W^{\%}(C)}(\omega) = i_1^{\%}(\nu) - i_0^{\%}(\nu)$.
  A chain homotopy $h\colon f_0 \simeq f_1$ induces the chain homotopy

$$W^{\%}(C) \otimes I \xrightarrow{-\otimes\omega} W^{\%}(C) \otimes W^{\%}(I) \xrightarrow{-\otimes-} W^{\%}(C \otimes I) \xrightarrow{h^{\%}} W^{\%}(D)$$

L-groups?

We have learned what is an equivalent of a non-degenerate symmetric bilinear form in the world of chain complexes.
We have learned what is an equivalent of a non-degenerate quadratic form in the world of chain complexes.

Question

How to define L-groups?

Idea

As cobordism groups.

Motivation

Wall, Surgery on compact manifolds, Chapter 9.

Cobordisms? Pairs?

In analogy with the previous slogan:

An $(n+1)$-dimensional symmetric pair should be like a pair of spaces (X, A), a choice of a cycle $c \in C(X, A)$, and choices of cap products and higher homotopies.

Relative cap products!

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C^k(X,A) & \longrightarrow & C^k(X) & \longrightarrow & C^k(A) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle -\cap c} & & \downarrow{\scriptstyle -\cap c} & & \downarrow{\scriptstyle -\cap \partial c} & & \\
\cdots & \longrightarrow & C_{n+1-k}(X) & \longrightarrow & C_{n+1-k}(X,A) & \longrightarrow & C^{n-k}(A) & \longrightarrow & \cdots
\end{array}$$

Pairs!

Recall that a chain map $f : C \to D$ induces $f^{\%} : W^{\%}(C) \to W^{\%}(D)$

Warning: $\mathcal{C}(f^{\%}) \neq W^{\%}(\mathcal{C}(f))$ in general.

Definition

An $(n+1)$-dimensional symmetric algebraic pair over R is a chain map $f\colon C \to D$ together with an $(n+1)$-dimensional cycle $(\delta\varphi, \varphi) \in \mathcal{C}(f^{\%})$.

An $(n+1)$-dimensional quadratic algebraic pair over R is a chain map $f\colon C \to D$ together with an $(n+1)$-dimensional cycle $(\delta\psi, \psi) \in \mathcal{C}(f_{\%})$.

This means:

$$d_{W^{\%}(D)}(\delta\varphi) = (-1)^n f^{\%}(\varphi)$$
$$d_{W_{\%}(D)}(\delta\psi) = (-1)^n f^{\%}(\psi)$$

Evaluation maps in the relative setting I

Notice $\mathcal{C}(f^{\%}) = \mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(W, \mathcal{C}(f \otimes f))$ so that we obtain

$$\begin{array}{ccccc} W^{\%}(C) & \longrightarrow & W^{\%}(D) & \longrightarrow & \mathcal{C}(f^{\%}) \\ \downarrow & & \downarrow & & \downarrow \\ C \otimes_R C & \longrightarrow & D \otimes_R D & \longrightarrow & \mathcal{C}(f \otimes_R f) \end{array}$$

Consider

$$C \xrightarrow{f} D \xrightarrow{e} \mathcal{C}(f), \qquad j : ef \simeq 0 \ (C \to \mathcal{C}(f))$$

Evaluation maps in the relative setting II

Let ev_l be induced as follows:

$$\begin{array}{ccccc} C \otimes_R C & \longrightarrow & D \otimes_R D & \longrightarrow & \mathcal{C}(f \otimes_R f) \\ & \searrow{\scriptstyle j\otimes f} & & \searrow & \Big\downarrow{\scriptstyle \mathrm{ev}_l} \\ & & & & \mathcal{C}(f) \otimes_R D \end{array}$$

Similarly ev_r is induced as follows:

$$\begin{array}{ccccc} C \otimes_R C & \longrightarrow & D \otimes_R D & \longrightarrow & \mathcal{C}(f \otimes_R f) \\ & \searrow{\scriptstyle f\otimes j} & & \searrow & \Big\downarrow{\scriptstyle \mathrm{ev}_r} \\ & & & & D \otimes_R \mathcal{C}(f) \end{array}$$

A (homotopy) commutative ladder of a symmetric pair

Hence we have

$$\mathrm{ev}_l : \mathcal{C}(f^{\%}) \to \mathcal{C}(f) \otimes_R D \cong \mathrm{Hom}_R(\mathcal{C}(f)^{-*}, D)$$
$$\mathrm{ev}_r : \mathcal{C}(f^{\%}) \to D \otimes_R \mathcal{C}(f) \cong \mathrm{Hom}_R(D^{-*}, \mathcal{C}(f))$$

and

$$\begin{array}{ccccccc}
C^{n-*} & \xrightarrow{\partial^*} & \mathcal{C}(f)^{n+1-*} & \xrightarrow{e^*} & D^{n+1-*} & \xrightarrow{f^*} & C^{n+1-*} \\
\downarrow{\scriptstyle \mathrm{ev}(\varphi)} & & \downarrow{\scriptstyle \mathrm{ev}_l(\delta\varphi,\varphi)} & \overset{\Phi_h(\delta\varphi,\varphi)}{\leadsto} & \downarrow{\scriptstyle \mathrm{ev}_r(\delta\varphi,\varphi)} & & \downarrow{\scriptstyle \mathrm{ev}(\varphi)} \\
C & \xrightarrow[f]{} & D & \xrightarrow[e]{} & \mathcal{C}(f) & \xrightarrow[\partial]{} & \Sigma C
\end{array}$$

where $\Phi_h(\delta\varphi, \varphi)$ is as chain homotopy between $\mathrm{ev}_r(\delta\varphi, \varphi) \circ e^*$ and $e \circ \mathrm{ev}_l(\delta\varphi, \varphi)$ obtained using $j \otimes j$.

Poincaré pairs

Definition

An $(n+1)$-dimensional symmetric algebraic Poincaré pair (SAPP) over R is a symmetric pair $(f\colon C \to D, (\delta\varphi, \varphi))$ such that

$$\mathrm{ev}_l(\delta\varphi, \varphi)\colon D^{n+1-*} \to \mathcal{C}(f)$$

is a chain equivalence.

An $(n+1)$-dimensional quadratic algebraic Poincaré pair (QAPP) over R is a quadratic pair $(f\colon C \to D, (\delta\psi, \psi))$ such that

$$\mathrm{ev}_l(1+T)\cdot(\delta\psi, \psi)\colon D^{n+1-*} \to \mathcal{C}(f)$$

is a chain equivalence.

Cobordisms

Definition

A cobordism of n-dimensional SAPCs $(C, \varphi), (C', \varphi')$ over R is an $(n+1)$-dimensional SAPP over R

$$((f\ f') \colon C \oplus C' \to E, (\delta\varphi, \varphi \oplus -\varphi'))$$

A cobordism of n-dimensional QAPCs $(C, \psi), (C', \psi')$ over R is an $(n+1)$-dimensional QAPP over R

$$((f\ f') \colon C \oplus C' \to E, (\delta\psi, \psi \oplus -\psi'))$$

Proposition

Cobordism is an equivalence relation on SAPCs and QAPCs. Homotopy equivalent SAPCs (QAPCs) are cobordant.

L-groups

Definition

The symmetric L-groups of R are

$$L^n(R)_{chain} := \{\text{cobordism classes of } n\text{-dimensional SAPCs in } R\}$$

The quadratic L-groups of R are

$$L_n(R)_{chain} := \{\text{cobordism classes of } n\text{-dimensional QAPCs in } R\}$$

The group operation is the direct sum.
The inverse of (C, φ) is $(C, -\varphi)$, the inverse of (C, ψ) is $(C, -\psi)$.

Remark

The groups $L_n(R)_{chain}$ are isomorphic to the surgery obstruction groups $L_n(R)$ from Spiros' lectures, but the groups $L^n(R)_{chain}$ are in general not isomorphic to $L^n(R)$.

Wrap-up

A symmetric structure is a cycle $\varphi \in W^{\%}(C)_n = \mathrm{Hom}_{\mathbb{Z}_2}(W, C \otimes C)$.

It is Poincaré if $\mathrm{ev}(\varphi) = \varphi_0 \colon C^{n-*} \xrightarrow{\simeq} C$

The symmetric construction gives $\varphi_X \colon C(X) \to W^{\%}(C(X))$.

A symmetric pair is a cycle $(\delta\varphi, \varphi) \in \mathcal{C}(f^{\%})$.

It is Poincaré if $\mathrm{ev}_r(\delta\varphi, \varphi) \colon C^{n-*} \xrightarrow{\simeq} \mathcal{C}(f)$

A cobordism is a Poincaré pair $((f\ f') \colon C \oplus C' \to D, (\delta\varphi, \varphi \oplus -\varphi'))$.

The L-groups are cobordism groups.

For X an n-GPC we obtain the symmetric signature

$$\mathbf{sign}^{\mathbf{L}^\bullet}(X) = (C(\widetilde{X}), \varphi_{\widetilde{X}}([X])) \in L^n(\mathbb{Z}[\pi_1(X)]).$$